

On the Structure of the Domain of a Symmetric Jump-type Dirichlet Form

by

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Abstract

We characterize the structure of the domain of a pure jump-type Dirichlet form which is given by a Beurling–Deny formula. In particular, we obtain sufficient conditions in terms of the jumping kernel guaranteeing that the test functions are a core for the Dirichlet form and that the form is a Silverstein extension. As an application we show that for recurrent Dirichlet forms the extended Dirichlet space can be interpreted in a natural way as a homogeneous Dirichlet space. For reflected Dirichlet spaces this leads to a simple purely analytic proof that the active reflected Dirichlet space (in the sense of Chen, Fukushima and Kuwae) coincides with the extended active reflected Dirichlet space.

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§1. Introduction

Let $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$ be a closable Markovian (symmetric) form on the space $L^2(\mathbb{R}^d) = L^2(\mathbb{R}^d; dx)$, where dx denotes Lebesgue measure on \mathbb{R}^d . It is well known that the closure $(\mathcal{E}, \mathcal{F})$ with respect to the norm $\sqrt{\mathcal{E}_1(u, u)} := (\mathcal{E}(u, u) + \|u\|_{L^2}^2)^{1/2}$ is a regular Dirichlet form. On the other hand, on the set

$$\mathcal{D}[\mathcal{E}] := \{u \in L^2(\mathbb{R}^d) : \mathcal{E}(u, u) < \infty\}$$

the pair $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ becomes a (not necessarily regular) Dirichlet form on $L^2(\mathbb{R}^d)$.

It is a natural question to ask whether \mathcal{F} and $\mathcal{D}[\mathcal{E}]$ coincide. If $\mathcal{E}(u, v) = \frac{1}{2}\mathcal{D}(u, v) = \frac{1}{2}\langle \nabla u, \nabla v \rangle_{L^2}$ is the classical Dirichlet form, then it is known that

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$\mathcal{F} = W_0^{1,2}(\mathbb{R}^d) = \overline{C_0^\infty(\mathbb{R}^d)}^{\|\cdot\|_{1,2}}$ whereas $\mathcal{D}[\mathcal{E}] = W^{1,2}(\mathbb{R}^d)$. Here $W^{1,2}(\mathbb{R}^d)$ denotes the usual L^2 -Sobolev space of order 1 equipped with the norm $\|u\|_{1,2} := \|\nabla u\|_{L^2} + \|u\|_{L^2}$. Since $W^{1,2}(\mathbb{R}^d) = W_0^{1,2}(\mathbb{R}^d)$ (see e.g. Adams and Hedberg [AH] or Stein [St70]), we have $\mathcal{F} = \mathcal{D}[\mathcal{E}]$.

The identification of the domains has been studied for reflected Dirichlet forms using harmonic functions in [Ch92] or [Kuw02]. In this paper, we will discuss the problem for the jump-type form

$$(1.1) \quad \mathcal{E}(u, v) = \frac{1}{2} \iint_{x \neq y} (u(x) - u(y))(v(x) - v(y)) N(dx, dy),$$

for a measure $N(dx, dy)$. Since the expression under the integral is symmetric in u and v , we can always assume that the measure $N(dx, dy) = N(dy, dx)$ is symmetric too. Recently one of us has investigated this problem for jump measures N which have a symmetric density: $N(dx, dy) = k(x, y) dx dy$. Under a rather restrictive assumption on k , it is shown in [U07] that $\mathcal{F} = \mathcal{D}[\mathcal{E}]$. We will now extend this result to a more general setting.

The structure of the domain of a Dirichlet form has been studied for certain self-adjoint extensions or Markov extensions of the generator associated with the form (see e.g. [RZ, T96, KaT96]). In particular, it has been shown that for local Dirichlet forms and diffusion processes such extensions are trivial.

Our paper is organized as follows: In Section 2, we study the L^2 -maximal domain and Silverstein extensions of the form (1.1); the homogeneous domain and reflected Dirichlet spaces are considered in Section 3, while the active reflected Dirichlet space is introduced in Section 4. The Appendix, Section 5, contains a brief survey on basic elements of the theory of Dirichlet forms.

§2. L^2 -maximal domains

In this section, we formulate our setting and prove one of the main theorems.

Let $\mu(x, dy)$ be a positive kernel on $\mathbb{R}^d \times \mathcal{B}(\mathbb{R}^d)$ which generates on $\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta$, $\Delta = \{(x, x) : x \in \mathbb{R}^d\}$, a symmetric measure $N(dx, dy) := \mu(x, dy) dx$; see Remark 2 below for some comments on this assumption. Consider the following symmetric quadratic form $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ defined on $L^2(\mathbb{R}^d)$:

$$\begin{cases} \mathcal{E}(u, v) := \frac{1}{2} \iint_{x \neq y} (u(x) - u(y))(v(x) - v(y)) N(dx, dy), \\ \mathcal{D}[\mathcal{E}] := \{u \in L^2(\mathbb{R}^d, dx) : \mathcal{E}(u, u) < \infty\}. \end{cases}$$

Set

$$(2.1) \quad \Phi(x) := \int_{0 < |x-y| \leq 1} |x-y|^2 \mu(x, dy) = \int_{0 < |h| \leq 1} |h|^2 \mu(x, dh + x)$$

and

$$(2.2) \quad \Psi(x) := \int_{|x-y|>1} \mu(x, dy) = \int_{|h|>1} \mu(x, dh + x).$$

If $\Psi, \Phi \in L^1_{\text{loc}}(\mathbb{R}^d)$ Example 1.2.4 in [FOT94] shows that $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$ is a closable Markovian (symmetric) form on $L^2(\mathbb{R}^d)$ and that the closure $(\mathcal{E}, \mathcal{F})$ is a regular (symmetric) Dirichlet form. The associated Markov process \mathbf{M} is of pure jump-type.

The assumption that $\Psi, \Phi \in L^1_{\text{loc}}(\mathbb{R}^d)$ is equivalent to saying that the test functions $C_0^\infty(\mathbb{R}^d)$ are contained in the form domain $\mathcal{D}[\mathcal{E}]$ (see Remark 2 at the end of this section).

For $x \in \mathbb{R}^d$ and $A \in \mathcal{B}(\mathbb{R}^d)$, put

$$(2.3) \quad \nu(x, A) := \mu(x, A + x).$$

Obviously, ν defines again a kernel. Note that the jump kernel $\mu(x, B)$ represents the rate of jumps starting from x and jumping into the set B , while $\nu(x, A)$ stands for the rate of jumps of size A starting from x . With this convention we can rewrite the form in the following way: for $u \in \mathcal{F}$,

$$\begin{aligned} \mathcal{E}(u, v) &= \frac{1}{2} \iint_{x \neq y} (u(x) - u(y))(v(x) - v(y)) \mu(x, dy) dx \\ &= \frac{1}{2} \iint_{h \neq 0} (u(x) - u(x+h))(v(x) - v(x+h)) \mu(x, dh + x) dx \\ &= \frac{1}{2} \iint_{h \neq 0} (u(x) - u(x+h))(v(x) - v(x+h)) \nu(x, dh) dx. \end{aligned}$$

We will also need the concept of shift-bounded measures which is common in harmonic analysis (see e.g. [BF]).

Definition 2.1 (locally shift-bounded kernel). A kernel $n(x, dy)$ defined on $\mathbb{R}^d \times \mathcal{B}(\mathbb{R}^d)$ is said to be *locally shift-bounded* if there exists a constant $c > 0$ such that

$$(2.4) \quad n(x+z, A) \leq cn(x, A) \quad \text{for all } x, z \in \mathbb{R}^d \text{ with } |z| \leq 1, A \subset B_1(0),$$

where $B_1(0)$ is the open ball with centre 0 and radius 1.

If $n(x, dy)$ is defined on $D \times \mathcal{B}(E)$ where $D, E \subset \mathbb{R}^d$ we call $n(x, dy)$ *locally shift-bounded* if the trivial extension $\bar{n}(x, dy)$ of $n(x, dy)$ onto $\mathbb{R}^d \times \mathcal{B}(\mathbb{R}^d)$,

$$\bar{n}(x, B) := \begin{cases} n(x, B \cap E) & \text{if } x \in D, \\ 0 & \text{if } x \notin D, \end{cases}$$

is locally shift-bounded.

Examples of locally shift-bounded kernels are kernels which are absolutely continuous with respect to some measure m on \mathbb{R}^d , say $n(x, dh) = n(x, h) m(dh)$, and where the density satisfies

$$0 < c := \inf_{x \in \mathbb{R}^d, |h| \leq 1} n(x, h) \leq \sup_{x \in \mathbb{R}^d, |h| \leq 1} n(x, h) =: C < \infty.$$

Obviously,

$$\frac{n(x+z, A)}{n(x, A)} \leq \frac{Cm(A)}{cm(A)} = \frac{C}{c} < \infty \quad \text{for all } x, z \in B_1(0), A \subset B_1(0).$$

This is, for example, the case if $n(x, dh)$ does not depend on x (i.e. if the underlying stochastic process is a Lévy process) or if the process is Lévy-like in the sense that $c\nu(h) \leq n(x, h) \leq C\nu(h)$ for all $|h| \leq 1$ and where $\nu(dh) := \nu(h) dh$ is the jump measure of some fixed Lévy process.

Let $\rho \in C_0^\infty(\mathbb{R}^d)$ be a smooth function satisfying $\rho(x) = \rho(-x) \geq 0$, $\text{supp } \rho = \overline{B_1(0)}$ and $\int \rho dx = 1$. For $\varepsilon > 0$, set $\rho_\varepsilon(\cdot) := \varepsilon^{-d} \rho(\cdot/\varepsilon)$. Clearly, $\rho_\varepsilon \geq 0$, $\text{supp } \rho_\varepsilon = \overline{B_1(\varepsilon)}$ and $\int \rho_\varepsilon dx = 1$. Denote by $J_\varepsilon[u]$ the *Friedrichs mollifier*, i.e.

$$(2.5) \quad J_\varepsilon[u](x) := u \star \rho_\varepsilon(x) = \int u(x-z) \rho_\varepsilon(z) dz, \quad x \in \mathbb{R}^d.$$

It is well known that $\|J_\varepsilon[u]\|_{L^2} \leq \|u\|_{L^2}$ and $\lim_{\varepsilon \rightarrow 0} \|J_\varepsilon[u] - u\|_{L^2} = 0$ for all $u \in L^2$.

Lemma 2.2. *Assume that $\Phi \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $\Psi \in L^\infty(\mathbb{R}^d)$ and that the kernel ν is locally shift-bounded. Then there exists a constant $c > 0$ such that, for all $u \in \mathcal{D}[\mathcal{E}]$,*

$$\mathcal{E}(J_\varepsilon[u], J_\varepsilon[u]) \leq c\mathcal{E}(u, u) + 4\|u\|_{L^2}^2 \cdot \|\Psi\|_{L^\infty}.$$

Proof. We split the integral of the form $\mathcal{E}(J_\varepsilon[u], J_\varepsilon[u])$ into two parts:

$$\begin{aligned} \mathcal{E}(J_\varepsilon[u], J_\varepsilon[u]) &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} (J_\varepsilon[u](x) - J_\varepsilon[u](y))^2 \mu(x, dy) dx \\ &= \left(\iint_{|x-y| \leq 1} + \iint_{|x-y| > 1} \right) (J_\varepsilon[u](x) - J_\varepsilon[u](y))^2 \mu(x, dy) dx \\ &=: \text{(I)} + \text{(II)}. \end{aligned}$$

We will estimate the two expressions separately. Since $\rho_\varepsilon(z) dz$ is a probability measure, Jensen's inequality and Tonelli's theorem yield

$$\begin{aligned}
(\text{I}) &= \iint_{|x-y|\leq 1} \left(\int (u(x-z) - u(y-z)) \rho_\varepsilon(z) dz \right)^2 \mu(x, dy) dx \\
&\leq \int \left(\iint_{|x-y|\leq 1} (u(x-z) - u(y-z))^2 \mu(x, dy) dx \right) \rho_\varepsilon(z) dz \\
&= \int \left(\iint_{|h|\leq 1} (u(x) - u(x+h))^2 \mu(x+z, dh+x+z) dx \right) \rho_\varepsilon(z) dz \\
&= \int \left(\iint_{|h|\leq 1} (u(x) - u(x+h))^2 \nu(x+z, dh) dx \right) \rho_\varepsilon(z) dz.
\end{aligned}$$

Since $\text{supp } \rho_\varepsilon \subset B_1(0)$ is compact and ν is locally shift-bounded we see that

$$\begin{aligned}
(\text{I}) &\leq c \int \left(\iint_{|h|\leq 1} (u(x) - u(x+h))^2 \nu(x, dh) dx \right) \rho_\varepsilon(z) dz \\
&= c \iint_{|h|\leq 1} (u(x) - u(x+h))^2 \nu(x, dh) dx \leq c\mathcal{E}(u, u).
\end{aligned}$$

As for the term (II), we see that

$$\begin{aligned}
(\text{II}) &\leq 2 \iint_{|x-y|\geq 1} ((J_\varepsilon[u](x))^2 + (J_\varepsilon[u](y))^2) \mu(x, dy) dx \\
&= 4 \int (J_\varepsilon[u](x))^2 \left(\int_{|x-y|\geq 1} \mu(x, dy) \right) dx \\
&\leq 4\|\Psi\|_{L^\infty} \int (J_\varepsilon[u](x))^2 dx = 4\|\Psi\|_{L^\infty} \cdot \|J_\varepsilon[u]\|_{L^2}^2 \leq 4\|\Psi\|_{L^\infty} \cdot \|u\|_{L^2}^2.
\end{aligned}$$

If we combine both calculations, our claim follows. \square

Remark. With some simple modifications in the proof of Lemma 2.2 we can show that $\mathcal{E}(J_\varepsilon[u], J_\varepsilon[u]) \leq c(K)\mathcal{E}(u, u)$ for all $u \in \mathcal{F}$ with $\text{supp } u \subset K$ for some fixed compact set K . In this case, we do not need to split the integral into two parts (I) + (II), but we can estimate the whole expression directly. This means that we only have to assume $\Phi, \Psi \in L^1_{\text{loc}}$.

If we require that (2.4) holds for all $A \subset \mathbb{R}^d$ and not just for $A \subset B_1(0)$, the above modification of the proof of Lemma 2.2 gives $\mathcal{E}(J_\varepsilon[u], J_\varepsilon[u]) \leq c\mathcal{E}(u, u)$ for all $u \in \mathcal{F}$. Again, we only need to assume that $\Phi, \Psi \in L^1_{\text{loc}}$.

Lemma 2.3. For $\varepsilon > 0$, define a real function T_ε as follows:

$$T_\varepsilon(x) := (-1/\varepsilon) \vee (x - (-\varepsilon) \vee x \wedge \varepsilon) \wedge 1/\varepsilon, \quad x \in \mathbb{R}.$$

Then T_ε is a normal contraction, i.e. T_ε satisfies

$$|T_\varepsilon(x)| \leq |x| \quad \text{and} \quad |T_\varepsilon(x) - T_\varepsilon(y)| \leq |x - y|, \quad x, y \in \mathbb{R}.$$

Moreover, for any $x \in \mathbb{R}$, $T_\varepsilon(x)$ converges to x as $\varepsilon \rightarrow 0$.

Proof. The second assertion is obvious from the definition of T_ε . So we show the first assertion. Let $f(x) = x$ and take $\varepsilon > 0$. Then it is easy to see that the functions $f_\varepsilon := (-\varepsilon) \vee f \wedge \varepsilon$ and $f - f_\varepsilon$ are normal contractions. Hence the lemma follows from the fact that the composition of two normal contractions is again a normal contraction. \square

We can now show the main result of this section.

Theorem 2.4. *Assume that $\Phi \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $\Psi \in L^\infty(\mathbb{R}^d)$ hold and that the kernel ν is locally shift-bounded. Then*

$$\mathcal{D}[\mathcal{E}] = \{u \in L^2(\mathbb{R}^d) : \mathcal{E}(u, u) < \infty\} = \mathcal{F};$$

this means that every element from $\mathcal{D}[\mathcal{E}]$ can be approximated by a sequence of functions in $C_0^\infty(\mathbb{R}^d)$ with respect to \mathcal{E}_1 .

Proof. For $u \in \mathcal{D}[\mathcal{E}]$ we define the Friedrichs mollifier $J_\varepsilon[u] = u \star \rho_\varepsilon$ as in (2.5). Note that $J_\varepsilon[u] \in C_\infty(\mathbb{R}^d)$ is a continuous function vanishing at infinity. Indeed, for the Fourier transform $\mathcal{F}f$ we know that $\mathcal{F}u \in L^2(\mathbb{R}^d)$ and $\mathcal{F}\rho_\varepsilon \in \mathcal{S} \subset L^2(\mathbb{R}^d)$; here \mathcal{S} denotes the Schwartz space of rapidly decreasing functions. By the convolution theorem

$$\mathcal{F}(J_\varepsilon[u]) = \mathcal{F}(u \star \rho_\varepsilon) = \mathcal{F}u \cdot \mathcal{F}\rho_\varepsilon \in L^1(\mathbb{R}^d)$$

and the Riemann–Lebesgue lemma tells us that $J_\varepsilon[u] = \mathcal{F}^{-1}\mathcal{F}(J_\varepsilon[u]) \in C_\infty(\mathbb{R}^d)$.

Since $J_\varepsilon[u]$ vanishes at infinity, $T_\varepsilon(J_\varepsilon[u])$ has compact support, and therefore $w_\varepsilon := J_\varepsilon[T_\varepsilon(J_\varepsilon[u])] \in C_0^\infty(\mathbb{R}^d) \subset \mathcal{F}$. Moreover, $L^2\text{-}\lim_{\varepsilon \rightarrow 0} w_\varepsilon = u$ since

$$\begin{aligned} \|w_\varepsilon - u\|_{L^2} &\leq \|J_\varepsilon[T_\varepsilon(J_\varepsilon[u])] - J_\varepsilon[T_\varepsilon(u)]\|_{L^2} + \|J_\varepsilon[T_\varepsilon(u)] - J_\varepsilon[u]\|_{L^2} + \|J_\varepsilon[u] - u\|_{L^2} \\ &\leq \|T_\varepsilon(J_\varepsilon[u]) - T_\varepsilon(u)\|_{L^2} + \|T_\varepsilon(u) - u\|_{L^2} + \|J_\varepsilon[u] - u\|_{L^2} \\ &\leq 2\|J_\varepsilon[u] - u\|_{L^2} + \|T_\varepsilon(u) - u\|_{L^2} \xrightarrow{\varepsilon \rightarrow 0} 0. \end{aligned}$$

The second estimate follows from Jensen's inequality since $\rho_\varepsilon(z) dz$ is a probability measure, while the third inequality results from the normal contraction property of T_ε . For the limits we use the fact that the Friedrichs mollifier converges to u in L^2 and, for the second expression, we use the dominated convergence theorem.

According to Lemma 2.2 we see

$$\mathcal{E}(w_\varepsilon, w_\varepsilon) \leq c\mathcal{E}(T_\varepsilon(J_\varepsilon[u]), T_\varepsilon(J_\varepsilon[u])) + 4\|T_\varepsilon(J_\varepsilon[u])\|_{L^2}^2 \|\Psi\|_{L^\infty}.$$

Since T_ε is a normal contraction, we get

$$\mathcal{E}(T_\varepsilon(J_\varepsilon[u]), T_\varepsilon(J_\varepsilon[u])) \leq \mathcal{E}(J_\varepsilon[u], J_\varepsilon[u])$$

and

$$\|T_\varepsilon(J_\varepsilon[u])\|_{L^2} \leq \|J_\varepsilon[u]\|_{L^2} \leq \|u\|_{L^2}.$$

A further application of Lemma 2.2 shows

$$\mathcal{E}(w_\varepsilon, w_\varepsilon) \leq c(c\mathcal{E}(u, u) + 4\|u\|_{L^2}^2\|\Psi\|_{L^\infty}) + 4\|u\|_{L^2}^2\|\Psi\|_{L^\infty}.$$

This means that the family $\{\mathcal{E}(w_\varepsilon, w_\varepsilon)\}_{\varepsilon>0}$, hence $\{\mathcal{E}_1(w_\varepsilon, w_\varepsilon)\}_{\varepsilon>0}$, is uniformly bounded. Therefore, we can use the Banach–Saks theorem to deduce that for a subsequence $\{\varepsilon(n)\}_{n \in \mathbb{N}}$ with $\lim_{n \rightarrow \infty} \varepsilon(n) = 0$ the Cesàro means

$$\frac{1}{n} \sum_{k=1}^n w_{\varepsilon(k)}$$

converge to a function $\tilde{u} \in \mathcal{F}$ with respect to $\sqrt{\mathcal{E}_1}$ and, in particular, in L^2 . On the other hand, we know that w_ε , hence any subsequence and any convex combination, converges to $u \in L^2(\mathbb{R}^d)$. Since L^2 -limits are unique, we conclude that $u = \tilde{u} \in \mathcal{F}$. \square

Recall that a symmetric Dirichlet form $(\eta, \mathcal{D}[\eta])$ on $L^2(\mathbb{R}^d)$ is said to be an *extension* of the Dirichlet form $(\mathcal{E}, \mathcal{F})$ if $\mathcal{D}[\eta] \supset \mathcal{F}$ and if $\eta(u, u) = \mathcal{E}(u, u)$ whenever $u \in \mathcal{F}$. By $\mathcal{A}(\mathcal{E}, \mathcal{F})$ we denote the family of all possible extensions of the form $(\mathcal{E}, \mathcal{F})$. Clearly, $(\mathcal{E}, \mathcal{D}[\mathcal{E}]) \in \mathcal{A}(\mathcal{E}, \mathcal{F})$. An element $(\eta, \mathcal{D}[\eta])$ of $\mathcal{A}(\mathcal{E}, \mathcal{F})$ is called a *Silverstein extension* if \mathcal{F}_b is an algebraic ideal in $\mathcal{D}[\eta]_b$. (The subscript b indicates that we consider only bounded elements of the respective set.) Most papers dealing with Silverstein extensions of Dirichlet forms consider only local Dirichlet forms (see e.g. [T96]).

The following theorem is, in an abstract setting of regular Dirichlet forms, contained in Kuwae [Kuw02, §5]. Since that paper is quite technical, we give a very short alternative proof based on our techniques. Note that our assumptions entail regularity.

Theorem 2.5. *Assume that $\Phi \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $\Psi \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ and that the kernel ν is locally shift-bounded. Then the Dirichlet form $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ is a Silverstein extension of the form $(\mathcal{E}, \mathcal{F})$, i.e. \mathcal{F}_b is an ideal in $\mathcal{D}[\mathcal{E}]_b$.*

Proof. As in [U07], it is enough to show that $u \cdot \varphi \in \mathcal{F}_b$ whenever $u \in \mathcal{D}[\mathcal{E}]_b$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$. Using the fact that u is bounded and φ is compactly supported, we see that $u\varphi$ belongs to $L^2(\mathbb{R}^d)$, $J_\varepsilon[u\varphi]$ belongs to $C_0^\infty(\mathbb{R}^d) \subset \mathcal{F}$ and $J_\varepsilon[u\varphi]$ converges to $u\varphi$ in $L^2(\mathbb{R}^d)$. As before $J_\varepsilon[u\varphi] = (u\varphi) \star \rho_\varepsilon$ is the Friedrichs mollifier.

We will prove that the family $\{\mathcal{E}(J_\varepsilon[u\varphi], J_\varepsilon[u\varphi])\}_{0 < \varepsilon < 1}$ is uniformly bounded. Indeed, let us denote by K the compact support of φ , $K := \text{supp } \varphi$. Then using an

estimate similar to the one in Lemma 2.2 and putting $K_1 := \{x + y \in \mathbb{R}^d : x \in K, |y| < 1\}$, we see that

$$\mathcal{E}(J_\varepsilon[u\varphi], J_\varepsilon[u\varphi]) \leq c\mathcal{E}(u\varphi, u\varphi) + 4\|u\|_{L^\infty}^2 \|\varphi\|_{L^2}^2 \|\Psi\|_{L^\infty(K_1)}$$

and

$$\begin{aligned} \mathcal{E}(u\varphi, u\varphi) &= \iint_{x \neq y} (u(x)\varphi(x) - u(y)\varphi(y))^2 \mu(x, dy) dx \\ &\leq 2 \iint_{x \neq y} u(x)^2 (\varphi(x) - \varphi(y))^2 \mu(x, dy) dx \\ &\quad + 2 \iint_{x \neq y} \varphi(y)^2 (u(x) - u(y))^2 \mu(x, dy) dx \\ &\leq 2\|u\|_{L^\infty}^2 \mathcal{E}(\varphi, \varphi) + 2\|\varphi\|_{L^\infty}^2 \mathcal{E}(u, u). \end{aligned}$$

Since $\{\mathcal{E}(J_\varepsilon[u\varphi], J_\varepsilon[u\varphi])\}_{0 < \varepsilon < 1}$ is uniformly bounded, we may argue as in the second part of the proof of Theorem 2.4 and take a subsequence of $\{J_{\varepsilon(k)}[u\varphi]\}_{k \in \mathbb{N}}$, for which $\varepsilon(k)$ goes to 0 as $k \rightarrow \infty$, such that the Cesàro means converge in $\sqrt{\mathcal{E}_1}$ to some $v \in \mathcal{F}$. Because of the uniqueness of L^2 -limits $v = u\varphi$ and the proof is then complete. \square

We will now consider the case where the Dirichlet form is defined on an arbitrary open set $D \subset \mathbb{R}^d$. By λ_D we denote Lebesgue measure on D and we assume that $N_D(dx, dy) := \mu_D(x, dy) \lambda_D(dx)$ is a symmetric measure on $D \times D \setminus \Delta$. Set

$$(2.6) \quad \begin{cases} \mathcal{E}_D(u, v) := \iint_{D \times D \setminus \Delta} (u(x) - u(y))(v(x) - v(y)) \mu(x, dy) \lambda_D(dx), \\ \mathcal{D}[\mathcal{E}_D] := \{u \in L^2(D, \lambda_D) : \mathcal{E}_D(u, u) < \infty\}. \end{cases}$$

Define, with the obvious changes, Φ and Ψ as in (2.1) and (2.2). If $\Phi, \Psi \in L^1_{\text{loc}}(D)$, then $(\mathcal{E}_D, \mathcal{D}[\mathcal{E}_D])$ is a Dirichlet form on $L^2(D, \lambda_D) (= L^2(\bar{D}, \lambda_D))$. As usual, \bar{D} is the closure of D , $C_0^\infty(D) = \{u \in C_0^\infty(\mathbb{R}^d) : \text{supp } u \subset D\}$, and $C_0^\infty(\bar{D}) = \{u|_{\bar{D}} : u \in C_0^\infty(\mathbb{R}^d)\}$.

We write $\mathcal{F}_{\bar{D}}$ (resp. \mathcal{F}_D^0) for the closure of $C_0^\infty(\bar{D})$ (resp. $C_0^\infty(D)$) with respect to $\mathcal{E}_{D,1}(u) := \mathcal{E}_D(u, u) + \|u\|_{L^2(D)}^2$. Clearly, $\mathcal{F}_{\bar{D}}, \mathcal{F}_D^0 \subset \mathcal{D}[\mathcal{E}_D]$ and $(\mathcal{E}_D, \mathcal{F}_{\bar{D}})$ (resp. $(\mathcal{E}_D^0, \mathcal{F}_D^0)$) are regular symmetric Dirichlet forms on $L^2(\bar{D}, \lambda_D)$ (resp. $L^2(D, \lambda_D)$), where \mathcal{E}_D^0 denotes the restriction of \mathcal{E}_D to $\mathcal{F}_D^0 \times \mathcal{F}_D^0$. Moreover, according to Theorem 4.4.3 in [FOT94], we have the identity

$$\mathcal{F}_D^0 = \{u \in \mathcal{F}_{\bar{D}} : \tilde{u} = 0, \mathcal{E}_D\text{-quasi everywhere on } \partial D\},$$

where \tilde{u} denotes the \mathcal{E}_D -quasi-continuous modification of $u \in \mathcal{F}_{\bar{D}}$.

We are interested in the relation of $\mathcal{D}[\mathcal{E}_D]$ and $\mathcal{F}_{\bar{D}}$. For this, we extend a function u defined on D to the whole space \mathbb{R}^d by setting $u = 0$ on $\mathbb{R}^d \setminus D$.

Proposition 2.6. *Let $(\mathcal{E}_D, \mathcal{D}[\mathcal{E}_D])$ be as in (2.6) where $D \subset \mathbb{R}^d$ is an open set. Assume that $\Phi \in L^1_{\text{loc}}(D)$ and $\Psi \in L^\infty(D)$ hold and that the kernel $\nu(x, A) := \mu(x, A + x)$, $x \in D$, $B \in \mathcal{B}(D)$, is locally shift-bounded. Then*

$$\mathcal{D}[\mathcal{E}_D] := \{u \in L^2(D, \lambda_D) : \mathcal{E}_D(u, u) < \infty\} = \mathcal{F}_{\bar{D}}.$$

Proof. Note that $\mathcal{D}[\mathcal{E}_D] = \{u \in L^2(\bar{D}, \lambda_D) : \mathcal{E}_D(u, u) < \infty\}$ since $L^2(D; \lambda_D) = L^2(\bar{D}; \lambda_D)$. Let $u \in \mathcal{D}[\mathcal{E}_D]$ and extend u by zero outside of \bar{D} ; in particular $u \in L^2(\mathbb{R}^d)$. Consider the mollifier $J_\varepsilon[u]$ as in (2.5). As in the proof of Theorem 2.4, we see $w_\varepsilon = J_\varepsilon[T_\varepsilon(J_\varepsilon[u])] \in C_0^\infty(\mathbb{R}^d)$, hence $w_\varepsilon|_{\bar{D}} \in C_0^\infty(\bar{D})$. We also see that $w_\varepsilon|_{\bar{D}}$ converges to u in $L^2(D, \lambda_D)$ and the family $\{\mathcal{E}_D(w_\varepsilon|_{\bar{D}}, w_\varepsilon|_{\bar{D}})\}_{0 < \varepsilon < 1}$ is uniformly bounded. The remaining part of the proof is now exactly as in Theorem 2.4. \square

Example 2.7 (Censored stable process in an open set; [BBC03]). Let $0 < \alpha < 2$ and $\mu(x, dy) = c_{d,\alpha}|x - y|^{-d-\alpha}dy$ where $c_{d,\alpha}$ is a positive constant depending on d and α . In this case, the assumptions of Proposition 2.6 are satisfied. Therefore,

$$\begin{cases} \mathcal{E}_D(u, v) = c_{d,\alpha} \iint_{D \times D \setminus \Delta} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+\alpha}} dx dy \\ \mathcal{D}[\mathcal{E}_D] = \{u \in L^2(D) : \mathcal{E}_D(u, u) < \infty\} \end{cases}$$

is a regular Dirichlet form on $L^2(\bar{D})$. The corresponding stochastic process is called a *censored stable process* on D and $\mathcal{D}[\mathcal{E}_D]$ is also obtained by taking the $\mathcal{E}_{D,1}$ -closure of $C_0^\infty(\bar{D})$. Moreover $\mathcal{D}[\mathcal{E}_D]$ is the *active reflected Dirichlet space* (see Section 4).

We conclude this section with a few remarks on our assumptions on $N(dx, dy)$.

Remark. Often the Dirichlet form \mathcal{E} is given in terms of its Beurling–Deny representation (1.1) with a jump kernel $N(dx, dy)$ which is not necessarily given as $\mu(x, dy) dx$. This is, e.g., the case when we start with a stochastic process admitting a Lévy system. Since the integrand in (1.1) is symmetric, we can always assume that $N(dx, dy)$ is symmetric. In order to make (1.1) convergent, one usually requires that

$$(2.7) \quad \iint_{K \times K} |x - y|^2 N(dx, dy) + \iint_{K \times L^c} N(dx, dy) < \infty, \quad \begin{array}{l} \text{for all } K \subset \mathring{L} \subset \mathbb{R}^d, \\ K, L \text{ compact.} \end{array}$$

Obviously, this is equivalent to saying that

$$(2.8) \quad \iint_{K \times \mathbb{R}^d} (|x - y|^2 \wedge 1) N(dx, dy) < \infty \quad \text{for all compact sets } K \subset \mathbb{R}^d.$$

Consider now the finite measure $M(dx, dy) := (|x - y|^2 \wedge 1)N(dx, dy)$ and set

$$m(K) := M(K \times \mathbb{R}^d) = \iint_{K \times \mathbb{R}^d} (|x - y|^2 \wedge 1)N(dx, dy), \quad K \subset \mathbb{R}^d \text{ compact.}$$

By a standard technique (cf. [EK, Appendix 8] or [K, Chapter 5]), we can disintegrate the bi-measure M and find

$$M(dx, dy) = \mu(x, dy) m(dx) \text{ and, by symmetry, } M(dx, dy) = \mu(y, dx) m(dy).$$

Thus,

$$N(dx, dy) = \frac{1}{|x - y|^2 \wedge 1} \mu(x, dy) m(dx)$$

and we have $m(dx) \ll dx$ —which allows us to use Lebesgue measure as reference measure—if, and only if, $M(dx \times \mathbb{R}^d) \ll dx$, i.e. if

$$\text{Leb}(X) = 0 \Rightarrow \iint_{(K \cap X) \times \mathbb{R}^d} (|x - y|^2 \wedge 1) N(dx, dy) = 0 \text{ for all compact } K \subset \mathbb{R}^d.$$

This is clearly equivalent to saying that $N((K \cap X) \times \mathbb{R}^d) = 0$ for all compact sets $K \subset \mathbb{R}^d$.

Finally, (2.8) or (2.7) is the same as $\Phi, \Psi \in L^1_{\text{loc}}$ since

$$\begin{aligned} \Phi(x) + \Psi(x) &= \int_{|x-y| \leq 1} |x - y|^2 \mu(x, dy) + \int_{|x-y| > 1} \mu(x, dy) \\ &= \int_{\mathbb{R}^d} (|x - y|^2 \wedge 1) \mu(x, dy). \end{aligned}$$

In particular, $C_0^\infty(\mathbb{R}^d) \subset \mathcal{D}[\mathcal{E}]$. Conversely, if $C_0^\infty(\mathbb{R}^d) \subset \mathcal{D}[\mathcal{E}]$, we fix any two compact sets $K \subset \mathring{L}$ and we pick $\chi \in C_0^\infty(\mathbb{R}^d)$ such that $\mathbb{1}_K \leq \chi(x) \leq \mathbb{1}_L$. Since the functions $u_j(x) := x_j \chi(x)$, $j = 1, \dots, d$, are of class $C_0^\infty(\mathbb{R}^d)$, and since $\sum_{j=1}^d \mathcal{E}(u_j, u_j) + \mathcal{E}(\chi, \chi) < \infty$, it is easy to deduce (2.7).

§3. Homogeneous domains and reflected Dirichlet spaces

In analogy to (homogeneous) Sobolev spaces we call

$$\dot{\mathcal{D}}[\mathcal{E}] := \{u : \mathbb{R}^d \xrightarrow{\text{measurable}} \mathbb{R} : \mathcal{E}(u, u) < \infty\}$$

the *homogeneous domain* (see [U07]). Strictly speaking, the symbol \mathcal{E} appearing on the right hand side is an extension of the original form. Here we do not need to stress this fact since \mathcal{E} is pure-jump given by the integral expression (1.1) which is *a priori* defined on all measurable functions. It is well known that for the extended Dirichlet space \mathcal{F}_e (cf. [FOT94]),

$$\mathcal{F} = \mathcal{F}_e \cap L^2(\mathbb{R}^d) \subset \mathcal{F}_e \subset \dot{\mathcal{D}}[\mathcal{E}].$$

In general, it is not clear whether $\dot{\mathcal{D}}[\mathcal{E}]$ coincides with \mathcal{F}_e . In [U07], where we assumed the existence of a jump *density*, we obtained a restrictive sufficient condition for $\dot{\mathcal{D}}[\mathcal{E}] = \mathcal{F}_e$ in terms of the density; this condition also entailed that the form is recurrent. In the present context we can give a more practical and more relaxed condition on jump *kernels*.

We begin with a simple lemma which holds for general Dirichlet forms.

Lemma 3.1. *Let X be a locally compact separable metric space and m a positive Radon measure on X with full support. Assume that $(\mathfrak{q}, \mathcal{Q})$ is a regular symmetric Dirichlet form on $L^2(X; m)$ which is recurrent. Then there exists a decreasing sequence $(U^\ell)_{\ell \in \mathbb{N}}$ of open sets with $m(U^\ell) < 1/\ell$ and a sequence $\{\varphi_n\}_{n \in \mathbb{N}} \subset C_0(X) \cap \mathcal{Q}$ so that $\sup_{n \in \mathbb{N}} \mathfrak{q}(\varphi_n, \varphi_n) < \infty$, $0 \leq \varphi_n(x) \leq 1$ for all x , and φ_n converges to 1 uniformly on all sets of the form $K \setminus U^\ell$ where $\ell \in \mathbb{N}$ and $K \subset X$ is compact.*

Proof. Since $(\mathfrak{q}, \mathcal{Q})$ is recurrent, we can find a sequence $\{\psi_n\}_{n \in \mathbb{N}} \subset \mathcal{Q}$ so that

$$\lim_{n \rightarrow \infty} \psi_n = 1 \text{ a.e.} \quad \text{and} \quad \lim_{n \rightarrow \infty} \mathfrak{q}(\psi_n, \psi_n) = 0$$

(see Theorem 1.6.3 in [FOT94]). Denote the exceptional set by N . Since m is a Radon measure, there exists a decreasing sequence of open sets $U^\ell \supset N$ such that $m(U^\ell) < 1/\ell$. Because of the regularity of the Dirichlet form we can assume that the sequence $\{\psi_n\}_{n \in \mathbb{N}}$ is actually from $C_0(X) \cap \mathcal{Q}$. Moreover, we can assume that $0 \leq \psi_n \leq 1$; otherwise we replace ψ_n by $0 \vee \psi_n \wedge 1$ and remark that normal contractions operate on the Dirichlet space $(\mathcal{Q}, \mathfrak{q})$. As $\lim_{n \rightarrow \infty} \mathfrak{q}(\psi_n, \psi_n) = 0$, we can extract a subsequence $\{\psi_{n(k)}\}_{k \in \mathbb{N}}$ satisfying $\mathfrak{q}(\psi_{n(k)}, \psi_{n(k)}) < 2^{-2k}$ for each k . For $k \geq 1$, we define

$$\varphi_k := (\psi_{n(1)} + \cdots + \psi_{n(k)}) \wedge 1.$$

Then $\varphi_k \in C(X) \cap \mathcal{Q}$, $0 \leq \varphi_k \leq 1$ and $\sup_{k \in \mathbb{N}} \varphi_k = 1$. Moreover,

$$\begin{aligned} \mathfrak{q}(\varphi_k, \varphi_k) &\leq \mathfrak{q}(\psi_{n(1)} + \cdots + \psi_{n(k)}, \psi_{n(1)} + \cdots + \psi_{n(k)}) \\ &\leq \sum_{\ell=1}^k 2^\ell \mathfrak{q}(\psi_{n(\ell)}, \psi_{n(\ell)}) \leq \sum_{\ell=1}^k 2^{-\ell} < \infty \end{aligned}$$

where we used the contraction property of Dirichlet forms and, repeatedly, the estimate $\mathfrak{q}(f+g, f+g) \leq 2\mathfrak{q}(f, f) + 2\mathfrak{q}(g, g)$. Since $\{1 - \varphi_k \mathbb{1}_{X \setminus U^\ell}\}_{k \in \mathbb{N}}$ is a decreasing sequence of upper semicontinuous functions with limit 0, we can use Dini's theorem (cf. [R, pp. 195–196]) to conclude that the convergence $\varphi_k \xrightarrow{k \rightarrow \infty} 1$ is uniform on all sets of the form $K \setminus U^\ell$ where $K \subset X$ is compact. \square

Theorem 3.2. *Assume that $\Phi \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $\Psi \in L^\infty(\mathbb{R}^d)$ and that the kernel ν is locally shift-bounded. Assume further that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is recurrent. Then the homogeneous domain $\dot{\mathcal{D}}[\mathcal{E}]$ coincides with the extended Dirichlet space \mathcal{F}_e .*

Proof. Let $(U^\ell)_{\ell \in \mathbb{N}}$ be the decreasing sequence of open sets from the lemma above. Denote $V^\ell := X \setminus U^\ell$. Note that $\dot{\mathcal{D}}[\mathcal{E}]_b = \{u : \mathbb{R}^d \xrightarrow{\text{m}^\text{ble}} \mathbb{R} : \mathcal{E}(u, u) < \infty\} \cap L^\infty(\mathbb{R}^d)$ is dense in $\dot{\mathcal{D}}[\mathcal{E}]$ with respect to the seminorm $\sqrt{\mathcal{E}}$.

Fix $u \in \dot{\mathcal{D}}[\mathcal{E}]_b$. It is not hard to see that $\varphi \cdot u \in \dot{\mathcal{D}}[\mathcal{E}]_b \cap L^2(\mathbb{R}^d)$ for all $\varphi \in C_0^\infty(\mathbb{R}^d)$. From Theorem 2.4 we know that $u \cdot \varphi \in \mathcal{F} = \mathcal{D}[\mathcal{E}]$.

Since the form $(\mathcal{E}, \mathcal{F})$ is recurrent, Lemma 3.1 guarantees the existence of a nonnegative and bounded sequence $\{\varphi_n\}_{n \in \mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$ such that φ_n converges to 1 uniformly on compact sets in V^ℓ and $\sup_{n \in \mathbb{N}} \mathcal{E}(\varphi_n, \varphi_n) < \infty$. But then $u\varphi_n$ (which is an element in \mathcal{F}) also converges to u uniformly on compact sets in V^ℓ , because the function u is bounded. Since φ_n is bounded by 1, we find

$$\begin{aligned} \mathcal{E}(u\varphi_n, u\varphi_n) &= \iint_{x \neq y} (u(x)\varphi_n(x) - u(y)\varphi_n(y))^2 \mu(x, dy) dx \\ &\leq 2 \|u\|_{L^\infty}^2 \iint_{x \neq y} (\varphi_n(x) - \varphi_n(y))^2 \mu(x, dy) dx \\ &\quad + 2 \iint_{x \neq y} (u(x) - u(y))^2 \mu(x, dy) dx \\ &= 2 \|u\|_{L^\infty}^2 \mathcal{E}(\varphi_n, \varphi_n) + 2\mathcal{E}(u, u). \end{aligned}$$

This means that $\mathcal{E}(u\varphi_n, u\varphi_n)$ is uniformly bounded. Thus, the sequence

$$\bar{w}_n(x, y) := u(x)\varphi_n(x) - u(y)\varphi_n(y), \quad x, y \in \mathbb{R}^d,$$

is a bounded sequence in $L^2(\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta; \mu(x, dy) dx)$. An application of the Banach–Saks theorem shows that there is a subsequence $\{\bar{w}_{n(k)}\}_{k \in \mathbb{N}}$ such that the convex combinations $k^{-1} \sum_{\ell=1}^k \bar{w}_{n(\ell)}$ converge to some element $\bar{w} \in L^2(\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta; \mu(x, dy) dx)$. On the other hand, the sequence $\{k^{-1} \sum_{\ell=1}^k \varphi_{n(\ell)}\}_{k \in \mathbb{N}}$ converges to 1 uniformly on compact sets in V^ℓ , and we know that for $\mu(x, dy) dx$ -a.a. $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \Delta$,

$$\begin{aligned} \frac{1}{k} \sum_{\ell=1}^k \bar{w}_{n(\ell)}(x, y) &= \frac{1}{k} \sum_{\ell=1}^k (u(x)\varphi_{n(\ell)}(x) - u(y)\varphi_{n(\ell)}(y)) \\ &= u(x) \left(\frac{1}{k} \sum_{\ell=1}^k \varphi_{n(\ell)}(x) \right) - u(y) \left(\frac{1}{k} \sum_{\ell=1}^k \varphi_{n(\ell)}(y) \right). \end{aligned}$$

Therefore, we get, for any compact set $K \subset V^\ell$,

$$\begin{aligned}
& \iint_{K \times K \setminus \Delta} (\bar{w}(x, y) - (u(x) - u(y)))^2 \mu(x, dy) dx \\
&= \iint_{K \times K \setminus \Delta} \left(\bar{w}(x, y) - \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{\ell=1}^k \bar{w}_{n(\ell)}(x, y) \right)^2 \mu(x, dy) dx \\
&= \lim_{k \rightarrow \infty} \iint_{K \times K \setminus \Delta} \left(\bar{w}(x, y) - \frac{1}{k} \sum_{\ell=1}^k \bar{w}_{n(\ell)}(x, y) \right)^2 \mu(x, dy) dx \\
&\leq \lim_{k \rightarrow \infty} \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} \left(\bar{w}(x, y) - \frac{1}{k} \sum_{\ell=1}^k \bar{w}_{n(\ell)}(x, y) \right)^2 \mu(x, dy) dx = 0
\end{aligned}$$

where the second equality follows from dominated convergence. Since $\ell \in \mathbb{N}$ was arbitrary, this estimate proves that

$$\bar{w}(x, y) = u(x) - u(y) =: \bar{u}(x, y) \quad \mu(x, dy) \text{ dx-a.e.}$$

Set $\psi_k(x) = u(x) \cdot (k^{-1} \sum_{\ell=1}^k \varphi_{n(\ell)}(x))$, $x \in \mathbb{R}^d$, for each k . Then $\psi_k \in \mathcal{F}$ and ψ_k converges to u a.e. In order to show that $u \in \mathcal{F}_e$ it is enough to prove that $\{\psi_k\}_{k \in \mathbb{N}}$ is an $\sqrt{\mathcal{E}}$ -Cauchy sequence. Now

$$\begin{aligned}
\mathcal{E}(\psi_k - \psi_{k'}, \psi_k - \psi_{k'}) &= \iint_{x \neq y} ((\psi_k(x) - \psi_{k'}(x)) - (\psi_k(y) - \psi_{k'}(y)))^2 \mu(x, dy) dx \\
&= \iint_{x \neq y} \left(\frac{1}{k} \sum_{\ell=1}^k \bar{w}_{n(\ell)}(x, y) - \frac{1}{k'} \sum_{\ell=1}^{k'} \bar{w}_{n(\ell)}(x, y) \right)^2 \mu(x, dy) dx \\
&\leq \left(\sqrt{\iint_{x \neq y} \left(\frac{1}{k} \sum_{\ell=1}^k \bar{w}_{n(\ell)}(x, y) - \bar{u}(x, y) \right)^2 \mu(x, dy) dx} \right. \\
&\quad \left. + \sqrt{\iint_{x \neq y} \left(\bar{u}(x, y) - \frac{1}{k'} \sum_{\ell=1}^{k'} \bar{w}_{n(\ell)}(x, y) \right)^2 \mu(x, dy) dx} \right)^2 \\
&\xrightarrow{k, k' \rightarrow \infty} 0.
\end{aligned}$$

This proves $u \in \mathcal{F}_e$. □

Theorem 3.2 can also be shown by using a recent characterization of reflected Dirichlet spaces due to Chen and Fukushima [CF11].

As usual we write $u \in \mathcal{F}_{\text{loc}}$ if for every relatively compact, open set $G \Subset \mathbb{R}^d$ there exists some $u_G \in \mathcal{F}$ such that $u|_G = u_G|_G$. Using the Beurling–Deny representation of the (quasi-)regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ we can extend \mathcal{E} to \mathcal{F}_{loc}

by

$$\tilde{\mathcal{E}}(u, u) := \sup_{G \in \mathbb{R}^d} \frac{1}{2} \iint_{G \times G \setminus \Delta} (u_G(x) - u_G(y))^2 N(dx, dy)$$

and we define the *reflected Dirichlet space* as

$$\mathcal{F}^{\text{ref}} := \left\{ u : \mathbb{R}^d \xrightarrow{\text{measurable}} \mathbb{R} : \begin{array}{l} \Theta_k u \in \mathcal{F}_{\text{loc}} \text{ for each } k \in \mathbb{N} \\ \text{and } \sup_k \tilde{\mathcal{E}}(\Theta_k u, \Theta_k u) < \infty \end{array} \right\},$$

where $\Theta_k u := (-k) \vee (u \wedge k)$.

Fukushima and Chen (compare also Chen [Ch92] and Kuwae [Kuw02] for related, less complete results) prove in [CF11] that for recurrent (quasi-)regular Dirichlet spaces $\mathcal{F}^{\text{ref}} = \mathcal{F}_e$; their method uses techniques from stochastic analysis, it is less direct than our approach, but also more general.

Lemma 3.3. *Assume that $\Phi \in L^1_{\text{loc}}$, $\Psi \in L^\infty$ and that the kernel ν is locally shift-bounded. Then $\mathcal{F}^{\text{ref}} = \dot{\mathcal{D}}[\mathcal{E}]$.*

Proof. Under our assumptions, Theorem 2.4 shows that $(\mathcal{E}, \mathcal{F})$ is regular. This means that \mathcal{F}^{ref} is well-defined.

Since $\Theta_k(x) = (-k) \vee x \wedge k$ is a normal contraction, we find for $u_k := \Theta_k u$ that $\tilde{\mathcal{E}}(u_k, u_k) \leq \tilde{\mathcal{E}}(u_{k+1}, u_{k+1})$ and $\sup_{k \in \mathbb{N}} \tilde{\mathcal{E}}(u_k, u_k) = \lim_{k \rightarrow \infty} \tilde{\mathcal{E}}(u_k, u_k)$. If we use Lebesgue's convergence theorem on the product space $(G \times G \setminus \Delta, N(dx, dy))$ and for any relatively compact open set G , we see that $\tilde{\mathcal{E}}(u, u) = \lim_{k \rightarrow \infty} \tilde{\mathcal{E}}(u_k, u_k)$ and $\mathcal{F}^{\text{ref}} \subset \dot{\mathcal{D}}[\mathcal{E}]$.

Note that this inclusion depends on the representation (1.1) of \mathcal{E} nor on $\Phi \in L^1_{\text{loc}}$ and $\Psi \in L^\infty$.

Now assume that $u \in \dot{\mathcal{D}}[\mathcal{E}]$ and choose for a fixed relatively compact set $G \Subset \mathbb{R}^d$ some $\phi = \phi_G \in C_0^\infty(\mathbb{R}^d)$ such that $\mathbb{1}_G \leq \phi \leq 1$. Then we see as in the proof of Theorem 2.5 that for each $k \in \mathbb{N}$ and $u_k := \Theta_k u$,

$$\begin{aligned} \mathcal{E}(J_\varepsilon[u_k \phi], J_\varepsilon[u_k \phi]) &\leq c\mathcal{E}(u_k \phi, u_k \phi) + 4k^2 \|\Psi\|_{L^\infty} \\ &\leq 2c\mathcal{E}(u_k, u_k) + 2ck^2 \mathcal{E}(\phi, \phi) + 4k^2 \|\Psi\|_{L^\infty} \\ &\leq 2c(\mathcal{E}(u, u) + k^2 \mathcal{E}(\phi, \phi) + 2k^2 \|\Psi\|_{L^\infty}). \end{aligned}$$

This means that the family $\{J_\varepsilon[u_k \phi]\}_{\varepsilon > 0}$ is \mathcal{E}_1 -bounded and, as in the proof of Theorem 2.4, we find a subsequence $\{J_{\varepsilon(\ell)}[u_k \phi]\}_{\ell \in \mathbb{N}}$ whose Cesàro means converge in $\sqrt{\mathcal{E}_1}$ to $u_k \phi$. This shows that $\Theta_k u \cdot \phi \in \mathcal{F}$, hence $u \in \mathcal{F}^{\text{ref}}$. Consequently, $\dot{\mathcal{D}}[\mathcal{E}] \subset \mathcal{F}^{\text{ref}}$, and we see $\dot{\mathcal{D}}[\mathcal{E}] = \mathcal{F}^{\text{ref}}$. \square

§4. The active reflected Dirichlet space and its extension

We will finally study the relation between active reflected Dirichlet spaces and the associated extended reflected spaces. We still assume that $(\mathcal{E}, \mathcal{F})$ is a pure jump Dirichlet form on $L^2(\mathbb{R}^d, m)$ with representation (1.1). It is, however, easy to adapt the definition of the extension $\tilde{\mathcal{E}}$ (just before Lemma 3.3) to the general case including local and killing parts; since this adds nothing new, we refrain from doing so. Let us, nevertheless, begin with an example in the local case. Recall that

$$\mathcal{F}_a^{\text{ref}} := \mathcal{F}^{\text{ref}} \cap L^2(\mathbb{R}^d)$$

is the *active reflected Dirichlet space* (cf. Chen [Ch92]).

Example 4.1 (Chen and Fukushima [CF09]). Consider Brownian motion in \mathbb{R}^3 , its generator $\frac{1}{2}\Delta$ and the classical Dirichlet integral $\mathcal{D}(u, u) = \int (\nabla u)^2 dx$ with Lebesgue measure as reference measure m . Recently Chen and Fukushima [CF09] showed the following characterization of the reflected Dirichlet space in terms of the Beppo-Levi space:

$$\mathcal{F}^{\text{ref}} = BL(\mathbb{R}^3) := \{u \in L^2_{\text{loc}}(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}.$$

By definition, $\mathcal{F}_a^{\text{ref}} = \mathcal{F}^{\text{ref}} \cap L^2(\mathbb{R}^3)$, which is the Sobolev space $W^1(\mathbb{R}^3) = H^1(\mathbb{R}^3)$. Since three-dimensional Brownian motion is transient, $(H_e^1(\mathbb{R}^3), \frac{1}{2}\mathcal{D})$ is a Hilbert space; as such it does not contain the function $x \mapsto 1$ while the space $BL(\mathbb{R}^3)$ clearly contains 1. This shows that, in general, $(\mathcal{F}_a^{\text{ref}})_e \subsetneq \mathcal{F}^{\text{ref}}$.

We will see that Example 4.1 is typical in the sense that one needs some additional condition to ensure $(\mathcal{F}_a^{\text{ref}})_e = \mathcal{F}^{\text{ref}}$.

Note that \mathcal{F}_{loc} is a lattice: fix $G \Subset \mathbb{R}^d$ and $u, v \in \mathcal{F}_{\text{loc}}$. If $u_G, v_G \in \mathcal{F}$ with $u|_G = u_G|_G$ and $v|_G = v_G|_G$ then $u_G \wedge v_G \in \mathcal{F}$ and

$$(u_G \wedge v_G)|_G = u_G|_G \wedge v_G|_G = u|_G \wedge v|_G = (u \wedge v)|_G.$$

Moreover, we find, for $u, v \in \mathcal{F}^{\text{ref}}$,

$$\begin{aligned} \tilde{\mathcal{E}}(u+v, u+v) &= \tilde{\mathcal{E}}((u \vee v) + (u \wedge v), (u \vee v) + (u \wedge v)), \\ \tilde{\mathcal{E}}(u-v, u-v) &\geq \tilde{\mathcal{E}}(|u-v|, |u-v|) = \tilde{\mathcal{E}}((u \vee v) - (u \wedge v), (u \vee v) - (u \wedge v)). \end{aligned}$$

Expanding the expression on either side and adding the resulting (in)equalities yields

$$\tilde{\mathcal{E}}(u, u) + \tilde{\mathcal{E}}(v, v) \geq \tilde{\mathcal{E}}(u \wedge v, u \wedge v) + \tilde{\mathcal{E}}(u \vee v, u \vee v) \geq \tilde{\mathcal{E}}(u \wedge v, u \wedge v).$$

This shows that \mathcal{F}^{ref} is a lattice too.

Clearly, $(\tilde{\mathcal{E}}, \mathcal{F}_a^{\text{ref}} = \mathcal{F}^{\text{ref}} \cap L^2(\mathbb{R}^d))$ is a Dirichlet form extending $(\mathcal{E}, \mathcal{F})$; we denote by \tilde{A} its generator. As usual,

$$\begin{aligned} (\mathcal{F}_a^{\text{ref}})_e &:= \left\{ u : \mathbb{R}^d \xrightarrow{\text{measurable}} \mathbb{R} : \exists (u_k)_{k \in \mathbb{N}} \subset \mathcal{F}_a^{\text{ref}}, \right. \\ &\quad \left. u_k \xrightarrow{\text{a.e.}} u \text{ and } \lim_{j, k \rightarrow \infty} \tilde{\mathcal{E}}(u_k - u_j, u_k - u_j) = 0 \right\} \\ &= \left\{ u : \mathbb{R}^d \xrightarrow{\text{measurable}} \mathbb{R} : \exists (u_k)_{k \in \mathbb{N}} \subset \mathcal{F}_a^{\text{ref}}, \right. \\ &\quad \left. u_k \xrightarrow{\text{a.e.}} u \text{ and } \sup_{k \in \mathbb{N}} \tilde{\mathcal{E}}(u_k, u_k) < \infty \right\}, \end{aligned}$$

denotes the extended Dirichlet space of $(\mathcal{F}_a^{\text{ref}}, \tilde{\mathcal{E}})$. The inclusion ‘ \subset ’ in the second equality is trivial. The converse, ‘ \supset ’ follows from a Banach–Saks argument applied to the seminorm $\tilde{\mathcal{E}}(u_k, u_k) = \|(-\tilde{A})^{1/2} u_k\|_{L^2}^2$.

We can now state the main theorem of this section.

Theorem 4.2. *Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form and let $\mathcal{F}_a^{\text{ref}}$ and $(\mathcal{F}_a^{\text{ref}})_e$ be the active reflected Dirichlet space and the extended active reflected Dirichlet space, respectively. Then $(\mathcal{F}_a^{\text{ref}})_e \subset \mathcal{F}^{\text{ref}}$.*

If $1 \in (\mathcal{F}_a^{\text{ref}})_e$, then the converse inclusion also holds; in particular, $\mathcal{F}^{\text{ref}} = (\mathcal{F}_a^{\text{ref}})_e$.

If $(\mathcal{E}, \mathcal{F})$ or $(\tilde{\mathcal{E}}, \mathcal{F}_a^{\text{ref}})$ is recurrent, then $1 \in (\mathcal{F}_a^{\text{ref}})_e$ and $\mathcal{E}(1, 1) = 0$. This means, in particular, that for recurrent Dirichlet forms the sets $(\mathcal{F}_a^{\text{ref}})_e = \mathcal{F}^{\text{ref}}$ coincide; if we deal with a jump-type Dirichlet form with shift-bounded kernel, then $(\mathcal{F}_a^{\text{ref}})_e = \mathcal{F}^{\text{ref}} = \mathcal{F}_e = \dot{\mathcal{D}}[\mathcal{E}]$.

Proof of Theorem 4.2. We begin with the first inclusion. Let $u \in (\mathcal{F}_a^{\text{ref}})_e$ and let $(u_\ell)_{\ell \in \mathbb{N}} \subset \mathcal{F}_a^{\text{ref}}$ be any approximating sequence such that $u_\ell \xrightarrow[\ell \rightarrow \infty]{\text{a.e.}} u$ and $\sup_{\ell \in \mathbb{N}} \tilde{\mathcal{E}}(u_\ell, u_\ell) < \infty$.

Since $u_\ell \in \mathcal{F}^{\text{ref}}$ we see that $\Theta_k u_\ell \in \mathcal{F}_{\text{loc}}$ for all k . Using the contraction property we get $\tilde{\mathcal{E}}(\Theta_k u_\ell, \Theta_k u_\ell) \leq \tilde{\mathcal{E}}(u_\ell, u_\ell)$, hence

$$(4.1) \quad \sup_{\ell, k \in \mathbb{N}} \tilde{\mathcal{E}}(\Theta_k u_\ell, \Theta_k u_\ell) \leq \sup_{\ell \in \mathbb{N}} \tilde{\mathcal{E}}(u_\ell, u_\ell) < \infty.$$

Therefore, we may use the Banach–Saks theorem to get a subsequence $(u_{\ell(j)})_{j \in \mathbb{N}}$ satisfying

$$(4.2) \quad \frac{1}{n} \sum_{j=1}^n \Theta_k u_{\ell(j)} \xrightarrow[n \rightarrow \infty]{\tilde{\mathcal{E}}} \Theta_k u.$$

In particular, $\Theta_k u \in \mathcal{F}_a^{\text{ref}}$ and $\Theta_m \Theta_k u \in \mathcal{F}_{\text{loc}}$ for all $m \in \mathbb{N}$. For $m \geq k$ this shows $\Theta_k u \in \mathcal{F}_{\text{loc}}$.

From (4.2), a convexity argument and (4.1) we also conclude that

$$\begin{aligned}\tilde{\mathcal{E}}(\Theta_k u, \Theta_k u) &= \lim_{n \rightarrow \infty} \tilde{\mathcal{E}}\left(\sum_{j=1}^n \frac{1}{n} \Theta_k u_{n(j)}, \sum_{j=1}^n \frac{1}{n} \Theta_k u_{n(j)}\right) \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \tilde{\mathcal{E}}(\Theta_k u_{n(j)}, \Theta_k u_{n(j)}) \leq \sup_{\ell \in \mathbb{N}} \tilde{\mathcal{E}}(u_\ell, u_\ell) < \infty.\end{aligned}$$

This shows that $u \in \mathcal{F}^{\text{ref}}$ and $(\mathcal{F}_a^{\text{ref}})_e \subset \mathcal{F}^{\text{ref}}$.

Now we assume that $1 \in (\mathcal{F}_a^{\text{ref}})_e$. This means that there exists a sequence $(\phi_k)_{k \in \mathbb{N}} \subset \mathcal{F}_a^{\text{ref}}$ with $\phi_k \xrightarrow[k \rightarrow \infty]{\text{a.e.}} 1$, $\phi_k \geq 0$ and $\sup_{k \in \mathbb{N}} \tilde{\mathcal{E}}(\phi_k, \phi_k) < \infty$.

For $u \in \mathcal{F}^{\text{ref}} \cap L^\infty$ we set $\psi_k := \|u\|_{L^\infty} \phi_k$. Clearly

$$u^\pm \wedge \psi_k \xrightarrow[k \rightarrow \infty]{\text{a.e.}} u^\pm \quad \text{and} \quad u^\pm \wedge \psi_k \in L^2.$$

Since $u^\pm \wedge \psi_k \in \mathcal{F}_{\text{loc}}$ we conclude that $u^\pm \wedge \psi_k \in \mathcal{F}_a^{\text{ref}}$. Moreover,

$$\begin{aligned}\sup_{k \in \mathbb{N}} \tilde{\mathcal{E}}(u^\pm \wedge \psi_k, u^\pm \wedge \psi_k) &\leq \tilde{\mathcal{E}}(u^\pm, u^\pm) + \sup_{k \in \mathbb{N}} \tilde{\mathcal{E}}(\psi_k, \psi_k) \\ &= \tilde{\mathcal{E}}(u^\pm, u^\pm) + \|u\|_{L^\infty}^2 \tilde{\mathcal{E}}(\phi_k, \phi_k) < \infty,\end{aligned}$$

which proves that $u^\pm \in (\mathcal{F}_a^{\text{ref}})_e$. Therefore, if $1 \in (\mathcal{F}_a^{\text{ref}})_e$, then

$$\mathcal{F}^{\text{ref}} \cap L^\infty \subset (\mathcal{F}_a^{\text{ref}})_e \subset \mathcal{F}^{\text{ref}} \quad \text{and} \quad \mathcal{F}^{\text{ref}} \cap L^\infty = (\mathcal{F}_a^{\text{ref}})_e \cap L^\infty.$$

If $u \in \mathcal{F}^{\text{ref}}$ then $\Theta_\ell u \in \mathcal{F}^{\text{ref}} \cap L^\infty$ (note that \mathcal{F}_{loc} is a lattice), and consequently $\Theta_\ell u \in (\mathcal{F}_a^{\text{ref}})_e$. Moreover,

$$\Theta_\ell u \xrightarrow[\ell \rightarrow \infty]{\text{a.e.}} u \quad \text{and} \quad \sup_{\ell \in \mathbb{N}} \tilde{\mathcal{E}}(\Theta_\ell u, \Theta_\ell u) \leq \tilde{\mathcal{E}}(u, u) < \infty,$$

which means that $u \in (\mathcal{F}_a^{\text{ref}})_e$. This finally proves $\mathcal{F}^{\text{ref}} \subset (\mathcal{F}_a^{\text{ref}})_e$. \square

§5. Appendix

For the benefit of the readers we summarize in this appendix some basic definitions and notions on Dirichlet forms which have been used in the previous sections.

Let (X, d) be a locally compact separable metric space and m a positive Radon measure on X with full support. Let \mathcal{F} be a dense subspace of $L^2(X; m)$. A symmetric bilinear form \mathcal{E} defined on $\mathcal{F} \times \mathcal{F}$ is called a *symmetric Dirichlet form* on $L^2(X; m)$ if

(E.1) (nonnegativity) $\mathcal{E}(u, u) \geq 0$ for all $u \in \mathcal{F}$.

(\mathcal{E} .2) (closedness) $(\mathcal{F}, \mathcal{E}_1)$ is a real Hilbert space with

$$\mathcal{E}_1(u, v) := \mathcal{E}(u, v) + \langle u, v \rangle_{L^2(m)}, \quad u, v \in \mathcal{F},$$

where $\langle u, v \rangle_{L^2(m)}$ denotes the inner product of u and v in $L^2(X; m)$.

(\mathcal{E} .3) (Markov property) If $u \in \mathcal{F}$, then $v := 0 \vee u \wedge 1 \in \mathcal{F}$ and $\mathcal{E}(v, v) \leq \mathcal{E}(u, u)$.

For a symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$, there exists a unique non-positive self-adjoint operator A such that

$$\mathcal{E}(u, v) = \langle \sqrt{-A}u, \sqrt{-A}v \rangle_{L^2(m)}, \quad u, v \in \mathcal{F},$$

and the semigroup $\{e^{tA} : t > 0\}$ generated by A is a Markov semigroup, i.e.,

$$0 \leq e^{tA}u \leq 1 \quad \text{whenever} \quad u \in L^2(X; m), \quad 0 \leq u \leq 1.$$

The Dirichlet form $(\mathcal{E}, \mathcal{F})$ is said to be *regular* if $C_0(X) \cap \mathcal{F}$ is dense in $C_0(X)$ —the space of compactly supported, continuous functions on X —with respect to the uniform norm and dense in \mathcal{F} with respect to the Hilbert norm $\sqrt{\mathcal{E}_1(\cdot, \cdot)}$.

The Beurling–Deny decomposition ([FOT94, Theorem 3.2.1]) says that any regular symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ can be expressed for $u, v \in C_0(X) \cap \mathcal{F}$ as follows:

$$\begin{aligned} \mathcal{E}(u, v) &= \mathcal{E}^{(c)}(u, v) + \int_X u(x)v(x) k(dx) \\ &\quad + \iint_{x \neq y} (u(x) - u(y))(v(x) - v(y)) J(dx, dy). \end{aligned}$$

Here $\mathcal{E}^{(c)}$ is a symmetric form with domain $\mathcal{D}[\mathcal{E}^{(c)}] = \mathcal{F} \cap C_0(X)$ satisfying the *strong local property*: $\mathcal{E}^{(c)}(u, v) = 0$ for $u \in \mathcal{D}[\mathcal{E}^{(c)}]$ and all $v \in \mathcal{F}$ which are constant on a neighbourhood of $\text{supp } u$, the support of u . J is symmetric positive Random measure on $X \times X \setminus \Delta$ and k is a positive Radon measure on X . Note that $\mathcal{E}^{(c)}$, J and k are uniquely determined by \mathcal{E} . We call J the *jumping measure* and k the *killing measure* of \mathcal{E} .

Fukushima’s existence theorem shows that, for a regular symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$, there exists an m -symmetric Hunt process $\mathbf{M} = (X_t, \mathbb{P}_x)$ on X whose transition function defines a semigroup on $L^2(X; m)$ such that

$$e^{tA}u(x) = \mathbb{E}_x[u(X_t)] \quad m\text{-a.e. for all } u \in L^2(E; m) \cap \mathcal{B}(X), \quad t > 0.$$

Here $C(X)$ (resp. $\mathcal{B}(X)$) denotes the set of continuous (resp. Borel) functions on X . \mathbf{M} is unique up to an appropriate equivalence (see [FOT94] for more information). The jumping measure J explains size and intensity of the jumps of the sample paths, while the measure k governs the killing of the sample paths inside X (cf. [FOT94, Theorem 4.5.2]).

Example 5.1. Let $X = (M, g)$ be a smooth Riemannian manifold and consider a regular symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ defined by

$$\begin{cases} \mathcal{E}(u, v) = \int_M \langle \text{grad } u, \text{grad } v \rangle dv_g, \\ \mathcal{F} \text{ is the closure of } C_0^\infty(M) \text{ under } \mathcal{E}_1. \end{cases}$$

If M is complete, it is well-known that the Laplace–Beltrami operator, the generator of the form, is essentially self-adjoint on $C_0^\infty(M)$ (see [D]).

In general we do not know whether the domain of the generator contains nice functions like $C_0^\infty(M)$ —not even for regular, strongly local, symmetric Dirichlet forms. Therefore, we cannot discuss the extensions of Dirichlet forms in the context of self-adjoint extensions.

Given a regular symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$, we can consider the following class of extensions:

$$\mathcal{A}_s(\mathcal{E}, \mathcal{F}) := \left\{ (\tilde{\mathcal{E}}, \tilde{\mathcal{F}}) : \begin{array}{l} (\tilde{\mathcal{E}}, \tilde{\mathcal{F}}) \text{ is a Dirichlet form with } \tilde{\mathcal{F}} \supset \mathcal{F}, \tilde{\mathcal{E}}(u, u) = \mathcal{E}(u, u) \\ \text{for } u \in \mathcal{F}, \text{ and } u \cdot v \in \mathcal{F}_b \text{ whenever } u \in \tilde{\mathcal{F}}_b \text{ and } v \in \mathcal{F}_b \end{array} \right\},$$

where \mathcal{F}_b (resp. $\tilde{\mathcal{F}}_b$) means $\mathcal{F} \cap L^\infty(X; m)$ (resp. $\tilde{\mathcal{F}} \cap L^\infty(X; m)$). This class is introduced by Silverstein in [Sil74a, Sil74b] in order to classify Markov processes which are the extensions of the Markov process associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$. We call an element of $\mathcal{A}_s(\mathcal{E}, \mathcal{F})$ an *extension of $(\mathcal{E}, \mathcal{F})$ in Silverstein's sense*. For the precise meaning of this extension, see Theorem 20.1 in [Sil74b] or A.4.4 in [FOT94].

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