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Affine Nil-Hecke Algebras and Braided Differential Structure on Affine Weyl Groups

by

Anatol N. Kirillov and Toshiaki Maeno

Abstract

We construct a model of the affine nil-Hecke algebra as a subalgebra of the Nichols–Woronowicz algebra associated to a Yetter–Drinfeld module over the affine Weyl group. We also discuss the Peterson isomorphism between the homology of the affine Grassmannian and the small quantum cohomology ring of the flag variety in terms of the braided differential calculus.

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Introduction

The cohomology ring of the flag variety is a fundamental object of research in the study of the Schubert calculus. Fomin and the first author [4] gave a combinatorial model of the cohomology ring $H^*(Fl_n)$ of the flag variety of type A as a commutative subalgebra of a quadratic algebra $\mathcal{E}_n$. It is remarkable that the algebra $\mathcal{E}_n$ has a natural quantum deformation $\mathcal{E}_n^q$ so that $\mathcal{E}_n^q$ contains the quantum cohomology ring $QH^*(Fl_n)$ as a commutative subalgebra.

It has been observed by Milinski and Schneider [12] and by Majid [11] that the defining relations of the Fomin–Kirillov quadratic algebra $\mathcal{E}_n$ are understandable from the viewpoint of a certain kind of braided Hopf algebra called the Nichols–Woronowicz algebra. Bazlov [2] constructed a model of the coinvariant algebra of the finite Coxeter groups as a commutative subalgebra of the Nichols–Woronowicz

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A. N. Kirillov: Research Institute for Mathematical Sciences, Kyoto University,
Sakyo-ku, Kyoto 606-8502, Japan;
e-mail: kirillov@kurims.kyoto-u.ac.jp
T. Maeno: Department of Electrical Engineering, Kyoto University,
Sakyo-ku, Kyoto 606-8501, Japan;
e-mail: maeno@kuee.kyoto-u.ac.jp

algebra. At the same time, the nil-Coxeter algebra, which is dual to the coinvariant algebra, is also realized as a subalgebra of the Nichols–Woronowicz algebra.

The braided analogue of the exterior algebra was introduced by Woronowicz [15] for the study of differential forms on quantum groups. For a given braided vector space M over a field K of characteristic zero, the braided analogue $\mathcal{B}(M)$ of the symmetric algebra of M is similarly defined to be the quotient of the free tensor algebra of M by the kernel of the braided symmetrizer. It is known that the algebra $\mathcal{B}(M)$ is a braided graded Hopf algebra characterized by the following conditions:

(1) $\mathcal{B}^0(M) = K$,
(2) $\mathcal{B}^1(M) = M = \{\text{primitive elements in } \mathcal{B}(M)\}$,
(3) $\mathcal{B}^1(M)$ generates $\mathcal{B}(M)$ as an algebra.

The algebra is related to earlier ideas in [13]. The Hopf algebra generated by the primitive elements has been studied by Nichols [13] and named the Nichols algebra by Andruskiewitsch and Schneider [1]. The actual braided Hopf algebra structure of $\mathcal{B}(M)$ has its origins in the work of Majid, notably [9], [10]. The tensor algebras TM and TM^* are primitively generated braided Hopf algebras [9] and are dually paired as braided Hopf algebras by extending the pairing in degree 1. This pairing is typically degenerate and the quotient by the kernel on each side yields braided Hopf algebras $\mathcal{B}(M)$, $\mathcal{B}(M^*)$ which are now nondegenerately paired and canonically associated to M. Majid used this construction in [10]. In this paper we will call $\mathcal{B}(M)$ the Nichols–Woronowicz algebra following [2].

The aim of this paper is to construct the nil-Hecke algebra as a subalgebra of an extension of the Nichols–Woronowicz algebra $\mathcal{B}_{\mathrm{aff}}$ associated to a Yetter–Drinfeld module over the affine Weyl groups. Our construction is analogous to the one in [2, Section 6].

It is known that the affine Grassmannian $\widehat{\mathrm{Gr}} := G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$ of a semisimple Lie group G is homotopic to the loop group ΩK of the maximal compact subgroup $K \subset G$. The homology $H_*(\widehat{\mathrm{Gr}}) \cong H_*(\Omega K)$ carries an associative algebra structure induced by the Pontryagin product. The strucuture of the Pontryagin ring $H_*(\Omega K)$ has been determined by Bott [3]. The Schubert calculus for Kac–Moody flag varieties was studied by Kostant and Kumar [6] by using the nil-Hecke algebra. Peterson [14] stated that the torus-equivariant homology $H_*^T(\widehat{\mathrm{Gr}})$ of the affine Grassmannian is isomorphic to the so-called Peterson subalgebra of the affine nil-Hecke algebra. So our construction gives a model of $H_*^T(\widehat{\mathrm{Gr}})$ as a commutative subalgebra of the Nichols–Woronowicz algebra $\mathcal{B}_{\mathrm{aff}}(S)$ (see Theorem 3.4).

Peterson [14] also pointed out that the Pontryagin ring $H_*^T(\widehat{\mathrm{Gr}})$ is isomorphic to the small quantum cohomology ring $QH_T^*(G/B)$ of the corresponding flag variety G/B as an algebra after a suitable localization. The affine Bruhat operator acting on $H_*^T(\widehat{\mathrm{Gr}})$ introduced by Lam and Shimozono [7] gives an explicit comparison between the multiplicative structure of $H_*^T(\widehat{\mathrm{Gr}})$ and that of $QH_T^*(G/B)$. In this paper, we will realize the affine Bruhat operator as a braided differential operator (see Section 3 for details) acting on our algebra $\mathcal{B}_{\mathrm{aff}}$.

§1. Affine nil-Hecke algebra

Let G be a simply-connected semisimple complex Lie group and W its Weyl group. Denote by Δ the set of roots. We fix the set Δ_+ of positive roots by choosing a set of simple roots $\alpha_1, \dots, \alpha_r$. The Weyl group W acts on the weight lattice P and the coroot lattice $Q^\vee$ of G. The affine Weyl group W_{aff} is generated by the affine reflections $s_{\alpha,k}$, $\alpha \in \Delta$, $k \in \mathbb{Z}$, with respect to the affine hyperplanes $H_{\alpha,k} := \{\lambda \in P \otimes \mathbb{R} \mid \langle \lambda, \alpha \rangle = k\}$. The affine Weyl group is the semidirect product of W and $Q^\vee$, i.e., $W_{\mathrm{aff}} = W \ltimes Q^\vee$. The affine Weyl group W_{aff} is generated by the simple reflections $s_1 := s_{\alpha_1,0}, \dots, s_r := s_{\alpha_r,0}$ and $s_0 := s_{\theta,1}$ where $\theta = -\alpha_0$ is the highest root. The affine Weyl group W has the presentation as a Coxeter group as follows:

$$W_{\mathrm{aff}} = \langle s_0, \dots, s_r \mid s_0^2 = \cdots = s_r^2 = 1,\ (s_i s_j)^{m_{ij}} = 1 \rangle.$$

Definition 1.1. The *affine nil-Coxeter algebra* $\mathbb{A}_0$ is the associative $\mathbb{Q}$-algebra generated by $\tau_0, \dots, \tau_r$ subject to the relations

$$\tau_0^2 = \cdots = \tau_r^2 = 0, \qquad (\tau_i \tau_j)^{[m_{ij}/2]} \tau_i^{\nu_{ij}} = (\tau_j \tau_i)^{[m_{ij}/2]} \tau_j^{\nu_{ij}},$$

where $\nu_{ij} := m_{ij} - 2[m_{ij}/2]$.

For a reduced expression $x = s_{i_1} \cdots s_{i_l}$ of an element $x \in W_{\mathrm{aff}}$, the element $\tau_x := \tau_{i_1} \cdots \tau_{i_l} \in \mathbb{A}_0$ is independent of the choice of the reduced expression of x. It is known that $\{\tau_x\}_{x \in W_{\mathrm{aff}}}$ form a linear basis of $\mathbb{A}_0$.

The nil-Coxeter algebra $\mathbb{A}_0$ acts on $S := \mathrm{Sym}\, P_{\mathbb{Q}}$ via

$$\begin{aligned} \tau_0(f) &:= \partial_{\alpha_0}(f) = -(f - s_{\theta,0} f)/\theta, \\ \tau_i(f) &:= \partial_{\alpha_i}(f) = (f - s_{\alpha_i,0} f)/\alpha_i, \qquad i = 1, \dots, r, \end{aligned}$$

for $f \in S$.

Definition 1.2 ([6]). The *nil-Hecke algebra* $\mathbb{A}$ is defined to be the cross product $\mathbb{A}_0 \ltimes S$, where the cross relation is given by

$$\tau_i f = \partial_{\alpha_i}(f) + s_i(f) \tau_i, \qquad f \in S,\ i = 1, \dots, r.$$

Here, we summarize some known results on the homology of the affine Grassmannian. The *affine Grassmannian* $\widehat{\mathrm{Gr}} := G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$ is homotopic to the loop group ΩK of the maximal compact subgroup $K \subset G$. Let $T \subset G$ be the maximal torus. We consider the T-equivariant homology group $H_*^T(\widehat{\mathrm{Gr}})$ over $\mathbb{Q}$. An associative algebra structure on $H_*^T(\widehat{\mathrm{Gr}}) \cong H_*^T(\Omega K)$ is induced from the group multiplication

$$\Omega K \times \Omega K \to \Omega K.$$

It is known that the algebra $H_*^T(\widehat{\mathrm{Gr}})$ is commutative. The algebra $H_*^T(\Omega K)$ is called the *Pontryagin ring*.

We regard the T-equivariant homology $H_*^T(\widehat{\mathrm{Gr}})$ as an S-algebra by identifying $S = H_T^*(pt)$. The diagonal embedding

$$\Omega K \to \Omega K \times \Omega K$$

induces a coproduct on $H_*^T(\widehat{\mathrm{Gr}})$. Let $J \subset \mathbb{A}$ be the left ideal of $\mathbb{A}$ generated by the elements τ_w, $w \in W \setminus \{\mathrm{id}\}$. The centralizer $Z_{\mathbb{A}}(S)$ of S in $\mathbb{A}$ is called the *Peterson subalgebra* of $\mathbb{A}$.

Proposition 1.1 ([14]). *Let $\{\xi_x \mid x \in W_{\mathrm{aff}}\}$ be the Schubert basis of the T-equivariant homology $H_*^T(\widehat{\mathrm{Gr}})$. The T-equivariant homology $H_*^T(\widehat{\mathrm{Gr}})$ is naturally identified with the Peterson subalgebra $Z_{\mathbb{A}}(S)$ via the S-algebra isomorphism $j : H_*^T(\widehat{\mathrm{Gr}}) \to Z_{\mathbb{A}}(S)$ characterized by the following conditions:*

(1) $j(\xi_x) = \tau_x \mod J$ *for* $x \in W_{\mathrm{aff}}$,
(2) $j(\xi)\xi' = \xi\xi'$ *for* $\xi, \xi' \in H_*^T(\widehat{\mathrm{Gr}})$.

§2. Nichols–Woronowicz algebra for affine Weyl groups

We briefly recall the construction of the Nichols–Woronowicz algebra associated to a braided vector space. Let M be a vector space over a field of characteristic zero and $\psi : M^{\otimes 2} \to M^{\otimes 2}$ be a fixed linear endomorphism satisfying the braid relations $\psi_i\psi_{i+1}\psi_i = \psi_{i+1}\psi_i\psi_{i+1}$ where $\psi_i : M^{\otimes n} \to M^{\otimes n}$ is a linear endomorphism obtained by applying ψ to the i-th and $(i+1)$-st components. Denote by s_i the simple transposition $(i, i+1) \in S_n$. For any reduced expression $w = s_{i_1} \cdots s_{i_l} \in S_n$, the endomorphism $\Psi_w = \psi_{i_1} \cdots \psi_{i_l} : M^{\otimes n} \to M^{\otimes n}$ is well-defined. The *Woronowicz symmetrizer* (cf. [15]) is given by $\sigma_n := \sum_{w \in S_n} \Psi_w$. The operator σ_n is also called the *braided integer* in [9].

Definition 2.1 (cf. [15]). The *Nichols–Woronowicz algebra* associated to a braided vector space M is defined by

$$\mathcal{B}(M) := \bigoplus_{n\geq 0} M^{\otimes n}/\mathrm{Ker}(\sigma_n),$$

where $\sigma_n : M^{\otimes n} \to M^{\otimes n}$ is the Woronowicz symmetrizer.

Definition 2.2. A vector space M is called a *Yetter–Drinfeld module* over a group Γ if the following conditions are satisfied:

(1) M is a Γ-module,

(2) M is Γ-graded, i.e. $M = \bigoplus_{g\in\Gamma} M_g$, where M_g is a linear subspace of M,

(3) for $h \in \Gamma$ and $v \in M_g$, $h(v) \in M_{hgh^{-1}}$.

The Yetter–Drinfeld module M over a group Γ is naturally braided with the braiding $\psi : M^{\otimes 2} \to M^{\otimes 2}$ defined by $\psi(a \otimes b) = g(b) \otimes a$ for $a \in M_g$ and $b \in M$.

In the following we are interested in the Yetter–Drinfeld module over the affine Weyl group W_{aff}. Denote by $t_\lambda \in W_{\mathrm{aff}}$ the translation by $\lambda \in Q^\vee$. We define a Yetter–Drinfeld module V_{aff} over W_{aff} by

$$V_{\mathrm{aff}} := \bigoplus_{\alpha\in\Delta,\, k\in\mathbb{Z}} \mathbb{Q}\cdot[\alpha,k]/([\alpha,k]+[-\alpha,-k]),$$

where W_{aff} acts on V_{aff} by

$$w[\alpha,k] := [w(\alpha),k], \quad w \in W, \qquad t_\lambda[\alpha,k] := [\alpha, k+(\alpha,\lambda)], \quad \lambda \in Q^\vee.$$

The W_{aff}-grading is given by $\deg_{W_{\mathrm{aff}}}([\alpha,k]) := s_{\alpha,k}$. Then it is easy to check the conditions in Definition 2.1. Now we have the Nichols–Woronowicz algebra $\mathcal{B}_{\mathrm{aff}} := \mathcal{B}(V_{\mathrm{aff}})$ associated to the Yetter–Drinfeld module V_{aff}.

Let $\mathcal{B}_W$ be the Nichols–Woronowicz algebra associated to the Yetter–Drinfeld module $V = \bigoplus_{\alpha\in\Delta} \mathbb{Q}\cdot[\alpha]/([\alpha]+[-\alpha])$ as in [2, Section 4].

Lemma 2.3. (1) *We have a surjective homomorphism* $\pi : \mathcal{B}_{\mathrm{aff}} \to \mathcal{B}_W$ *given by* $\pi([\alpha,k]) := [\alpha]$.

(2) *The algebra* $\mathcal{B}_{\mathrm{aff}}$ *acts on* S *via* $[\alpha,k]f = \partial_\alpha(f)$ *for all* $k \in \mathbb{Z}$.

Proof. (1) Denote by ψ and $\bar\psi$ the braidings on V_{aff} and V respectively. Let $\tilde\pi : \bigoplus_n {V_{\mathrm{aff}}}^{\otimes n} \to \bigoplus_n V^{\otimes n}$ be the lift of π. Since

$$\psi([\alpha,k]\otimes[\beta,l]) = [s_\alpha(\beta), l - \langle\alpha^\vee,\beta\rangle k] \otimes [\alpha,k]$$

and $\bar\psi([\alpha]\otimes[\beta]) = [s_\alpha(\beta)]\otimes[\alpha]$, the map $\tilde\pi$ sends the kernel of the braided symmetrizer σ_n of ${V_{\mathrm{aff}}}^{\otimes n}$ to that of $V^{\otimes n}$.

(2) In [2], it is shown that the algebra $\mathcal{B}_W$ acts on the coinvariant algebra S_W via $[\alpha] \mapsto \partial_\alpha$. Let S^W be the W-invariant subalgebra of S. Then we have the decomposition $S = S^W \otimes S_W$. The operator ∂_α extends S^W-linearly to an operator on S. Hence $\mathcal{B}_W$ acts on S. We have seen the existence of the natural projection π from $\mathcal{B}_{\mathrm{aff}}$ to $\mathcal{B}$, so π induces the action of $\mathcal{B}_{\mathrm{aff}}$ on S. □

Let us define the extension $\mathcal{B}_{\mathrm{aff}}(S) = \mathcal{B}_{\mathrm{aff}} \ltimes S$ by the cross relation

$$[\alpha, k]f = \partial_\alpha f + s_{\alpha,0}(f)[\alpha, k], \qquad [\alpha, k] \in V_{\mathrm{aff}},\ f \in S.$$

Proposition 2.1. *There exists a homomorphism $\varphi : \mathbb{A} \to \mathcal{B}_{\mathrm{aff}}(S)$ given by $\tau_0 \mapsto [\alpha_0, -1]$, $\tau_i \mapsto [\alpha_i, 0]$, $i = 1, \dots, r$, and $f \mapsto f$, $f \in S$.*

Proof. It is enough to check the Coxeter relations among $\varphi(\tau_0), \dots, \varphi(\tau_r)$ in $\mathcal{B}_{\mathrm{aff}}(S)$ based on the classification of the affine root systems. This is done by the direct computation of the symmetrizer for the subsystems of rank 2 in a similar manner to [2, Section 6]. □

Example 2.4. We list the Coxeter relations in $\mathcal{B}_{\mathrm{aff}}$ involving $[\theta, 1] = -[\alpha_0, -1]$ for the root systems of rank 2. Let $(\varepsilon_1, \dots, \varepsilon_r)$ be an orthonormal basis of the r-dimensional Euclidean space. Put $[ij, k] := [\varepsilon_i - \varepsilon_j, k]$, $[\overline{ij}, k] := [\varepsilon_i + \varepsilon_j, k]$, $[i, k] := [\varepsilon_i, k]$ and $[\alpha] := [\alpha, 0]$.

(i) (Type A_2 case)

$$[13, 1][23][13, 1] + [23][13, 1][23] = 0, \qquad [13, 1][12][13, 1] + [12][13, 1][12] = 0.$$

(ii) (Type B_2 case)

$$[\overline{12}, 1][2][\overline{12}, 1][2] = [2][\overline{12}, 1][2][\overline{12}, 1].$$

(iii) (Type G_2 case) Let α_1, α_2 be the simple roots for the G_2-system. We assume that α_1 is a short root and α_2 is a long one. Then we have $\theta = 3\alpha_1 + 2\alpha_2$, and

$$[\theta, 1][\alpha_2][\theta, 1] + [\alpha_2][\theta, 1][\alpha_2] = 0.$$

§3. Model of nil-Hecke algebra

The connected components of $P \otimes \mathbb{R} \setminus \bigcup_{\alpha \in \Delta_+,\, k \in \mathbb{Z}} H_{\alpha,k}$ are called *alcoves*. The affine Weyl group W_{aff} acts on the set of alcoves simply transitively.

Definition 3.1 ([8]). (1) A sequence $(A_0, \dots, A_l)$ of alcoves A_i is called an *alcove path* if A_i and A_{i+1} have a common wall and $A_i \neq A_{i+1}$.

(2) An alcove path $(A_0, \dots, A_l)$ is called *reduced* if the length l of the path is minimal among all alcove paths connecting A_0 and A_l.

(3) We use the symbol $A_i \xrightarrow{\beta,k} A_{i+1}$ when A_i and A_{i+1} have a common wall of the form $H_{\beta,k}$ and the direction of the root β is from A_i to A_{i+1}.

The alcove A° defined by the inequalities $\langle\lambda,\alpha_0\rangle \geq -1$ and $\langle\lambda,\alpha_i\rangle \geq 0$, $i=1,\dots,r$, is called the *fundamental alcove*. For a reduced alcove path $\gamma : A_0 = A^\circ \xrightarrow{\beta_1,k_1} \cdots \xrightarrow{\beta_l,k_l} A_l$, we define an element $[\gamma] \in \mathcal{B}_{\mathrm{aff}}$ by

$$[\gamma] := [-\beta_1,-k_1]\cdots[-\beta_l,-k_l].$$

When $A_l = x^{-1}(A^\circ)$ for $x \in W_{\mathrm{aff}}$, we will also use the symbol $[x]$ instead of $[\gamma]$, since $[\gamma]$ depends only on x thanks to the Yang–Baxter relations listed in Example 2.4.

For a braided vector space M, it is known that an element $a \in M$ acts on $\mathcal{B}(M^*)$ as a braided differential operator (see [2], [9], [11]). Let us identify M^* with M via the W_{aff}-invariant inner product (,) given by

$$([\alpha,k],[\beta,l]) = \begin{cases} 1 & \text{if } \alpha=\beta \text{ and } k=l, \\ 0 & \text{otherwise,} \end{cases}$$

for $\alpha,\beta \in \Delta_+$ and $k,l \in \mathbb{Z}$. In our case, the differential operator $\overleftarrow{D}_{[\alpha,k]}$, $[\alpha,k] \in V_{\mathrm{aff}}$, acting from the right is determined by the following conditions:

(0) $(c)\overleftarrow{D}_{[\alpha,k]} = 0$ for $c \in \mathbb{Q}$,

(1) $([\alpha,k])\overleftarrow{D}_{[\beta,l]} = ([\alpha,k],[\beta,l])$,

(2) $(FG)\overleftarrow{D}_{[\alpha,k]} = F(G\overleftarrow{D}_{[\alpha,k]}) + (F\overleftarrow{D}_{[\alpha,k]})s_{\alpha,k}(G)$,

for $\alpha,\beta \in \Delta$, $k,l \in \mathbb{Z}$ and $F,G \in \mathcal{B}_{\mathrm{aff}}$. The operator $\overleftarrow{D}_{[\alpha,k]}$ extends to one acting on $\mathcal{B}_{\mathrm{aff}}(S)$ by the commutation relation $f \cdot \overleftarrow{D}_{[\alpha,k]} = \overleftarrow{D}_{[\alpha,k]} \cdot s_{\alpha,k}(f)$, $f \in S$.

We use the abbreviations $\overleftarrow{D}_0 := \overleftarrow{D}_{[\alpha_0,-1]}$ and $\overleftarrow{D}_i := \overleftarrow{D}_{[\alpha_i,0]}$, $i=1,\dots,r$. For $x \in W_{\mathrm{aff}}$, fix a reduced decomposition $x = s_{i_1}\cdots s_{i_l}$. We define the corresponding braided differential operator $\overleftarrow{D}_x$ acting on $\mathcal{B}_{\mathrm{aff}}$ by the formula

$$\overleftarrow{D}_x := \overleftarrow{D}_{i_l}\cdots\overleftarrow{D}_{i_1},$$

which is also independent of the choice of the reduced decomposition of x because of the braid relations.

Lemma 3.2. *For $x \in W_{\mathrm{aff}}$, take a reduced alcove path γ from the fundamental alcove A° to $x^{-1}(A^\circ)$. Then $([\gamma])\overleftarrow{D}_x = 1$.*

Proof. Take a reduced path

$$\gamma : A_0 = A^\circ \xrightarrow{\beta_1,k_1} A_1 \xrightarrow{\beta_2,k_2} \cdots \xrightarrow{\beta_l,k_l} A_l = x^{-1}(A^\circ).$$

Define a sequence $\sigma_1, \ldots, \sigma_l \in W_{\text{aff}}$ inductively by

$$\sigma_1 := s_{\beta_1,k_1}, \qquad \sigma_{j+1} := \sigma_j\sigma_{j-1}\cdots\sigma_1 \cdot s_{\beta_{j+1},k_{j+1}} \cdot \sigma_1 \cdots \sigma_{j-1}\sigma_j.$$

Then it is easy to see that $\sigma_\nu \cdots \sigma_1(A_j) \neq A^\circ$, $1 \leq \nu \leq j-1$, $\sigma_j \cdots \sigma_1(A_j) = A^\circ$ and the walls $\sigma_j \cdots \sigma_1(H_{\beta_{j+1},k_{j+1}})$ correspond to simple roots. Hence, $\sigma_1, \ldots, \sigma_l$ are simple reflections. This sequence gives a reduced expression $x = \sigma_l \cdots \sigma_1$. Put $\sigma_i = s_{\alpha_{i_j}}$. Since the direction of β_{j+1} is chosen to be from A_j to A_{j+1}, we have

$$[\gamma]\overleftarrow{D}_x = ([\beta_1, k_1])\overleftarrow{D}_{i_1} \cdot (\sigma_1([\beta_2, k_2]))\overleftarrow{D}_{i_2} \cdots (\sigma_{l-1} \cdots \sigma_1([\beta_l, k_l]))\overleftarrow{D}_{i_l} = 1. \qquad \square$$

Example 3.3 (A_2-case). The standard realization is given by $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_3$, $\alpha_0 = \varepsilon_3 - \varepsilon_1$. Consider the translation t_{α_1} by the simple root α_1. If we take a reduced path

$$\gamma : A_0 = A^\circ \xrightarrow{-\alpha_2,0} A_1 \xrightarrow{\alpha_1,1} A_2 \xrightarrow{-\alpha_0,1} A_3 \xrightarrow{\alpha_1,2} A_4 = t_{\alpha_1}(A^\circ),$$

then we have $[\gamma] = [23][21,-1][31,-1][21,-2]$. On the other hand, the differential operator corresponding to $t_{-\alpha_1}$ is given by $\overleftarrow{D}_2\overleftarrow{D}_0\overleftarrow{D}_2\overleftarrow{D}_1$, where $\overleftarrow{D}_0 = \overleftarrow{D}_{[31,-1]}$, $\overleftarrow{D}_1 = \overleftarrow{D}_{[12]}$, $\overleftarrow{D}_2 = \overleftarrow{D}_{[23]}$. It is easy to check by direct computation that

$$([23][21,-1][31,-1][12,2])\overleftarrow{D}_2\overleftarrow{D}_0\overleftarrow{D}_2\overleftarrow{D}_1 = 1.$$

Theorem 3.4. *The algebra homomorphism $\varphi : \mathbb{A} \to \mathcal{B}_{\text{aff}}(S)$ is injective.*

Proof. The nil-Hecke algebra $\mathbb{A}$ is also W_{aff}-graded by assigning the W_{aff}-degree to each generator by $\deg_{W_{\text{aff}}}(\tau_i) = s_i$. Since the homomorphism $\varphi : \mathbb{A} \to \mathcal{B}_{\text{aff}}(S)$ preserves the W_{aff}-grading, it is enough to check $\varphi(\tau_x) \neq 0$ for $x \in W_{\text{aff}}$ in order to show the injectivity of φ. On the other hand, ${\mathcal{B}_{\text{aff}}}^{\text{op}}$ acts on $\mathcal{B}_{\text{aff}}$ itself via the braided differential operators. Let γ be a reduced alcove path from A° to $x^{-1}(A^\circ)$. Then $([\gamma])\overleftarrow{D}_x = 1$ from Lemma 3.2. This shows $\overleftarrow{D}_x \neq 0$, so $\varphi(\tau_x) \neq 0$. $\square$

This theorem implies the following (see Proposition 1.1):

Corollary 3.5. *The T-equivariant Pontryagin ring $H_*^T(\widehat{\mathrm{Gr}})$ is a subalgebra of $\mathcal{B}_{\text{aff}}(S)$.*

By taking the non-equivariant limit, we also have:

Corollary 3.6. *The Pontryagin ring $H_*(\widehat{\mathrm{Gr}})$ is a subalgebra of $\mathcal{B}_{\text{aff}}$.*

§4. Affine Bruhat operators

We denote by $x \to y$ the cover relation in the Bruhat ordering of W_{aff}, i.e. $y = xs_{\alpha,k}$ for some $\alpha \in \Delta$ and $k \in \mathbb{Z}$, and $l(y) = l(x) + 1$.

We will use some terminology from [7]. Denote by $\tilde{Q}$ the set of antidominant elements in $Q^\vee$. An element $x \in W_{\mathrm{aff}}$ can be uniquely expressed as a product of the form $x = wt_{v\lambda} \in W_{\mathrm{aff}}$ with $v, w \in W$, $\lambda \in \tilde{Q}$. We say that $x = wt_{v\lambda}$ *belongs to the v-chamber*. An element $\lambda \in \tilde{Q}$ is called *superregular* when $|\langle \lambda, \alpha\rangle| > 2(\#W)+2$ for all $\alpha \in \Delta_+$. If $\lambda \in \tilde{Q}$ is superregular, then $x = wt_{v\lambda}$ is also called superregular. The subset of superregular elements in W_{aff} is denoted by ${W_{\mathrm{aff}}}^{\mathrm{sreg}}$. We say that a property holds *for sufficiently superregular elements* ${W_{\mathrm{aff}}}^{\mathrm{ssreg}} \subset W_{\mathrm{aff}}$ if there is a positive constant $k \in \mathbb{Z}$ such that the property holds for all $x \in {W_{\mathrm{aff}}}^{\mathrm{sreg}}$ satisfying the following condition:

$$y \in W_{\mathrm{aff}},\ y < x, \text{ and } l(x) - l(y) < k \ \Rightarrow\ y \in {W_{\mathrm{aff}}}^{\mathrm{sreg}}.$$

The meaning of ${W_{\mathrm{aff}}}^{\mathrm{ssreg}}$ depends on the context (see [7, Section 4] for the details). For $v \in W$, consider the S-submodule M_v^{ssreg} in $\mathcal{B}_{\mathrm{aff}}$ generated by the sufficiently superregular elements $[x]$ where x belongs to the v-chamber.

Lemma 4.1. *Let $x \in W_{\mathrm{aff}}$. For $\alpha \in \Delta$ and $k \in \mathbb{Z}_{>0}$, we have*

$$[x]\overleftarrow{D}_{[\alpha,k]} = \begin{cases} [xs_{\alpha,k}] & \textit{if } l(x) = l(xs_{\alpha,k}) + 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The fundamental alcove A° is contained in the region $\{\lambda \in P \otimes \mathbb{R} \mid \langle \lambda, \alpha\rangle < k\}$ for $\alpha \in \Delta$ and $k \in \mathbb{Z}_{>0}$. Choose any reduced path $\gamma : A_0 \xrightarrow{\beta_1,k_1} \cdots \xrightarrow{\beta_l,k_l} A_l = x^{-1}(A^\circ)$ with $k_i \geq 0$. If $l(x) > l(xs_{\alpha,k})$, then $(\beta_i, k_i) = (\alpha, k)$ for some i. Take the largest i and consider the path

$$\begin{aligned}\gamma' : A_0 \xrightarrow{\beta_1,k_1} \cdots \xrightarrow{\beta_{i-1},k_{i-1}} A_{i-1} \xrightarrow{\beta'_{i+1},k'_{i+1}} s_{\alpha,k}(A_{i+1}) \xrightarrow{\beta'_{i+2},k'_{i+2}} \cdots \\ \cdots \xrightarrow{\beta'_l,k'_l} s_{\alpha,k}(A_l) = s_{\alpha,k}x^{-1}(A^\circ) = (xs_{\alpha,k})^{-1}(A^\circ),\end{aligned}$$

where (β'_j, k'_j) is determined by the condition $s_{\alpha,k}(H_{\beta_j,k_j}) = H_{\beta'_j,k'_j}$. If $l(x) = l(xs_{\alpha,k}) + 1$, then γ' is a reduced path. In this case, $[x]\overleftarrow{D}_{[\alpha,k]} = [xs_{\alpha,k}]$. If $l(x) > l(xs_{\alpha,k}) + 1$, then γ' is not reduced and $[x]\overleftarrow{D}_{[\alpha,k]} = 0$. When $l(x) < l(xs_{\alpha,k})$, the element $[\alpha, k]$ does not appear in the monomial $[\gamma]$, so $[x]\overleftarrow{D}_{[\alpha,k]} = 0$. □

Proposition 4.1 ([7, Proposition 4.1]). *Let $\lambda \in \tilde{Q}$ be superregular. For $x = wt_{v\lambda}$ and $y = xs_{v\alpha,-n}$ with $v, w \in W$, we have the cover relation $y \to x$ if and only if one of the following conditions holds:*

(1) *$l(wv) = l(wvs_\alpha) - 1$ and $n = \langle \lambda, \alpha\rangle$, giving $y = ws_{v(\alpha)}t_{v(\lambda)}$,*

(2) *$l(wv) = l(wvs_\alpha) + \langle \alpha^\vee, 2\rho\rangle - 1$ and $n = \langle \lambda, \alpha\rangle + 1$, giving $y = ws_{v(\alpha)}t_{v(\lambda+\alpha^\vee)}$,*

(3) $l(v) = l(vs_\alpha) + 1$ *and* $n = 0$, *giving* $y = ws_{v(\alpha)}t_{vs_\alpha(\lambda)}$,

(4) $l(v) = l(vs_\alpha) - \langle \alpha^\vee, 2\rho\rangle + 1$ *and* $n = -1$, *giving* $y = ws_{v(\alpha)}t_{vs_\alpha(\lambda+\alpha^\vee)}$.

In [7], the first parts of conditions (1) and (2) are called the *near relation* because x and y belong to the same chamber. In this paper we denote the near relation by $y \to_{\mathrm{near}} x$.

The affine Bruhat operator $B^\mu : S\langle {W_{\mathrm{aff}}}^{\mathrm{ssreg}}\rangle \to S\langle {W_{\mathrm{aff}}}^{\mathrm{sreg}}\rangle$, $\mu \in P$, due to Lam and Shimozono [7, Section 5] is an S-linear map defined by the formula

$$B^\mu(x) = (\mu - wv\mu)x + \sum_{\alpha\in\Delta_+} \sum_{xs_{v(\alpha),k}\to_{\mathrm{near}} x} \langle \alpha^\vee, \mu\rangle x s_{v(\alpha),k}$$

for $x = wt_{v\lambda} \in {W_{\mathrm{aff}}}^{\mathrm{ssreg}}$. We also introduce the operator β_v^μ, $\mu \in P$, acting on each M_v^{ssreg} by

$$\beta_v^\mu([x]) := (\mu - wv\mu)[x] + [x] \sum_{\alpha\in\Delta_+, k>1} \langle \alpha^\vee, \mu\rangle \overleftarrow{D}_{[v(\alpha),k]},$$

where $x = wt_{v\lambda} \in {W_{\mathrm{aff}}}^{\mathrm{ssreg}}$. Denote by ${W_{\mathrm{aff}}}^{\mathrm{ssreg}}(v)$ the subset of W_{aff} consisting of the superregular elements belonging to the v-chamber. Fix a left S-module isomorphism

$$\iota : S\langle {W_{\mathrm{aff}}}^{\mathrm{ssreg}}(v)\rangle \to M_v^{\mathrm{ssreg}}, \qquad x \mapsto [x].$$

Proposition 4.2. *For each* $v \in W$ *and a sufficiently superregular element* $x \in {W_{\mathrm{aff}}}^{\mathrm{ssreg}}(v)$,

$$\beta_v^\mu([x]) = \iota(B^\mu(x)).$$

Proof. This can be shown by using Lemma 4.1 and Proposition 4.1:

$$\begin{aligned} \beta_v^\mu([x]) &= (\mu - wv\mu)[x] + [x] \sum_{\alpha\in\Delta_+,\, k>1} \langle \alpha^\vee, \mu\rangle \overleftarrow{D}_{[v(\alpha),k]} \\ &= (\mu - wv\mu)[x] + \sum_{\alpha\in\Delta_+} \sum_{k>1,\, l(xs_{v(\alpha),k})=l(x)-1} \langle \alpha^\vee, \mu\rangle [xs_{v(\alpha),k}] \\ &= (\mu - wv\mu)[x] + \sum_{\alpha\in\Delta_+} \sum_{xs_{v(\alpha),k}\to_{near} x} \langle \alpha^\vee, \mu\rangle [xs_{v(\alpha),k}] = \iota(B^\mu(x)). \qquad \square \end{aligned}$$

Remark 4.2. In [5] the authors introduced the quantization operators η_α acting on the model of $H^*(G/B) \otimes \mathbb{C}[q_1, \ldots, q_r]$ realized as a subalgebra of $\mathcal{B}_W \otimes \mathbb{C}[q_1, \ldots, q_r]$. For a superregular element $\lambda \in \tilde{Q}$ and $w \in W$, consider a homomorphism θ_w^λ from the λ-small elements (see [7, Section 5]) of $H^*(G/B) \otimes \mathbb{C}[q]$ to $\mathcal{B}_{\mathrm{aff}}$ defined by

$$\theta_w^\lambda(q^\mu \sigma^v) := [vw^{-1}t_{w(\lambda+\mu)}],$$

where σ^v is the Schubert class of G/B corresponding to $v \in W$ and $q^\mu = q_1^{\mu_1} \cdots q_r^{\mu_r}$ for $\mu = \sum_{i=1}^r \mu_i \alpha_i^\vee$. The following is an interpretation of the formula of [7, Proposition 5.1] in our setting:

$$\theta_w^\lambda(\eta_\alpha(\sigma)) = \beta_w^{\varpi_\alpha}(\theta_w^\lambda(\sigma)).$$

§5. Peterson isomorphism

As seen in the proof of Lemma 3.1, we can read off two kinds of decomposition of an element $x \in W_{\mathrm{aff}}$ into products of reflections from an alcove path

$$\gamma : A_0 = A^\circ \xrightarrow{\beta_1, k_1} A_1 \xrightarrow{\beta_2, k_2} \cdots \xrightarrow{\beta_l, k_l} A_l = x^{-1}(A^\circ).$$

One of them is a reduced decomposition $x = \sigma_l \cdots \sigma_1$, and another is a decomposition $x = s_{\beta_l, k_l} \cdots s_{\beta_1, k_1}$ into a product of (not necessarily simple) reflections. Let M be a left S-submodule of $\mathcal{B}_{\mathrm{aff}}(S)$ generated by the elements $[x]$, $x \in W_{\mathrm{aff}}$. The correspondence between the above decompositions of x induces an isomorphism of S-modules:

$$\upsilon : M \to \varphi(\mathbb{A}) \cong \mathbb{A}, \qquad [x] = [-\beta_1, -k_1] \cdots [-\beta_l, -k_l] \mapsto \varphi(\tau_x) = [\sigma_l] \cdots [\sigma_1].$$

For a superregular antidominant element $\lambda \in Q$ and $\mu_1, \ldots, \mu_k \in P$, we see that $\upsilon(\sum_{w \in W} \beta_w^{\mu_k} \cdots \beta_w^{\mu_1}[t_{w\lambda}])$ belongs to $Z_{\mathbb{A}}(S)$ from [7, Theorem 7.2].

The Peterson isomorphism

$$\begin{aligned}\psi : H_*^T(\widehat{\mathrm{Gr}})_{\mathrm{loc}} &:= H_*^T(\widehat{\mathrm{Gr}})[\xi_{t_\lambda}^{-1} | \lambda \in \tilde{Q}] \\ &\to QH_T^*(G/B)_{\mathrm{loc}} := QH^*(G/B)[q_1^{-1}, \ldots, q_r^{-1}]\end{aligned}$$

is an S-algebra isomorphism given by

$$\xi_{wt_\lambda}\xi_{t_\mu} \mapsto q^{\lambda-\mu}\sigma^w, \qquad w \in W,\ \lambda, \mu \in \tilde{Q}.$$

Proposition 5.1. *The Peterson isomorphism ψ is characterized by the condition*

$$\psi\Big(j^{-1}\Big(\upsilon\Big(\sum_{w \in W} \beta_w^{\mu_k} \cdots \beta_w^{\mu_1}[t_{w\lambda}]\Big)\Big)\Big) = q^\lambda \eta_{\mu_1} \cdots \eta_{\mu_k} \sigma^{\mathrm{id}}$$

for a superregular element $\lambda \in \tilde{Q}$ and $\mu_1, \ldots \mu_k \in P$.

§6. Quadratic relations

For $\alpha \in \Delta_+$ and $v \in W$, define the operator $\mathbb{D}_v(\alpha)$ by

$$\mathbb{D}_v(\alpha) := \sum_{k > 1} \overleftarrow{D}_{[v(\alpha), k]}.$$

Then

$$\beta_v^\mu([x]) = (\mu - wv\mu)[x] + [x] \sum_{\alpha \in \Delta_+} \langle \alpha^\vee, \mu \rangle \mathbb{D}_v(\alpha).$$

In the following, we discuss the relations among the operators $\mathbb{D}_v(\alpha)$, $\alpha \in \Delta_+$, for a root system of type A_{n-1}. For simplicity, we consider only the non-equivariant case with $v = \mathrm{id}$. Take the standard realization of the A_{n-1}-system:

$$\Delta = \{\varepsilon_i - \varepsilon_j \mid 1 \le i, j \le n,\ i \ne j\}.$$

Put $\mathbb{D}(ij) := \mathbb{D}_{\mathrm{id}}(\varepsilon_i - \varepsilon_j)$ for $1 \le i < j \le n$, and $\mathbb{D}(ij) := -\mathbb{D}(ji)$ for $i > j$. In this situation, we have a formula for the non-equivariant limit $\bar{\beta}_{\mathrm{id}}^{\varepsilon_i}$ of the operator $\beta_{\mathrm{id}}^{\varepsilon_i}$:

$$\bar{\beta}_{\mathrm{id}}^{\varepsilon_i} = \sum_{j \ne i} \mathbb{D}(ij).$$

Note that this formula is analogous to the definition of the Dunkl elements in [4].

Let T_i, $1 \le i \le n-1$, be linear operators on M^{ssreg} defined by $T_i([x]) := [xt_{\alpha_i}]$, where $x \in W_{\mathrm{aff}}$ and $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. It is easy to check from Proposition 4.1 that $(T_i[x])\mathbb{D}(jk) = T_i([x]\mathbb{D}(jk))$. Our next goal is to show that the operators $\mathbb{D}(ij)$ satisfy the defining relations of the quantum deformation $\mathcal{E}_n^q$ of the Fomin–Kirillov quadratic algebra [4].

Proposition 6.1. (i) *For $1 \le i < j \le n$, we have*

$$\mathbb{D}(ij)^2 = \begin{cases} T_i & \text{if } j = i+1, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *If $\{i, j\} \cap \{k, l\} = \emptyset$, then $\mathbb{D}(ij)\mathbb{D}(kl) = \mathbb{D}(kl)\mathbb{D}(ij)$.*

(iii) *For $1 \le i, j \le n$, $i \ne j$,*

$$\mathbb{D}(ij)\mathbb{D}(jk) + \mathbb{D}(jk)\mathbb{D}(kl) + \mathbb{D}(ki)\mathbb{D}(ij) = 0.$$

Proof. First of all, let us check (i). We have

$$\mathbb{D}(ij)^2 = \sum_{k,l>1} \overleftarrow{D}_{[ij,k]} \overleftarrow{D}_{[ij,l]}.$$

Let $\lambda \in \tilde{Q}$ be sufficiently superregular. For $x = wt_\lambda \in W_{\mathrm{aff}}$, assume that

$$[x] \overleftarrow{D}_{[ij,k]} \overleftarrow{D}_{[ij,l]} \ne 0.$$

Then we have the arrows $xs_{ij,k} \to_{\mathrm{near}} x$ and $xs_{ij,k}s_{ij,l} \to_{\mathrm{near}} xs_{ij,k}$ in the Bruhat ordering. From conditions (1) and (2) of Proposition 4.1, one of the following conditions holds:

Case 1: $k = -\langle \lambda, \varepsilon_i - \varepsilon_j \rangle$ and $l(w) = l(ws_{ij}) - 1$.
Case 2: $k = -\langle \lambda, \varepsilon_i - \varepsilon_j \rangle - 1$ and $l(w) = l(ws_{ij}) + \langle \varepsilon_i - \varepsilon_j, 2\rho \rangle - 1$.

In Case 1, since the arrow $xs_{ij,k}s_{ij,l} = ws_{ij}t_\lambda s_{ij,l} \to_{\text{near}} xs_{ij,k}$ must come from condition (2) of Proposition 4.1, we have $\langle \varepsilon_i - \varepsilon_j, 2\rho \rangle - 1 = 1$. This equality implies that $\varepsilon_i - \varepsilon_j$ is a simple root α_i, and we get

$$[x]\mathbb{D}(i\ i+1)^2 = [x]\overleftarrow{D}_{[\alpha_i, -\langle \lambda, \alpha_i \rangle]}\overleftarrow{D}_{[\alpha_i, -\langle \lambda, \alpha_i \rangle - 1]} = [xt_{\alpha_i}] = T_i[x].$$

In Case 2, since the arrow $xs_{ij,k}s_{ij,l} = ws_{ij}t_{\lambda+\varepsilon_i-\varepsilon_j}s_{ij,l} \to_{\text{near}} xs_{ij,k}$ comes from condition (1) of Proposition 4.1, we again obtain $\langle \varepsilon_i - \varepsilon_j, 2\rho \rangle - 1 = 1$ and $\varepsilon_i - \varepsilon_j = \alpha_i$. Hence we get

$$[x]\mathbb{D}(i\ i+1)^2 = [x]\overleftarrow{D}_{[\alpha_i, -\langle \lambda, \alpha_i \rangle - 1]}\overleftarrow{D}_{[\alpha_i, -\langle \lambda, \alpha_i \rangle - 2]} = [xt_{\alpha_i}] = T_i[x].$$

If $j \neq i+1$, we have $\mathbb{D}(ij)^2 = 0$. Relations (ii) and (iii) follow from the identities $[ij, a][kl, b] = [kl, b][ij, a]$ for $\{i, j\} \cap \{k, l\} = \emptyset$, and

$$[ij, a][jk, b] + [jk, b][ki, -a-b] + [ki, -a-b][ij, a] = 0$$

in $\mathcal{B}_{\text{aff}}$. □

Remark 6.1. The operators $\mathbb{D}_v(\alpha)$ induce the quantum Bruhat representation of $\mathcal{E}_n^q$ via θ_v^λ.

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