Publ. RIMS Kyoto Univ. **48** (2012), 229–233
DOI 10.2977/PRIMS/68

Corrigendum: "Homogenized Spectral Problems for Exactly Solvable Operators: Asymptotics of Polynomial Eigenfunctions"

by

Julius BORCEA, Rikard BØGVAD and Boris SHAPIRO

Abstract

Here we provide a correct proof of Proposition 6 of [2]. No other results of the latter paper are affected.

2010 Mathematics Subject Classification: 30C15, 31A35, 34E05.
Keywords: asymptotic root-counting measure, Cauchy transform, homogenized spectral problem, exactly solvable operator.

§1. Necessary results and corrected proof

To make this note self-contained we briefly recall the basic set-up of [2]. Given a $(k+1)$-tuple of polynomials $(Q_k(z), Q_{k-1}(z), \dots, Q_0(z))$ with $\deg Q_i(z) \le i$ consider the *homogenized spectral pencil* of differential operators given by

$$T_\lambda = \sum_{i=0}^{k} Q_i(z)\lambda^{k-i}\frac{d^i}{dz^i}. \tag{1.1}$$

Introduce the algebraic curve Γ associated with T_λ and given by the equation

$$\sum_{i=0}^{k} Q_i(z)w^i = 0, \tag{1.2}$$

where the polynomials $Q_i(z) = \sum_{j=0}^{i} a_{i,j}z^j$ are the same as in (1.1).

The curve Γ and its associated pencil T_λ are called of *general type* if the following two nondegeneracy requirements are satisfied:

Communicated by M. Kashiwara. Received September 22, 2010.

Julius Borcea unexpectedly passed away on April 8, 2009 at the age of 40.
R. Bøgvad: Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden; e-mail: rikard@math.su.se
B. Shapiro: Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden; e-mail: shapiro@math.su.se

(i) $\deg Q_k(z) = k$ (i.e., $a_{k,k} \neq 0$),

(ii) no two roots of the (characteristic) equation

$$a_{k,k} + a_{k-1,k-1}t + \ldots + a_{0,0}t^k = 0 \tag{1.3}$$

lie on a line through the origin (in particular, 0 is not a root of (1.3)).

The first statement of [2] we need is as follows.

Proposition 1. *If the characteristic equation* (1.3) *has* k *distinct solutions* $\alpha_1, \ldots, \alpha_k$ *and satisfies the above nondegeneracy assumptions* (*in particular, these imply that* $a_{0,0} \neq 0$ *and* $a_{k,k} \neq 0$) *then*

(i) *for all sufficiently large* n *there exist exactly* k *distinct eigenvalues* $\lambda_{n,j}$, $j = 1, \ldots, k$, *such that the associated spectral pencil* T_λ *has a polynomial eigenfunction* $p_{n,j}(z)$ *of degree exactly* n,

(ii) *the eigenvalues* $\lambda_{n,j}$ *split into* k *distinct families labeled by the roots of* (1.3) *such that the eigenvalues in the* j*-th family satisfy*

$$\lim_{n\to\infty} \frac{\lambda_{n,j}}{n} = \alpha_j, \qquad j = 1, \ldots, k.$$

The main result of [2] is given below.

Theorem 1. *In the notation of Proposition* 1, *for any pencil* T_λ *of general type and every* $j = 1, \ldots, k$ *there exists a subsequence* $\{n_{i,j}\}$, $i = 1, 2, \ldots$, *such that the limits*

$$\Psi_j(z) := \lim_{i\to\infty} \frac{p'_{n_{i,j}}(z)}{\lambda_{n_{i,j}} p_{n_{i,j}}(z)}, \qquad j = 1, \ldots, k,$$

exist almost everywhere in $\mathbb{C}$ *and are analytic functions in some neighborhood of* ∞. *Each* $\Psi_j(z)$ *satisfies equation* (1.2), *i.e.,* $\sum_{i=0}^{k} Q_i(z)\Psi_j^i(z) = 0$ *almost everywhere in* $\mathbb{C}$, *and the functions* $\Psi_1(z), \ldots, \Psi_k(z)$ *are independent sections of* Γ *considered as a branched covering over* $\mathbb{CP}^1$ *in a sufficiently small neighborhood of* ∞.

The proof requires Lemma 1 and Proposition 2 below. (The proof of the latter proposition suggested in [2] was erroneous and is corrected below.)

Lemma 1 (cf. Lemma 8 of [1]). *Let* $\{q_m(z)\}$ *be a sequence of polynomials with* $\deg q_m(z) \to \infty$ *as* $m \to \infty$. *Denote by* μ_m *and* μ'_m *the root-counting measures of* $q_m(z)$ *and* $q'_m(z)$, *respectively, and assume that there exists a compact set* K *containing the supports of all measures* μ_m *and therefore also the supports of all measures* μ'_m. *If* $\mu_m \to \mu$ *and* $\mu'_m \to \mu'$ *as* $m \to \infty$, *and* u *and* u' *are the logarithmic potentials of* μ *and* μ', *respectively, then* $u' \leq u$ *in* $\mathbb{C}$ *with equality in the unbounded component of* $\mathbb{C} \setminus \operatorname{supp}(\mu)$.

Example 1. Consider the polynomial sequence $\{z^m - 1\}$. The measure μ is then the uniform distribution on the unit circle of total mass 1. Its logarithmic potential $u(z)$ equals $\log|z|$ if $|z| \geq 1$ and 0 in the disk $|z| \leq 1$. On the other hand, the sequence of derivatives is given by $\{mz^{m-1}\}$ and the corresponding (limiting) logarithmic potential $u'(z)$ equals $\log|z|$ in $\mathbb{C} \setminus \{0\}$. Obviously, $u(z) = u'(z)$ in $|z| \geq 1$ and $u'(z) < u(z)$ in $|z| < 1$.

In the notation of Theorem 1 consider the family of eigenpolynomials $\{p_{n,j}(z)\}$ for some arbitrarily fixed value of the index $j = 1, \ldots, k$. Assume that N_j is a subsequence of the natural numbers such that

$$\mu_j^{(i)} := \lim_{n\to\infty,\, n\in N_j} \mu_{n,j}^{(i)} \tag{1.4}$$

exists for $i = 0, \ldots, k$, where $\mu_{n,j}^{(i)}$ denotes the root-counting measure of $p_{n,j}^{(i)}(z)$. The existence of such N_j follows by Helly's theorem from the existence of a compact set K that contains the support of all $\mu_{n,j}^{(i)}$. Notice that for each i the logarithmic potential $u_j^{(i)}$ of $\mu_j^{(i)}$ satisfies a.e. the identity

$$u_j^{(i)}(z) - u_j^{(0)}(z) = \lim_{n\to\infty,\, n\in N_j} \frac{1}{n} \log\left|\frac{p_{n,j}^{(i)}(z)}{n(n-1)\ldots(n-i+1)p_{n,j}(z)}\right|.$$

The next proposition completes the proof of Theorem 1 and also shows the remarkable property that if one considers a sequence of eigenpolynomials for some spectral pencil then the situation $u'(z) < u(z)$ seen in Example 1 can never occur. In fact, for the validity of Proposition 2 one only needs two assumptions:

(a) $\deg Q_k(z) = k$ (i.e., $a_{k,k} \neq 0$, so that all α_j, $j = 1, \ldots, k$, are non-zero) and
(b) $Q_0 \neq 0$.

Proposition 2. *The measures $\mu_j^{(i)}$, $i = 0, \ldots, k$, are all equal and the scalar multiple $\widetilde{\Psi}_j = C_\mu/\alpha_j$ of the Cauchy transform of this common measure μ_j satisfies equation (1.2) almost everywhere.*

Proof. For $n \in N_j$ one has

$$\frac{p_{n,j}^{(i+1)}(z)}{(n-i)p_{n,j}^{(i)}(z)} \to C^{(i+1)}(z) := \int_{\mathbb{C}} \frac{d\mu_j^{(i)}(\zeta)}{z-\zeta} \quad \text{as } n \to \infty$$

with convergence in L^1_{loc}. The well-known property of convergence in L^1_{loc} implies that passing to a subsequence one can assume that the above convergence is actually the pointwise convergence almost everywhere in $\mathbb{C}$. It follows that

$$\frac{p_{n,j}^{(i)}(z)}{n^i p_{n,j}(z)} \to C^{(1)}(z)\ldots C^{(i)}(z), \tag{1.5}$$

pointwise almost everywhere in $\mathbb{C}$. We claim that this limit is non-zero a.e. Granted this, consider

$$u_j^{(k)}(z) - u_j^{(0)}(z) = \lim_{n\to\infty,\, n\in N_j} \frac{1}{n}\log\left|\frac{p_{n,j}^{(k)}(z)}{n(n-1)\dots(n-k+1)p_{n,j}(z)}\right| = 0$$

almost everywhere in $\mathbb{C}$. On the other hand, $u_j^{(0)} \geq u_j^{(1)} \geq \dots \geq u_j^{(k)}$ by Lemma 1. Hence the potentials $u_j^{(i)}$ are all equal and the corresponding measures $\mu_j^{(i)} = \Delta u_j^{(i)}/2\pi$ are equal as well.

It remains to settle the above claim. Recall that $p_{n,j}(z)$ satisfies the differential equation $T_{\lambda_{n,j}} p_{n,j}(z) = 0$, i.e.,

$$Q_k(z)p_{n,j}^{(k)}(z) + \lambda_{n,j}Q_{k-1}(z)p_{n,j}^{(k-1)}(z) + \dots + \lambda_{n,j}^k Q_0(z)p_{n,j}(z) = 0. \tag{1.6}$$

Therefore,

$$Q_k(z)\frac{p_{n,j}^{(k)}(z)}{n^k p_{n,j}(z)} + \frac{\lambda_{n,j}}{n}Q_{k-1}(z)\frac{p_{n,j}^{(k-1)}(z)}{n^{k-1}p_{n,j}(z)} + \dots + \frac{\lambda_{n,j}^k}{n^k}Q_0(z) = 0.$$

Using the asymptotics $\lambda_{n,j} \sim \alpha_j n$ and the pointwise convergence a.e. in (1.5) we get

$$\begin{aligned} Q_k(z)C^{(1)}(z)\dots C^{(k)}(z) + \alpha_j Q_{k-1}(z)C^{(1)}(z)\dots C^{(k-1)}(z)& \\ + \dots + \alpha_j^k Q_0(z) = 0.& \end{aligned} \tag{1.7}$$

Using the assumption that $Q_0 \neq 0 \neq \alpha_j$, we conclude that $C^{(1)}(z) \neq 0$ a.e. To prove that $C^{(2)}(z)$ is also non-zero a.e., we consider the differential equation satisfied by $p'_{n,j}(z)$,

$$\begin{aligned} Q_k(z)p_{n,j}^{(k+1)}(z) + (Q'_k(z) + \lambda_{n,j}Q_{k-1}(z))p_{n,j}^{(k)}(z)& \\ + \dots + (\lambda_{n,j}^{k-1}Q'_1(z) + \lambda_{n,j}^k Q_0(z))p'_{n,j}(z) = 0,& \end{aligned}$$

which is obtained by differentiating (1.6). Repeating the previous analysis we get

$$\begin{aligned} Q_k(z)\frac{p_{n,j}^{(k+1)}(z)}{n^k p'_{n,j}(z)} + \frac{Q'_k(z) + \lambda_{n,j}Q_{k-1}(z)}{n}\frac{p_{n,j}^{(k)}(z)}{n^{k-1}p'_{n,j}(z)}& \\ + \dots + \frac{\lambda_{n,j}^{k-1}Q'_1(z) + \lambda_{n,j}^k Q_0(z)}{n^k} = 0.& \end{aligned}$$

Hence in the limit we obtain

$$Q_k(z)C^{(2)}(z)\dots C^{(k+1)}(z) + \alpha_j Q_{k-1}(z)C^{(2)}(z)\dots C^{(k)}(z) + \dots + \alpha_j^k Q_0(z) = 0,$$

which implies that $C^{(2)}(z)$ is non-zero a.e. as well. Similarly, $C^{(i)}(z)$, $i \geq 3$, is non-zero a.e., which proves the claim.

The fact that the multiple $\widetilde{\Psi}_j = C_\mu/\alpha_j$ of the Cauchy transform of this common measure μ_j satisfies equation (1.2) almost everywhere follows by (1.7), since the equality of the measures implies that $C_\mu = C^{(1)} = C^{(2)} = \dots$, and thus

$$Q_k(z)(C_\mu(z))^k + \alpha_j Q_{k-1}(z)(C_\mu(z))^{k-1} + \cdots + \alpha_j^k Q_0(z) = 0,$$

which is equivalent to (1.2). □

Note that in Example 1, the polynomials $p_n(z) := z^n - 1$ satisfy the differential equation $zp_n''(z) - (n-1)p_n' = 0$. They may thus be thought of as eigenpolynomials of the pencil $z\frac{d^2}{dz^2} - (\lambda - 1)\frac{d}{dz}$ corresponding to positive integer values n of λ. The corresponding homogenized pencil $z\frac{d^2}{dz^2} - \lambda\frac{d}{dz}$ has $Q_0 = 0$, and so does not satisfy the hypothesis of the proposition.

References

[1] T. Bergkvist and H. Rullgård, On polynomial eigenfunctions for a class of differential operators, Math. Res. Lett. **9** (2002), 153–171. Zbl 1016.34083 MR 1909635

[2] J. Borcea, R. Bøgvad and B. Shapiro, Homogenized spectral problems for exactly solvable operators: asymptotics of polynomial eigenfunctions, Publ. RIMS Kyoto Univ. **45** (2009), 525–568. Zbl 1182.30008 MR 2510511