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Complex 2-Normed Linear Spaces and Extension of Linear 2-Functionals

S. N. Lal, S. Bhattacharya and C. Sreedhar

Abstract. The known concept of 2-normed real linear spaces is extended to 2-normed complex linear spaces. This extension is not trivial. A Hahn-Banach type extension theorem for complex linear 2-functionals is established and it is shown that it is not possible to get this result from the known Hahn-Banach type extension theorem for real linear 2-functionals using the Bohnenblust-Sobczyk technique directly as is done in the case of linear functionals. As an application of our extension theorem, a 2-norm version of the Ascoli-Mazur theorem on tangent functionals is established. Several examples and counter examples illustrate the results obtained in the paper.

Keywords: 2*-norms, linear, convex, 2-bounded and tangent* 2*-functionals, internal and bounding points*

AMS subject classification: Primary 46A22, secondary 46A70

1. Introduction

An area in the Euclidean plane is uniquely determined by three given points in the plane and thus each 2-simplex has its area. A 3-simplex is determined by a quadruple of distinct points in the Euclidean space and thus with each 3-simplex there is associated a non-negative real number known as its volume. Menger [43] in 1928 gave a logical generalisation of the distance function defined on pairs of points with the introduction of an n-metric structure on a set which is a function defined on $(n+1)$-tuples of points. There was not much activity on Menger's n-metric structure for a considerable length of time though from time to time some contributions were made to this theory, and its applications including connections with variational calculus were studied (for details, see Blumenthal [3] and Iseki [36]).

Parallel to the development of the notion of generalised metric, Vulich [44] introduced in 1937 a notion of higher dimensional norm on linear spaces and as in the case of Menger's, this work also failed to attract the attention of the analysts till 1958 when the work of Froda [30] appeared. However, significant developments began in 1962 when S. Gähler [31] introduced the concept of 2-metric spaces axiomatising the concept of area function of Euclidean triangles. In a subsequent paper Gähler [32] extended his idea of 2-metric spaces and introduced the concept of 2-normed real linear spaces. The 2-norm $\|x, y\|$ of x, y in a real linear space represents two times the area of the triangle with

All authors: Banaras Hindu Univ., Dept. Math., Varanasi 221005, India

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sides x and y. With a 2-norm, a real linear space becomes a topological linear space and Gähler proved the existence of 2-normed real linear spaces which are not normable.

The concepts of 2-metric and 2-normed real linear spaces have been investigated extensively by Diminnie, Gähler and White [15, 18], Ehret [23], Freese and Gähler [29], Gähler [31 - 34], Iseki [36], Lal and Das [38], Cho [5], Cho et al. [7 - 13], White [45] and several others [1, 4, 6, 14, 16, 17, 19 - 22, 24 - 28, 35, 37, 39 - 41, 46] from different points of view. In [45] White proved a Hahn-Banach type extension theorem for linear 2-functionals on 2-normed real linear spaces. It is interesting to note that until now the study of 2-normed spaces was restricted only to real linear spaces. Here we introduce the concept of 2-normed linear spaces on a field $\mathbb{K}$, $\mathbb{K}$ being the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers, and prove the Hahn-Banach theorem for these spaces. We then give an application of this extension theorem by establishing Theorem 5.1 embodying a necessary and sufficient condition for the existence of tangent 2-functionals. Our Theorem 5.1 is a complex 2-norm version of the Ascoli-Mazur theorem [2, 42]. We conclude the paper by giving an example illustrating Theorem 5.1 which itself demonstrates the necessity of development of the theory of 2-normed spaces.

It is important to note that the transition from real 2-normed linear spaces to complex 2-normed linear spaces is not that obvious as in the case of normed linear spaces. One of the reasons for this is that a complex normed linear space E remains a real normed linear space when E is considered as a linear space over the real field, but this does not hold in the case of complex 2-normed linear spaces E as in the latter case x and ix (where x is any non-zero element of E) are linearly dependent, but when E is considered over the real field, x and ix become linearly independent (see Definition 2.1).

2. Definition and examples

Let E be a linear space over the field $\mathbb{K}$ standing for the field of all real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$.

Definition 2.1. A mapping $\|\cdot,\cdot\| : E \times E \to \mathbb{R}$ is called a 2-*norm* on E if for all $x, y, z \in E$ and $\alpha \in \mathbb{K}$

(i) $\|x, y\| = 0$ if and only if x, y are linearly dependent.

(ii) $\|x, y\| = \|y, x\|$

(iii) $\|\alpha x, y\| = |\alpha|\, \|x, y\|$

(iv) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$.

The pair $(E, \|\cdot,\cdot\|)$ is called a 2-normed linear space over $\mathbb{K}$.

It may be noted that $\|x, y\| \geq 0$ for all $x, y \in E$. We also note the following obvious fact.

Lemma 2.1. *For* $\alpha \in \mathbb{K}$ *and* $x, y \in E$, $\|\alpha x + y, x\| = \|y, x\|$.

Before giving some examples of complex 2-normed linear spaces we note the following

Lemma 2.2. If $\alpha, \beta \in \mathbb{C}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$, then

$$\sum_{\substack{i,j=1 \\ i<j}}^{n} |\alpha_i \beta_j - \alpha_j \beta_i|^2 = \|\alpha\|^2 \|\beta\|^2 - |\langle \alpha, \beta \rangle|^2$$

where $\langle \alpha, \beta \rangle = \sum_{i=1}^n \alpha_i \bar{\beta}_i$ and $\|\alpha\|^2 = \langle \alpha, \alpha \rangle$.

Proof. Observe that

$$\begin{aligned} &\|\alpha\|^2 \|\beta\|^2 - |\langle \alpha, \beta \rangle|^2 \\ &\quad = \sum_{i=1}^{n} |\alpha_i|^2 |\beta_i|^2 + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} |\alpha_i|^2 |\beta_j|^2 - \sum_{i=j=1}^{n} \alpha_i \bar{\alpha}_j \bar{\beta}_i \beta_j - \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \alpha_i \bar{\beta}_i \bar{\alpha}_j \beta_j \\ &\quad = \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \left(|\alpha_i|^2 |\beta_j|^2 - \alpha_i \beta_j \overline{\alpha_j \beta_i} \right) \\ &\quad = \sum_{\substack{i,j=1 \\ 1<j}}^{n} \left| \alpha_i \beta_j - \alpha_j \beta_i \right|^2 \end{aligned}$$

and the statement follows ∎

Corollary. *If $\alpha, \beta, \gamma \in \mathbb{C}^n$, then*

$$\sqrt{\|\alpha+\beta\|^2 \|\gamma\|^2 - |\langle \alpha+\beta, \gamma \rangle|^2} \leq \sqrt{\|\alpha\|^2 \|\gamma\|^2 - |\langle \alpha, \gamma \rangle|^2} + \sqrt{\|\beta\|^2 \|\gamma\|^2 - |\langle \beta, \gamma \rangle|^2}.$$

Proof. The proof follows using Lemma 2.2 and the Minkowski inequality ∎

Example 2.1. Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{C}$ and define

$$\|x, y\| = \sqrt{\|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2} \qquad (x, y \in X).$$

Then $(X, \|\cdot, \cdot\|)$ is a complex 2-normed space.

Indeed, axioms (i) - (iii) of Definition 2.1 are easy to verify. In order to verify axiom (iv) take $x, y, z \in X$ and let $\mathbb{B} = \{e_i : i \in \Delta\}$ be a Hamel basis for X. Then x, y, z are spanned by finitely many elements of $\mathbb{B}$, say by $\{e_1, \ldots, e_n\} \subset \mathbb{B}$. Consider the linear subspace $[e_1, \ldots, e_n]$ spanned by $e_1, \ldots, e_n$. By the Graham-Schmidt orthonormalization process, there exists an orthonormal set of vectors $\{f_1, \ldots, f_n\}$ such that $[f_1, \ldots, f_n] = [e_1, \ldots, e_n]$. Let now $x = \sum_{i=1}^n \alpha_i f_i$, $y = \sum_{i=1}^n \beta_i f_i$, $z = \sum_{i=1}^n \gamma_i f_i$ and write $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$ where $\alpha, \beta, \gamma \in \mathbb{C}^n$. Then

$$\begin{aligned} \|x, z\| &= \sqrt{\|\alpha\|^2 \|\gamma\|^2 - |\langle \alpha, \gamma \rangle|^2} \\ \|y, z\| &= \sqrt{\|\beta\|^2 \|\gamma\|^2 - |\langle \beta, \gamma \rangle|^2} \\ \|x+y, z\| &= \sqrt{\|\alpha+\beta\|^2 \|\gamma\|^2 - |\langle \alpha+\beta, \gamma \rangle|^2}. \end{aligned}$$

Using the corollary of Lemma 2.2 it immediately follows that $\|x+y, z\| \leq \|x, z\| + \|y, z\|$ which is axiom (iv) of Definition 2.1. Thus it is established that $(X, \|\cdot, \cdot\|)$ is a 2-normed linear space ∎

Example 2.2. Let $H_1, \ldots, H_n$ be inner product spaces over $\mathbb{C}$, $E = H_1 \times \cdots \times H_n$, $\emptyset \neq D \subset \mathbb{C}$ any subset and $f_1, \ldots, f_n : D \to \mathbb{C}$ functions such that, for some $\theta_0 \in D$, $f_i(\theta_0) \neq 0 \quad (1 \leq i \leq n)$. Let $\theta_1, \ldots, \theta_m \in D$ and $a_0, a_1, \ldots, a_m \in \mathbb{R}_+$, $a_0 \neq 0$. For $x, y \in E$, $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, and $\theta \in D$ define $F(x, y; \theta)$ by

$$F(x,y;\theta) = \left(\sum_{i=1}^{n} \|x_i\|_{H_i}^2 |f_i(\theta)|^2\right)\left(\sum_{i=1}^{n} \|y_i\|_{H_i}^2 |f_i(\theta)|^2\right) - \left|\sum_{i=1}^{n} \langle x_i, y_i\rangle_{H_i} |f_i(\theta)|^2\right|^2 \tag{2.1}$$

where $\|\cdot\|_{H_i}$ and $\langle\cdot,\cdot\rangle_{H_i}$ are the norm and scalar product in H_i, respectively, and define $\|x, y\|$ by

$$\|x,y\| = a_0 \sup_{\theta \in D} \sqrt{F(x,y;\theta)} + \sum_{i=1}^{m} a_i \sqrt{F(x,y;\theta_i)}. \tag{2.2}$$

Then $(E, \|\cdot,\cdot\|)$ is a complex 2-normed linear space.

Indeed, the space

$$\left((f_1(\theta)H_1) \times \cdots \times (f_n(\theta)H_n), \sum_{i=1}^{n} \langle f_i(\theta)x_i, f_i(\theta)y_i\rangle_{H_i}\right)$$

is an inner product space for every $\theta \in D$. Applying the Cauchy-Schwarz inequality, it immediately follows that $F(x, y; \theta) \geq 0$ for every $x, y \in E$ and $\theta \in D$. Proceeding as in Example 2.1, it can be easily shown that for every $x, y, z \in E$ and $\theta \in D$

$$\sqrt{F(x+y,z;\theta)} \leq \sqrt{F(x,z;\theta)} + \sqrt{F(y,z;\theta)}$$

and therefore axiom (iv) in Definition 2.1 is satisfied. Axioms (ii) - (iii) are obviously satisfied. Now let us verify axiom (i). For this let $x, y \in E$ be linearly dependent. Then $x = \alpha y$ or $y = \alpha x$ for some $\alpha \in \mathbb{C}$. In both the cases, for every $\theta \in D$, $\big(f_1(\theta)x_1, \ldots, f_n(\theta)x_n\big)$ and $\big(f_1(\theta)y_1, \ldots, f_n(\theta)y_n\big)$ are linearly dependent in $(f_1(\theta)H_1) \times \cdots \times (f_n(\theta)H_n)$ and so $F(x, y; \theta) = 0$ for every $\theta \in D$ which implies $\|x, y\| = 0$. Conversely, let $\|x, y\| = 0$. Then (2.2) implies $F(x, y; \theta) = 0$ for every $\theta \in D$ and so in particular $F(x, y; \theta_0) = 0$ which implies that $\big(f_1(\theta_0)x_1, \ldots, f_n(\theta_0)x_n\big)$ and $\big(f_1(\theta_0)y_1, \ldots, f_n(\theta_0)y_n\big)$ are linearly dependent in $\big(f_1(\theta_0)H_1) \times \cdots \times (f_n(\theta_0)H_n\big)$. As $f_i(\theta_0) \neq 0 \quad (1 \leq i \leq n)$ it follows that $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are linearly dependent in E. This completes the verification of axiom (i) and it follows that $(E, \|\cdot,\cdot\|)$ is a complex 2-normed linear space.

Remarks.

(i) Let $\theta_i \quad (1 \leq i \leq m)$ be such that $\sum_{i=1}^{m} a_i \sqrt{F(x,y;\theta_i)} = 0$ ensures the linear dependence of x and y in E. Then we may drop the condition $f_i(\theta_0) \neq 0 \quad (1 \leq i \leq n)$ for some $\theta_0 \in D$. In this case we may also take $a_0 = 0$.

(ii) In the case of Example 2.1, $\|x+y, z\|^2 + \|x-y, z\|^2 = 2\|x, z\|^2 + 2\|y, z\|^2$ for all $x, y, z \in X$. Indeed,

$$\begin{aligned}\|x+y,z\|^2 + \|x-y,z\|^2 &= \langle x+y, x+y\rangle\langle z,z\rangle - \langle x+y,z\rangle\overline{\langle x+y,z\rangle} \\ &\quad + \langle x-y,x-y\rangle\langle z,z\rangle - \langle x-y,z\rangle\overline{\langle x-y,z\rangle} \\ &= 2\|x,z\|^2 + 2\|y,z\|^2\end{aligned}$$

and the statement is proved.

(iii) The 2-norm identity in statement (ii) above is not necessarily satisfied for the class of Example 2.2 if $n > 1$. Indeed, for this let us consider a special example in which $n = 3$, $H_1 = H_2 = H_3 = \mathbb{C}$, $D = [0, 2\pi]$, $f_1(\theta) = 1$, $f_2(\theta) = \cos\theta$, $f_3(\theta) = \sin\theta$, $a_0 = 1$ and no $\theta_1, \dots, \theta_m$ are fixed, i.e. the second sum in (2.2) is absent. Set $x = (1+i, 1, i)$, $y = (1, i, 1)$ and $z = (1-i, i, 1+i)$. Then

$$\begin{aligned} F(x+y, z; \theta) &= 13 - \sin^2\theta - 2\sin^4\theta \\ F(x-y, z; \theta) &= 5 + 7\sin^2\theta - 10\sin^4\theta \\ F(x, z; \theta) &= 8 - \sin^2\theta - 5\sin^4\theta \\ F(y, z; \theta) &= 1 + 4\sin^2\theta - \sin^4\theta. \end{aligned}$$

Now $\|x+y, z\| = \sup_{\theta\in D}\sqrt{F(x+y, z; \theta)} = \sqrt{13}$, $\|x-y, z\| = \sqrt{249/40}$, $\|x, z\| = \sqrt{8}$ and $\|y, z\| = 2$. Then, clearly,

$$\frac{769}{40} = \|x+y, z\|^2 + \|x-y, z\|^2 \neq 2\|x, z\|^2 + 2\|y, z\|^2 = 24$$

and the statement is shown by this special example.

Example 2.3. Let H be an inner product space over $\mathbb{C}$, $\emptyset \neq D \subset \mathbb{C}$ any subset, $\{f_n\}_{n=1}^{\infty}$ a sequence of complex-valued functions on D such that, for some $\theta_0 \in D$, $f_n(\theta_0) \neq 0$ for every $n \in \mathbb{N}$ and $\sup_{n\in\mathbb{N},\theta\in D}|f_n(\theta)|$ is finite. Let E be the set of all sequences $x = \{x_n\} \in H$ such that $\sup_{\theta\in D}\sum_{n=1}^{\infty}\|x_n\|^2|f_n(\theta)|^2$ is finite. For $x = \{x_n\}, y = \{y_n\} \in E$, $x = y$ if and only if $x_n = y_n$ for every n. Define $x + y = \{x_n + y_n\}$. Since for every $\theta \in D$ and every $n \in \mathbb{N}$

$$\sqrt{\sum_{i=1}^{n}\|x_i + y_i\|^2|f_i(\theta)|^2} \leq \sqrt{\sum_{i=1}^{n}\|x_i\|^2|f_i(\theta)|^2} + \sqrt{\sum_{i=1}^{n}\|y_i\|^2|f_i(\theta)|^2}$$

we have $x + y \in E$. Thus E is a linear space over $\mathbb{C}$. Define for every $\theta \in D$ and $x, y \in E$

$$F(x, y; \theta) = \left(\sum_{n=1}^{\infty}\|x_n\|^2|f_n(\theta)|^2\right)\left(\sum_{n=1}^{\infty}\|y_n\|^2|f_n(\theta)|^2\right) - \left|\sum_{n=1}^{\infty}\langle x_n, y_n\rangle|f_n(\theta)|^2\right|^2.$$

Let $a_0 > 0$, $a_1, \dots, a_m \in \mathbb{R}_+$ and $\theta_1, \dots, \theta_m \in D$. On $E \times E$, define $\|\cdot,\cdot\|$ by

$$\|x, y\| = a_0 \sup_{\theta\in D}\sqrt{F(x, y; \theta)} + \sum_{i=1}^{m} a_i\sqrt{F(x, y; \theta_i)}.$$

Then $(E, \|\cdot,\cdot\|)$ is a complex 2-normed linear space.

3. Failure of the Bohnenblust-Sobczyk technique

In this section we show that the well-known method of obtaining a complex version of the Hahn-Banach theorem for linear functionals fails in the case of linear 2-functionals.

Definition 3.1. Let M and N be two linear subspaces of a linear space E over the field $\mathbb{K}$.

(i) A mapping $f: M \times N \to \mathbb{K}$ such that

$$\begin{aligned} &f\big(\alpha_1 x_1 + \beta_1 y_1, \alpha_2 x_2 + \beta_2 y_2\big) \\ &\quad = \alpha_1\alpha_2 f(x_1, x_2) + \beta_1\alpha_2 f(y_1, x_2) + \alpha_1\beta_2 f(x_1, y_2) + \beta_1\beta_2 f(y_1, y_2) \end{aligned}$$

for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{K}$, all $x_1, y_1 \in M$ and $x_2, y_2 \in N$ is called a *linear 2-functional* with domain $M \times N$.

(ii) A mapping $p: M \times N \to \mathbb{R}$ such that

$$\begin{aligned} &p\big(\alpha\lambda x + (\alpha - \alpha\lambda)x', \beta\mu y + (\beta - \beta\mu)y'\big) \\ &\quad \le \alpha\beta|\lambda\mu| p(x, y) + \alpha|\lambda|(\beta - \beta\mu)p(x, y') \\ &\qquad + (\alpha - \alpha\lambda)\beta|\mu| p(x', y) + (\alpha - \alpha\lambda)(\beta - \beta\mu)p(x', y') \end{aligned}$$

for all $|\lambda|, |\mu| \le 1$, $\alpha, \beta \ge 0$, and $x, x' \in M$, $y, y' \in N$ is called a *convex 2-functional* with domain $M \times N$.

Definition 3.2. A linear 2-functional $f: M \times N \to \mathbb{K}$ is called 2-*bounded* if there exists $k > 0$ such that, for all $(x, y) \in M \times N$, $|f(x, y)| \le k\,\|x, y\|$. For a 2-bounded linear 2-functional f with domain $M \times N$ the infimum over all $k > 0$ satisfying the inequality above is called the *norm* of f and denoted by $\|f\|$.

Proposition 3.1. *Let E be a linear space over $\mathbb{K}$ and f a 2-bounded linear 2-functional with domain $M \times N$. Then $f(x, y) = 0$ for $(x, y) \in M \times N$ if x and y are linearly dependent in E.*

Proof. The proof follows directly from axioms Definition 2.1/(i) and Definition 3.2 ∎

Following arguments similar to those used in the proof of [45: Theorem 2.1] we can establish the following

Proposition 3.2. *Let E be a linear space over $\mathbb{K}$ and f a 2-bounded linear 2-functional with domain $M \times N$. Then*

$$\|f\| = \sup_{\substack{(x,y) \in M \times N \\ \|x,y\| \ne 0}} \frac{|f(x, y)|}{\|x, y\|}.$$

Hahn-Banach-type extension theorems for real linear functionals on real linear spaces have been established by White [45], Ehret [23], Lal and Das [38] and Mabizela [41]. We have shown in [37] that the result due to Mabizela [41] is essentially contained in a theorem due to Lal and Das [38]. The result due to White is as follows:

Theorem A (see [45]). *Let M be a linear subspace of a real 2-normed linear space E and let $z \in E$. Let f be a 2-bounded real linear 2-functional on $M \times [z]$. Then there exists a 2-bounded real linear 2-functional $\tilde{f}$ on $E \times [z]$ such that*

(i) $\tilde{f}(x, y) = f(x, y)$ *for every* $(x, y) \in M \times [z]$

(ii) $\|\tilde{f}\| = \|f\|$.

So far no result on the extension of complex linear 2-functionals on 2-normed complex linear spaces has been obtained. In Example 3.1 we show that the Bohnenblust-Sobczyk technique which is used for obtaining a complex version of the real Hahn-Banach theorem fails in the case of linear 2-functionals.

By $E_{\mathbb{R}}$ we denote the linear space E over the real field $\mathbb{R}$. Before proceeding further we note the following result due to Lal and Das [38].

Theorem B (see [38]). *Let $M_{\mathbb{R}}$ be a subspace of a real linear space $E_{\mathbb{R}}$ and let $z \in E_{\mathbb{R}}$. If p is a non-negative convex 2-functional on $E_{\mathbb{R}} \times E_{\mathbb{R}}$ and U is a linear 2-functional on $M_{\mathbb{R}} \times [z]_{\mathbb{R}}$ with $U(x, \alpha z) \le p(x, \alpha z)$ for every $(x, \alpha z) \in M_{\mathbb{R}} \times [z]_{\mathbb{R}}$, then there exists a linear 2-functional $\tilde{U}$ on $E_{\mathbb{R}} \times [z]_{\mathbb{R}}$ with*

(i) $\tilde{U}(x, \alpha z) \le p(x, \alpha z)$ *for every* $(x, \alpha z) \in E_{\mathbb{R}} \times [z]_{\mathbb{R}}$

(ii) $\tilde{U}(x, \alpha z) = U(x, \alpha z)$ *for every* $(x, \alpha z) \in M_{\mathbb{R}} \times [z]_{\mathbb{R}}$.

Example 3.1. In Example 2.2, let $n = 3$, $H_1 = H_2 = H_3 = \mathbb{C}$, $D = [0, \frac{1}{2}\pi]$, $f_1(\theta) = 1$, $f_2(\theta) = \cos\theta$, $f_3(\theta) = \sin\theta$, $a_0 = a_1 = 1$ and $\theta_1 = \frac{1}{2}\pi$. Consider the linear subspace $M = \{(a, b, 0) : a, b \in \mathbb{C}\} \subset E$ and let $z = (1, 0, 0)$. In Theorem B define the convex 2-functional p on $E_{\mathbb{R}} \times E_{\mathbb{R}}$ by taking $p(x, y) = \|x, y\|$. Note that $\|x, y\|$ is not a 2-norm on $E_{\mathbb{R}}$ as x and ix for any $x \ne 0$ in $E_{\mathbb{R}}$ are linearly independent although $\|x, ix\| = 0$.

Let $f : M \times [z] \to \mathbb{C}$ be defined by

$$f(x, y) = bd \qquad \big(x = (a, b, 0), y = dz\big) \tag{3.1}$$

where $a, b, d \in \mathbb{C}$. Clearly, f is a complex linear 2-functional on $M \times [z]$. For $x = (a, b, c)$ and $y = dz$ where $a = \theta + i\varphi, b = \alpha + i\beta, c = \gamma + i\delta, d = \xi + i\eta$ we first compute $\|x, y\|$. From (2.1) we have

$$\begin{aligned} &F(x, y; \theta) \\ &\quad = \Big[(\theta^2 + \varphi^2) + (\alpha^2 + \beta^2)\cos^2\theta + (\gamma^2 + \delta^2)\sin^2\theta\Big][\xi^2 + \eta^2] - |(\theta + i\varphi)(\xi - i\eta)|^2 \\ &\quad = (\xi^2 + \eta^2)\Big[(\alpha^2 + \beta^2)\cos^2\theta + (\gamma^2 + \delta^2)\sin^2\theta\Big] \\ &\quad = |d|^2\Big[|b|^2\cos^2\theta + |c|^2\sin^2\theta\Big] \end{aligned}$$

and therefore

$$\sup_{\theta \in D} F(x, y; \theta) = \begin{cases} |d|^2|b|^2 & \text{if } |b| \ge |c| \\ |d|^2|c|^2 & \text{if } |b| < |c|. \end{cases}$$

Now from (2.2) we have

$$\|x, y\| = \begin{cases} |d|(|b| + |c|) & \text{if } |b| \ge |c| \\ 2|d|\,|c| & \text{if } |b| < |c|. \end{cases} \tag{3.2}$$

From (3.1) - (3.2) it is clear that $|f(x,y)| = \|x,y\|$ for every $(x,y) \in M \times [z]$, that is, f is a 2-bounded complex linear 2-functional on $M \times [z]$ with $\|f\| = 1$. Bohnenblust-Sobczyk technique suggests that to get an extension $\tilde{f}$ on $E \times [z]$ using Theorem B, we get a real linear 2-functional U on $M_{\mathbb{R}} \times [z]_{\mathbb{R}}$ where $U(x,y) = \operatorname{Re} f(x,y) = \operatorname{Re} bd$ for $x = (a,b,0) \in M_{\mathbb{R}}$ and $y = dz \in [z]_{\mathbb{R}}$. In view of (3.2) it is clear that $|U(x,y)| \le \|x,y\| = p(x,y)$ for every $(x,y) \in M_{\mathbb{R}} \times [z]_{\mathbb{R}}$. Now using Theorem B we get an extension $\widetilde{U}$ on $E_{\mathbb{R}} \times [z]_{\mathbb{R}}$ of U.

To get the required extension $\tilde{f}$ on $E \times [z]$ define

$$\tilde{f}(x,y) = \widetilde{U}(x,y) - i\widetilde{U}(ix,y) \qquad \big((x,y) \in E \times [z]\big).$$

We claim that $\tilde{f}$ is not necessarily a complex linear 2-functional on $E \times [z]$. To see this we consider the extension $\widetilde{U}$ of U on $E_{\mathbb{R}} \times [z]_{\mathbb{R}}$ defined by

$$\widetilde{U}(x,y) = \alpha\xi - \beta\eta + \gamma\xi - \delta\eta \tag{3.7}$$

where $x = (\theta + i\varphi, \alpha + i\beta, \gamma + i\delta)$ and $y = (\xi + i\eta)z$. Note that

$$\begin{aligned} |\widetilde{U}(x,y)| &\le |\alpha\xi - \beta\eta| + |\gamma\xi - \delta\eta| \\ &\le \sqrt{\xi^2 + \eta^2}\big(\sqrt{\alpha^2 + \beta^2} + \sqrt{\gamma^2 + \delta^2}\big) \\ &= |d|(|b| + |c|) \\ &= \|x,y\| \end{aligned}$$

by (3.2) and thus $|\widetilde{U}(x,y)| \le p(x,y)$ for every $(x,y) \in E_{\mathbb{R}} \times [z]_{\mathbb{R}}$. Now take $x = (0,0,1)$ and $y = z$. Then

$$\begin{aligned} \tilde{f}(x,y) &= \widetilde{U}(x,y) - i\widetilde{U}(ix,y) = 1 \\ \tilde{f}(ix,iy) &= \widetilde{U}(ix,iz) - i\widetilde{U}(-x,iz) = 1 \end{aligned}$$

and this shows that $\tilde{f}(ix,iy) \ne -\tilde{f}(x,y)$ from which it follows that $\tilde{f}$ is not a complex linear 2-functional on $E \times [z]$.

4. The Hahn-Banach theorem for 2-functionals

The counter example in Example 3.1 demonstrated the failure of the Bohnenblust-Sobczyk technique in obtaining directly the extension of 2-bounded complex linear 2-functionals. We now establish the following complex version of White's extension theorem.

Theorem 4.1. *Let M be a linear subspace of a 2-normed linear space E over $\mathbb{K}$ and $z \in E$. If f is a 2-bounded linear 2-functional on $M \times [z]$ (or on $[z] \times M$), then there exists a 2-bounded linear 2-functional F on $E \times [z]$ (or on $[z] \times E$) satisfying*

$$\|F\| = \|f\| \tag{4.1}$$

$$F(x,\alpha z) = f(x,\alpha z) \ \text{ for all } (x,\alpha z) \in M \times [z] \tag{4.2}$$

(*or* $F(\alpha z, x) = f(\alpha z, x)$ *for all* $(\alpha z, x) \in [z] \times M$).

Before establishing the theorem we would like to remark that in its proof given below we have used the complex version of the Hahn-Banach theorem, and the latter is derived from the Hahn-Banach theorem for real linear spaces using the Bohnenblust-Sobczyk technique. Thus, as shown in the counter Example 3.1, although one is unable to use the Bohnenblust-Sobczyk technique directly to get the complex version of White's extension theorem, the Bohnenblust-Sobczyk technique enters into the proof through back doors.

Proof of Theorem 4.1. Let f be defined on $M \times [z]$ (the case with $[z] \times M$ as domain of f follows similarly).

If $z = 0$, then by Proposition 3.1 $f(x, zy) = 0$ for every $(x, zy) \in M \times [z]$ and by defining F on $E \times [z]$ by $F(x, \alpha z) = 0$ for every $(x, \alpha z) \in E \times [z]$, the theorem follows.

Now let $z \neq 0$. For an index set I, let $\{z\} \cup \{y_i : i \in I\}$ be a Hamel basis for E. If N is the linear subspace of E generated by $\{y_i : i \in I\}$, then $E = [z] \oplus N$. Denote a mapping $\| \cdot \|$ on N by $\|x\| = \|x, z\|$ for every $x \in N$. Then $(N, \| \cdot \|)$ is a normed linear space over $\mathbb{K}$. We now consider two cases separately.

Case 1: $z \notin M$. Clearly, $M \subset N$. The functional $\tilde{f}$ on M defined by $\tilde{f}(x) = f(x, z)$ is linear and $|\tilde{f}(x)| = |f(x, z)| \leq \|f\| \, \|x, z\| = \|f\| \, \|x\|$, that is, $\tilde{f}$ is bounded on M. Now by Proposition 3.2

$$\|f\| = \sup_{\substack{(x,\alpha z)\in M\times[z] \\ \|x,\alpha z\|\neq 0}} \frac{|f(x, \alpha z)|}{\|x, \alpha z\|} = \sup_{\substack{x\in M \\ \|x\|\neq 0}} \frac{|\tilde{f}(x)|}{\|x\|} = \|\tilde{f}\|$$

which means the norm of $\tilde{f}$ on M. Appealing to the Hahn-Banach theorem we get a bounded linear functional $\widetilde{F}$ on N with

$$\begin{aligned} &\|\widetilde{F}\| = \|\tilde{f}\| = \|f\| \\ &\widetilde{F} = \tilde{f} \quad \text{on } M. \end{aligned} \tag{4.3}$$

Define F on $E \times [z]$ by

$$F(x, \alpha z) = \begin{cases} \alpha \widetilde{F}(x) & \text{if } x \in N \\ \alpha \widetilde{F}(y) & \text{if } x = \beta z + y, \beta \in \mathbb{K} \text{ and } y \in N \text{ (as } E = [z] \oplus N). \end{cases}$$

This is a well-defined linear 2-functional on $E \times [z]$. For $(x, \alpha z) \in M \times [z]$, $F(x, \alpha z) = \alpha \widetilde{F}(x) = \alpha \tilde{f}(x) = f(x, \alpha z)$ and F satisfies (4.2). Again, for $x \in E$ let $x = \beta z + y$ where $\beta \in \mathbb{K}$ and $y \in N$. For $\alpha \in \mathbb{K}$, using (4.3) and Lemma 2.1 we have

$$\begin{aligned} |F(x, \alpha z)| &= |\alpha \widetilde{F}(y)| \leq |\alpha| \, \|f\| \, \|y\| \\ &= |\alpha| \, \|f\| \, \|y, z\| = |\alpha| \, \|f\| \, \|\beta z + y, z\| = \|f\| \, \|x, \alpha z\| \end{aligned}$$

and it follows that F is a 2-bounded linear 2-functional on $E \times [z]$ with $\|F\| = \|f\|$ and (4.1) holds. Thus, in Case 1, the theorem is established.

Case 2: $z \in M$. Clearly, $J = M \cap N$ is a linear subspace of E and $z \notin J$. Using the arguments of Case 1 for f restricted to $J \times [z]$, we get a bounded linear 2-functional F on $E \times [z]$ satisfying $\|F\| = \|f\|$ and

$$F(x, \alpha z) = f(x, \alpha z) \qquad \big((x, \alpha z) \in J \times [z]\big). \tag{4.4}$$

Now let $(x, \alpha z) \in M \times [z]$. Then as $E = [z] \oplus N$, $x = \beta z + y$ where $\beta \in \mathbb{K}$ and $y \in N$. Since $x, z \in M$, it follows that $y \in M$ and so $y \in J$ and therefore, using (4.4),

$$\begin{aligned} F(x, \alpha z) &= F(\beta z + y, \alpha z) = \alpha F(y, z) \\ &= \alpha\beta f(z, z) + \alpha f(y, z) = f(\beta z + y, \alpha z) = f(x, \alpha z) \end{aligned}$$

that is, $F(x, \alpha z) = f(x, \alpha z)$ for all $(x, \alpha z) \in M \times [z]$. This completes the proof of the theorem in Case 2 and the theorem is completely established ∎

5. A 2-norm version of the Ascoli-Mazur theorem

In this section we give an application of Theorem 4.1. For establishing the main result of this section viz. Theorem 5.1 which is a 2-norm version of the Ascoli-Mazur theorem (see, e.g., [2, 42]) using Theorem 4.1, we need some definitions and results which are given below.

Proposition 5.1 (cf. [11: Lemma 2.1]). *Let $(E, \|\cdot,\cdot\|)$ be a 2-normed linear space over $\mathbb{K}$. For all $x, y, z \in E$, $\lim_{h\to 0+} \frac{\|x+hy,z\|-\|x,z\|}{h}$ and $\lim_{h\to 0-} \frac{\|x+hy,z\|-\|x,z\|}{h}$ exist.*

Proof. Let $x, y, z \in E$. Observe that

$$\frac{\|x + h_2 y, z\| - \|x, z\|}{h_2} \le \frac{\|x + h_1 y, z\| - \|x, z\|}{h_1} \qquad \text{for } 0 < h_2 < h_1$$

and

$$\frac{\|x + hy, z\| - \|x, z\|}{h} \ge -\|y, z\| \qquad \text{for } h > 0.$$

Hence

$$\lim_{h\to 0+} \frac{\|x + hy, z\| - \|x, z\|}{h} = \inf_{h>0} \frac{\|x + hy, z\| - \|x, z\|}{h}.$$

The second part of the proposition follows similarly ∎

Definition 5.1. Let $(E, \|\cdot,\cdot\|)$ be a 2-normed linear space over $\mathbb{K}$. For $x, y, z \in E$ we define

$$T_{1\pm}(x, z)(y) = \lim_{h\to 0\pm} \frac{\|x + hy, z\| - \|x, z\|}{h}.$$

The following proposition can be easily established.

Proposition 5.2. *Let* $(E, \|\cdot,\cdot\|)$ *be a 2-normed linear space over* $\mathbb{K}$. *Then, for all* $x, y, z, y_1, y_2 \in E$ *and* $\alpha = |\alpha|e^{i\theta} \in \mathbb{K}$:

(i) $-\|y, z\| \leq T_{1+}(x, z)(y) \leq \|y, z\|$

(ii) $T_{1+}(x, z)(y) = -T_{1-}(x, z)(y)$

(iii) $T_{1+}(x, z)(y_1 + y_2) \leq T_{1+}(x, z)(y_1) + T_{1+}(x, z)(y_2)$

(iv) $T_{1+}(x, z)(\alpha y) = |\alpha| T_{1+}(x, z)(e^{i\theta} y)$

(v) $T_{1+}(x, z)(x) = \|x, z\|$

(vi) $T_{1+}(x, z)(\alpha x) = |\alpha| \cos\theta T_{1+}(x, z)(x)$.

Definition 5.2. Let E be a 2-normed linear space over $\mathbb{K}$ and $A \subset E$ a subset. Then $a \in A$ is called an

(i) *internal point* if for all $x \in E$ there exists an $\varepsilon_x > 0$ such that $a + \delta x \in A$ for $|\delta| \leq \varepsilon_x$.

(ii) *bounding point* if a is neither an internal point of A nor of $E \setminus A$.

Definition 5.3. Let A be a subset of a 2-normed linear space E, x a bounding point of A and $z \in E$ fixed. A 2-bounded linear 2-functional f on $E \times [z]$ is said to be *tangent to* A *at* x *along* z if there exists a number $c \in \mathbb{R}$ such that $\operatorname{Re} f(A \times [z]) \leq c$ and $\operatorname{Re} f(x, z) = c$.

In order to prove Theorem 5.1 we need the following

Lemma 5.1. *Let* $(E, \|\cdot,\cdot\|)$ *be a 2-normed linear space over* $\mathbb{K}$, *let* $0 \neq z \in E$ *and set* $A = \{x \in E : \|x, z\| \leq 1\}$. *Then:*

(i) *x is an internal point of A if and only if* $\|x, z\| < 1$

(ii) *x is a bounding point of A if and only if* $\|x, z\| = 1$.

Proof. If x is an internal point of A, then there exists $\varepsilon_x > 0$ such that $x + \varepsilon_x x \in A$ which gives $\|x + \varepsilon_x x, z\| \leq 1$ and then $\|x, z\| < 1$. Conversely, let x be such that $\|x, z\| < 1$. Write $\varepsilon = 1 - \|x, z\|$, let $y \in E$ and choose δ such that $|\delta|\,\|y, z\| < \frac{1}{2}\varepsilon$. Then $\|x + \delta y, z\| < 1 - \varepsilon + \frac{1}{2}\varepsilon < 1$ and so $x + \delta y \in A$. Thus for all $y \in E$ there exists $\varepsilon'_x > 0$ ($\varepsilon'_x = \frac{1}{2}\varepsilon\|y, z\|$ if $\|y, z\| \neq 0$) such that, for any δ with $|\delta| \leq \varepsilon'_x$, $x + \delta y \in A$. Note that when $\|y, z\| = 0$, then $\|x + \delta y, z\| \leq \|x, z\| < 1$ for all $\delta \in \mathbb{C}$ and so $x + \delta y \in A$. Hence for all x such that $\|x, z\| < 1$, x is an internal point of A.

Let now x be an internal point of $E \setminus A$. Then, by the definition of A, $\|x, z\| > 1$. Conversely, let $\|x, z\| > 1$. Let $y \in E$ be such that $\|y, z\| \neq 0$ and define $\varepsilon_y = \frac{\|x,z\|-1}{2\|y,z\|}$. Suppose, if possible, $x + \delta y \in A$ for $|\delta| \leq \varepsilon_y$. Then $\|x + \delta y, z\| \leq 1$ and we have

$$\|x, z\| \leq \frac{\|x, z\| - 1}{2\|y, z\|}\,\|y, z\| + 1$$

which yields $\|x, z\| \leq 1$, a result in contradiction to our hypothesis. Consequently, $x + \delta y \in E \setminus A$ for $|\delta| \leq \varepsilon_y$ and then x is an internal point of $E \setminus A$. When $y \in E$ is such that $\|y, z\| = 0$, then $y = \alpha z$ for some $\alpha \in \mathbb{K}$. In this case, for any $\delta \in \mathbb{C}$, $\|x + \delta y, z\| = \|x, z\| > 1$ which implies $x + \delta y \in E \setminus A$, and so also x is an internal point of $E \setminus A$.

From what we have established above and Definition 5.2 it is now clear that x is a bounding point of A if and only if $\|x, z\| = 1$. This completes the proof ∎

Theorem 5.1. *Let $(E, \|\cdot,\cdot\|)$ be a 2-normed linear space over $\mathbb{K}$, let $0 \neq z \in E$ (to avoid the trivial case) and set $A = \{x \in E : \|x, z\| \leq 1\}$. If x is a bounding point of A, then a 2-bounded linear 2-functional f on $E \times [z]$ with $f(x, z) = 1$ is tangent to A at x along z if and only if*

$$-T_{1+}(x, z)(-y) \leq \operatorname{Re} f(y, z) \leq T_{1+}(x, z)(y) \tag{5.1}$$

for all $y \in E$. Conversely, if x is a bounding point of A and y is any point in E such that

$$-T_{1+}(x, z)(-y) \leq c \leq T_{1+}(x, z)(y) \tag{5.2}$$

for some $c \in \mathbb{R}$, then there exists a 2-bounded linear 2-functional f on $E \times [z]$ which is tangent to A at x along z with $f(x, z) = 1$ and $\operatorname{Re} f(y, z) = c$.

Proof. Let x be a bounding point of A and let f be a 2-bounded linear 2-functional on $E \times [z]$ which is tangent to A at x along z with $f(x, z) = 1$. Then using Lemma 5.1 we have as $\|x, z\| = 1$

$$\begin{aligned} \operatorname{Re} f(y, z) &= \frac{1 + \operatorname{Re} hf(y, z) - 1}{h} \\ &= \frac{f(x, z) + \operatorname{Re} f(hy, z) - \|x, z\|}{h} \\ &= \frac{\operatorname{Re} f(x + hy, z) - \|x, z\|}{h} \\ &\leq \frac{\|x + hy, z\| - \|x, z\|}{h}. \end{aligned}$$

Taking limit as $h \to 0+$, the above inequality gives

$$\operatorname{Re} f(y, z) \leq T_{1+}(x, z)(y) \tag{5.3}$$

for all $y \in E$. Clearly, then $-T_{1+}(x, z)(-y) \leq \operatorname{Re} f(y, z)$. Combining this with (5.3), (5.1) follows.

Conversely, let f satisfy (5.1). Using Proposition 5.2/(i) we have $\operatorname{Re} f(y, z) \leq T_{1+}(x, z)(y) \leq \|y, z\| \leq 1$ for $y \in A$. Thus f is such that $f(x, z) = 1$ and $\operatorname{Re} f(y, z) \leq 1$ for all $y \in A$ and so f is tangent to A at x along z. This completes the proof of the first part of the theorem.

Now let x be a bounding point of A and $y \in E$ any point such that (5.2) holds. We are required to prove that there is a 2-bounded linear 2-functional f on $E \times [z]$ such that

$$\left.\begin{aligned} &f(x, z) = 1 = \|x, z\| \\ &\operatorname{Re} f(y, z) = c \\ &\operatorname{Re} f(y', z) \leq 1 \quad (y' \in A) \end{aligned}\right\}. \tag{5.4}$$

Consider the linear subspace $[x, y]$ spanned by x and y.

Case 1: x, y linearly dependent. Write $y = \alpha x$ where $\alpha = |\alpha| e^{i\theta}$. Then $[x, y] = [x]$ and by Proposition 5.2/(v)-(vi)

$$T_{1+}(x, z)(y) = |\alpha| \cos\theta T_{1+}(x, z)(x) = |\alpha| \cos\theta \|x, z\| = |\alpha| \cos\theta.$$

Using Lemma 5.1, similarly we can show that

$$-T_{1+}(x,z)(-y) = |\alpha|\cos\theta.$$

Hypothesis (5.2) now clearly implies $|\alpha|\cos\theta = c$. Define f_0 on $[x]\times[z]$ by $f_0(ax,bz) = ab$. Then f_0 is a linear 2-functional on $[x]\times[z]$ and 2-bounded with $\|f_0\| = 1$, as $|f_0(ax,bz)| = |a|\,|b|\,\|x,z\| = \|ax,bz\|$. Clearly, $f_0(x,z) = 1$ and $f_0(y,z) = f_0(\alpha x,z) = \alpha f_0(x,z) = |\alpha|e^{i\theta}$ which gives $\operatorname{Re} f_0(y,z) = |\alpha|\cos\theta = c$. Appealing to the extension Theorem 4.1, we get a 2-bounded linear 2-functional f on $E\times[z]$ such that $\|f\| = 1$, $f(ax,bz) = f_0(ax,bz) = ab$ for all $(ax,bz)\in[x]\times[z]$. This establishes that we have a 2-bounded linear 2-functional f on $E\times[z]$ such that $(5.4)_{1-2}$ hold. For $y'\in A$ we have $|f(y',z)| \le \|f\|\,\|y',z\| \le 1$ and $(5.4)_3$ follows.

Case 2: x,y linearly independent. Then define f_1 on $[x,y]_{\mathbb{R}}$ by

$$f_1(\alpha x+\beta y) = \alpha+\beta c \qquad (\alpha,\beta\in\mathbb{R}).$$

This is a real linear functional on $[x,y]_{\mathbb{R}}$. Define p on $[x,y,ix,iy]_{\mathbb{R}}$ by

$$p\big(\alpha x+\beta y+i\gamma x+i\delta y\big) = \big\|(\alpha+i\gamma)x+(\beta+i\delta)y,z\big\| \qquad (\alpha,\beta,\gamma,\delta\in\mathbb{R}).$$

Then p is a semi-norm on $[x,y,ix,iy]_{\mathbb{R}}$. We claim that

$$|f_1(\alpha x+\beta y)| \le p(\alpha x+\beta y) \qquad (\alpha,\beta\in\mathbb{R}).$$

Equivalently, we show that

$$|\alpha+\beta c| \le \|\alpha x+\beta y,z\| \qquad (\alpha,\beta\in\mathbb{R}). \tag{5.5}$$

For $\alpha = 0$, (5.5) is $|\beta|\,|c| \le |\beta|\,\|y,z\|$ which for $\beta = 0$ is trivially true and for $\beta\ne 0$ can be written as $|c| \le \|y,z\|$. Suppose, if possible, that $|c| > \|y,z\|$. Then, using the second half of Proposition 5.2/(i) and hypothesis (5.2), $|c| > T_{1+}(x,z)(y) \ge c$, that is, $|c| > c$ which implies that c is negative and can be written as $c = -d$ where $d > 0$. Then $d > \|y,z\| = \|-y,z\| \ge T_{1+}(x,z)(-y)$ and therefore $c < -T_{1+}(x,z)(-y)$. By hypothesis $-T_{1+}(x,z)(-y) \le c$ and so $c < c$ which is impossible. Thus we have shown that if $\alpha = 0$, then (5.5) holds for every $\beta\in\mathbb{R}$.

Let now $\alpha\ne 0$. Then (5.5) is equivalent to $|1+\frac{\beta}{\alpha}c| \le \|x+\frac{\beta}{\alpha}y,z\|$ for every $\beta\in\mathbb{R}$. Thus in order to establish (5.5) it is sufficient to show that

$$|1+\gamma c| \le \|x+\gamma y,z\| \qquad (\gamma\in\mathbb{R}).$$

If $\gamma = 0$, this holds trivially as $\|x,z\| = 1$. Now let $\gamma\ne 0$ and suppose that the inequality we have to show is false. Then there exists $\gamma\ne 0$ such that

$$|1+\gamma c| - 1 > \|x+\gamma y,z\| - \|x,z\|. \tag{5.6}$$

For $\gamma > 0$, using the fact that

$$T_{1+}(x,z)(y) = \inf_{h>0}\frac{\|x+hy,z\|-\|x,z\|}{h},$$

(5.6) implies

$$|1+\gamma c|-1>\gamma T_{1+}(x,z)(y)\geq \gamma c,$$

that is $|1+\gamma c|>1+\gamma c$. For $\gamma<0$, (5.6) implies

$$|1+\gamma c|-1>-\gamma T_{1+}(x,z)(-y)\geq (-\gamma)(-c),$$

that is $|1+\gamma c|>1+\gamma c$ again. Thus whether $\gamma>0$ or $\gamma<0$, always $|1+\gamma c|>1+\gamma c$ which implies $1+\gamma c<0$. Hence $|1+\gamma c|>\|x+\gamma y,z\|$ which first implies $-1-\gamma c>|\gamma|\,\|y,z\|-\|x,z\|$ and then, using Proposition 5.2(i) and (5.2),

$$-\gamma c>|\gamma|\,\|y,z\|\geq |\gamma|T_{1+}(x,z)(y)\geq |\gamma|c$$

which is possible only if $c<0$. Again observe that

$$-\gamma c>|\gamma|\,\|y,z\|=|\gamma|\,\|-y,z\|\geq |\gamma|T_{1+}(x,z)(-y)\geq -|\gamma|c,$$

that is, $\gamma(-c)>|\gamma|(-c)$ which implies that $\gamma>|\gamma|$ as $c<0$. But this inequality cannot hold. Hence (5.5) is true for every $\alpha,\beta\in\mathbb{R}$, that is, f_1 is a real linear functional on $[x,y]_{\mathbb{R}}$ satisfying

$$|f_1(\alpha x+\beta y)|\leq p(\alpha x+\beta y)\qquad (\alpha,\beta\in\mathbb{R}).$$

Appealing to the Hahn-Banach theorem we get a real linear functional f_2 on $[x,y,ix,iy]_{\mathbb{R}}$ satisfying

$$\left.\begin{array}{l} f_2(\alpha x+\beta y)=f_1(\alpha x+\beta y)=\alpha+\beta c\\ |f_2(\alpha x+\beta y+i\gamma x+i\delta y)|\leq p(\alpha x+\beta y+i\gamma x+i\delta y)\end{array}\right\}\quad (\alpha,\beta,\gamma,\delta\in\mathbb{R}),\tag{5.7}$$

that is

$$|f_2(\alpha x+\beta y)|\leq \|\alpha x+\beta y,z\|\qquad (\alpha,\beta\in\mathbb{C}).\tag{5.8}$$

Define now f_3 on $[x,y]$ by

$$f_3(u)=f_2(u)-if_2(iu).$$

This is a complex linear functional on $[x,y]$, and for $u\in[x,y]$, if $f_3(u)=e^{i\theta}|f_3(u)|$, we have

$$|f_3(u)|=e^{-i\theta}f_3(u)=f_3(e^{-i\theta}u)=f_2(e^{-i\theta}u)\leq \|e^{-i\theta}u,z\|$$

using (5.8). Hence

$$|f_3(u)|\leq \|u,z\|\qquad (u\in[x,y]).\tag{5.9}$$

Now define f_0 on $[x,y]\times[z]$ by

$$f_0(\alpha x+\beta y,\gamma z)=\gamma f_3(\alpha x+\beta y)\qquad (\alpha,\beta,\gamma\in\mathbb{C}).\tag{5.10}$$

Then f_0 is a complex linear 2-functional on $[x,y]\times[z]$ and

$$f_3(x)=f_2(x)-if_2(ix)=1-if_2(ix)$$

as $f_2(x) = 1$ by (5.7). Using (5.9) we get

$$|f_3(x)| = \sqrt{1 + f_2^2(ix)} \leq \|x, z\| = 1$$

which implies that $f_2(ix) = 0$ and so $f_3(x) = 1$. Hence

$$\left.\begin{array}{l} f_0(x,z) = f_3(x) = \|x,z\| = 1 \\ \operatorname{Re} f_0(y,z) = \operatorname{Re} f_3(y) = f_2(y) = c \quad (\text{by } (5.7), (5.10)) \\ |f_0(\alpha x + \beta y, \gamma z)| = |\gamma|\,|f_3(\alpha x + \beta y)| \leq \|\alpha x + \beta y, \gamma z\| \quad (\text{by } (5.9), (5.10)) \end{array}\right\} \qquad (5.11)$$

for every $\alpha, \beta, \gamma \in \mathbb{C}$. It is now clear that f_0 is a 2-bounded linear 2-functional on $[x, y] \times [z]$ and $\|f_0\| = 1$. Appealing to Theorem 4.1 we get a 2-bounded linear 2-functional f on $E \times [z]$ such that $\|f\| = 1$ and $f(\alpha x + \beta y, \gamma z) = f_0(\alpha x + \beta y, \gamma z)$. Furthermore, $f(x, z) = f_0(x, z) = 1$ using $(5.11)_1$, $\operatorname{Re} f(y, z) = \operatorname{Re} f_0(y, z) = c$ using $(5.11)_2$, and $|f(u, \gamma z)| \leq \|u, \gamma z\|$ for every $(u, \gamma z) \in E \times [z]$. Thus for every $u \in A$, $|f(u, z)| \leq 1$ and it follows that f is tangent to A at x along z with $f(x, z) = 1$ and $\operatorname{Re} f(y, z) = c$. This completes the proof of the theorem ∎

The following example illustrates Theorem 5.1.

Example 5.1. Consider the space $(\mathbb{R}^3, \|\cdot,\cdot\|)$ where for $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$

$$\|x, y\| = \sqrt{(x_1^2 + x_2^2 + x_3^2)(y_1^2 + y_2^2 + y_3^2) - (x_1y_1 + x_2y_2 + x_3y_3)^2}.$$

Write $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$ and take $z = e_3$. Then

$$A = \{x \in E : \|x, z\| \leq 1\} = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq 1\}$$

which is the right circular cylinder with its axis along the direction of e_3 and radius 1. A point of the type $x = (\sin\theta, \cos\theta, x_3)$ $(\theta \in [0, 2\pi])$ is a bounding point for A (see Lemma 5.1) as $\|x, z\| = \sqrt{x_1^2 + x_2^2} = 1$. Note that here x is any point on the surface of the cylinder A.

We now construct a 2-bounded linear 2-functional f on $\mathbb{R}^3 \times [z]$ with $\|f\| > 0$. Write $f(e_1, e_3) = a$ and $f(e_2, e_3) = b$ where $0 \neq a, b \in \mathbb{R}$. For $\alpha \in \mathbb{R}$ and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ define

$$f(x, \alpha z) = \alpha(x_1 a + x_2 b).$$

Note that f is a linear 2-functional on $\mathbb{R}^3 \times [z]$. We claim that $\|f\| = \sqrt{a^2 + b^2}$. Observe that

$$|f(x, \alpha z)| \leq |\alpha| \sqrt{x_1^2 + x_2^2} \sqrt{a^2 + b^2} = \sqrt{a^2 + b^2} \|x, \alpha z\|$$

and therefore f is a 2-bounded linear 2-functional with $\|f\| \leq \sqrt{a^2 + b^2}$. Taking $x_1 = a$ and $x_2 = b$ we have

$$|f(x, z)| = a^2 + b^2 \leq \|f\|\,\|x, z\| = \sqrt{a^2 + b^2}\|f\|$$

and therefore $\sqrt{a^2+b^2} \le \|f\|$. Thus it follows that $\|f\| = \sqrt{a^2+b^2} > 0$.

Now write $a = \gamma \sin\varphi$ and $b = \gamma\cos\varphi$ for some $\varphi \in [0, 2\pi]$. Then

$$f(x, \alpha z) = \alpha\gamma(x_1 \sin\varphi + x_2 \cos\varphi) \tag{5.12}$$

is the required 2-bounded linear 2-functional on $\mathbb{R}^3 \times [z]$ with $\|f\| = \sqrt{a^2+b^2} = \gamma > 0$. For the bounding point $x = (\sin\theta, \cos\theta, x_3)$ of A and $y = (y_1, y_2, y_3) \in \mathbb{R}^3$ we have

$$\begin{aligned} \|x + hy, z\| &= \sqrt{(\sin\theta + hy_1)^2 + (\cos\theta + hy_2)^2} \\ &= \sqrt{[1 + 2h(y_1 \sin\theta + y_2\cos\theta) + \text{ terms involving higher powers of } h]} \end{aligned}$$

and therefore (remember that $\|x, z\| = 1$)

$$T_{1+}(x,z)(y) = \lim_{h\to 0+} \frac{\|x+hy, z\| - \|x, z\|}{h} = y_1 \sin\theta + y_2\cos\theta.$$

Thus

$$\left.\begin{aligned} T_{1+}(x,z)(y) &= y_1\sin\theta + y_2\cos\theta \\ T_{1+}(x,z)(-y) &= -y_1\sin\theta - y_2\cos\theta \end{aligned}\right\}. \tag{5.13}$$

Let now $x = (\sin\theta, \cos\theta, x_3)$ $(\theta \in [0, 2\pi], x_3 \in \mathbb{R})$ be a bounding point of A, and let $f(x, z) = 1$. Then from (5.12) we get $\cos(\varphi - \theta) = \frac{1}{\gamma}$. Suppose now that f is tangent to A at x along z. Then, for every $y = (y_1, y_2, y_3) \in A$, $f(y, z) \le 1$ or $\gamma(y_1 \sin\varphi + y_2\cos\varphi) \le 1$. The second inequality is in particular true for $y_1 = \sin\varphi$ and $y_2 = \cos\varphi$ as $(\sin\varphi, \cos\varphi, y_3) \in A$ for every $y_3 \in \mathbb{R}$. Consequently, $\gamma \le 1$ which in view of $\cos(\varphi - \theta) = \frac{1}{\gamma}$ implies $\cos(\varphi-\theta) \ge 1$. As we always have $\cos(\varphi - \theta) \le 1$, it follows $\cos(\varphi - \theta) = \frac{1}{\gamma} = 1$, that is, $\gamma = 1$. Hence $\|f\| = \gamma = 1$ and $\varphi - \theta = 2n\pi$ where $n \in \mathbb{Z}$. Using (5.12) we get $f(y, z) = y_1 \sin\theta + y_2 \cos\theta$ and then from (5.13) it follows that

$$-T_{1+}(x,z)(-y) = f(y,z) = T_{1+}(x,z)(y) \qquad (y \in \mathbb{R}^3).$$

Note that for the right circular cylinder $y_1^2 + y_2^2 = 1$ the tangent plane at $(\sin\theta, \cos\theta, x_3)$ (the bounding point of A) is $y_1 \sin\theta + y_2\cos\theta = 1$.

Now let f be a 2-bounded linear 2-functional on $\mathbb{R}^3 \times [z]$ with $f(x, z) = 1$ and

$$-T_{1+}(x,z)(-y) \le f(y,z) \le T_{1+}(x,z)(y) \qquad (y \in \mathbb{R}^3). \tag{5.14}$$

We have already seen that $f(x, z) = 1$ implies $\cos(\varphi - \theta) = \frac{1}{\gamma}$. From (5.12) - (5.14) it follows that

$$y_1 \sin\theta + y_2\cos\theta = \gamma(y_1\sin\varphi + y_2\cos\varphi) \qquad \big(y = (y_1, y_2, y_3) \in \mathbb{R}^3\big).$$

Choosing $y_1 = \sin\varphi$, $y_2 = \cos\varphi$ and any $y_3 \in \mathbb{R}$, we have $\cos(\varphi-\theta) = \gamma$. From here and $\cos(\varphi-\theta) = \frac{1}{\gamma}$ it follows that $\gamma = 1$, and so $\|f\| = 1$ and $\varphi = 2n\pi + \theta$ where $n \in \mathbb{Z}$. From

(5.12) we have immediately $f(y,z) = y_1 \sin\theta + y_2 \cos\theta$ and so for $y = (y_1, y_2, y_3) \in A$, that is for $\sqrt{y_1^2 + y_2^2} \le 1$,

$$f(y,z) = y_1 \sin\theta + y_2 \cos\theta = \sqrt{y_1^2 + y_2^2}\,\sin(\theta + \psi) \le 1$$

where $\cos\psi = y_1/\sqrt{y_1^2 + y_2^2}$ and $\sin\psi = y_2/\sqrt{y_1^2 + y_2^2}$. Thus we have shown that the 2-bounded linear 2-functional f on $\mathbb{R}^3 \times [z]$ with $f(x,z) = 1$ and satisfying (5.14) for every $y \in \mathbb{R}^3$ satisfies $f(y,z) \le 1$ for every $y \in A$. Hence f is tangent to A at x along z. This completes the illustration of the first part of the theorem.

We now illustrate the converse part of the theorem. Hypothesis (5.2) in view of (5.13) implies that $y_1 \sin\theta + y_2 \cos\theta = c$. Now consider the 2-bounded linear 2-functional f from $\mathbb{R}^3 \times [z]$ defined by

$$f(y', \alpha z) = \alpha(y_1' \sin\theta + y_2' \cos\theta) \tag{5.15}$$

for every $y' = (y_1', y_2', y_3') \in \mathbb{R}^3$. Then

$$\left.\begin{aligned} f(y,z) &= y_1 \sin\theta + y_2 \cos\theta = c \\ f(x,z) &= \sin^2\theta + \cos^2\theta = 1 \end{aligned}\right\}$$

and for $y' \in A$, that is for $\sqrt{y_1'^2 + y_2'^2} \le 1$,

$$f(y', z) = y_1' \sin\theta + y_2' \cos\theta = \sqrt{y_1'^2 + y_2'^2}\,\sin(\theta + \psi) \le 1$$

where $\cos\psi = y_1'/\sqrt{y_1'^2 + y_2'^2}$ and $\sin\psi = y_2'/\sqrt{y_1'^2 + y_2'^2}$. Thus, if x is a bounding point of A and y is any point in $\mathbb{R}^3$ such that (5.2) holds, then the 2-bounded linear 2-functional f defined by (5.15) on $\mathbb{R}^3 \times [z]$ is tangent to A at x along z with $f(x,z) = 1$ and $f(y,z) = c$. This completes the illustration of the theorem ∎

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