

# The Existence of Non-Trivial Bounded Functionals Implies the Hahn-Banach Extension Theorem

W. A. J. Luxemburg and M. Väth

**Abstract.** We show that it is impossible to prove the existence of a linear (bounded or unbounded) functional on any  $L_\infty/C_0$  without an uncountable form of the axiom of choice. Moreover, we show that if on each Banach space there exists at least one non-trivial bounded linear functional, then the Hahn-Banach extension theorem must hold. We also discuss relations of non-measurable sets and the Hahn-Banach extension theorem.

**Keywords:** *Power of the Hahn-Banach theorem, linear functionals, axiom of choice, axiom of dependent choices, Shelah's model*

**AMS subject classification:** Primary 46A22, 46S30, secondary 03E25, 03E35, 03E65

From the very beginning, the axiom of choice (AC) in Zermelo-Fraenkel's set theory (ZF) was a topic of discussion, mostly due to its consequences on measure theory and functional analysis. Thus, one was looking for weaker axioms: The most important ones are the axiom of dependent choices (DC), the prime ideal theorem (PI), and the Hahn-Banach extension theorem

**(HB)** If  $X$  is a real linear space,  $p$  a sublinear functional on  $X$ , and  $f_0$  a linear functional defined on a subspace of  $X$  with  $f_0(x) \leq p(x)$ , then there exists a linear extension  $f$  of  $f_0$  to  $X$  such that  $f(x) \leq p(x)$ .

The axiom DC (see, e.g., [14]) allows recursive countable choices and is widely accepted: Most “standard” proofs use DC implicitly; a typical example is the classical proof of Baire's category theorem. For the other mentioned axioms, the implications

$$\text{AC} \implies \text{PI} \implies \text{HB}$$

in ZF are known (the latter follows from [19, 20], and was proved independently in [21]). Both implications can not be reversed, as has been shown in [12] and [31], respectively;

---

W. A. J. Luxemburg: California Inst. Techn., Lab. Math. & Phys., Mathematics 253–37, Pasadena, Cal 91125, USA; lux@cco.caltech.edu

M. Väth: University of Würzburg, Dept. Math., Am Hubland, D-97074 Würzburg, Germany vaeth@mathematik.uni-wuerzburg.de

The paper was written while the second author was visiting Caltech as a DAAD fellow. Financial support by the DAAD is gratefully acknowledged.

the second implication can not even be reversed in the presence of DC (see [32]). Thus, one might consider HB + DC as an essential weakening of AC. Nevertheless, it is known that the most striking consequences of the axiom of choice can already be obtained by using only ZF + HB: In particular, the measure extension theorem [24], the existence of non-Lebesgue measurable sets [9], and the Banach-Tarski (and Hausdorff) paradox [30]; see also the survey [7]. However, in ZF + DC, one can not prove the Banach-Tarski paradox [46: Theorem 13.2] and (if the existence of an inaccessible cardinal is consistent) one can also not prove the existence of a non-Lebesgue measurable set [38]. But ZF + DC still suffices to prove a weak form of HB (with certain separability assumptions), see [11: p. 183]. Moreover, for most “concrete” examples of spaces the dual space is always “rather rich” (for the large class of so-called ideal spaces, see, e.g., [44, 48, 50]). Thus, one is led to the question whether there is at least *some* weaker form of HB which can be proved in ZF + DC for *each* Banach space. The purpose of this note is to give a negative answer to this question:

**Theorem 1.** *It is not possible in ZF + DC to prove the existence of a non-trivial linear (bounded or unbounded) functional on any of the spaces  $L_\infty(S)/C_0(S)$ , where  $S$  is a locally compact Hausdorff space with a  $\sigma$ -finite Radon measure.*

Here,  $L_\infty(S)$  denotes the space of all (classes of) measurable and essentially bounded real functions on  $S$  with the ess sup-norm, and  $C_0(S)$  the closed subspace of all continuous functions vanishing at  $\infty$ .

We remark that for a slightly different set theory, a space with the same property was constructed in [17]; however, in the model used there even the countable form of the axiom of choice fails.

We can also prove a positive statement which in a certain sense extends Theorem 1 (at least for bounded functionals). The Hahn-Banach theorem is equivalent to an apparently much weaker statement:

**Theorem 2 (ZF).** *Assume that on each non-trivial Banach space there exists some non-trivial bounded linear functional. Then HB holds.*

We prove Theorem 2 even in ZF (i.e. the axiom DC is not needed). Moreover, the proof is even constructive: The extension  $f$  in HB can be given explicitly in terms of a functional on an appropriate Banach space.

Theorem 2 contains in particular the surprising fact that the existence of *one* non-trivial functional on *each* Banach space implies the existence of “many” such functionals on each Banach space. Be aware that the existence of one non-trivial functional on a *fixed* Banach space does not imply the existence of more functionals on that space. For example, on  $X = (l_\infty/c_0) \times \mathbb{R}$  there is the non-trivial bounded functional  $f(( [x], t )) = t$ , but in view of Theorem 1, the axioms of ZF + DC are not sufficient to prove the existence of another linear functional which is linear independent of  $f$ , because any such functional  $g$  would induce a non-trivial functional on  $l_\infty/c_0$  by means of  $[x] \mapsto g(( [x], 0 ))$ .

Note that we could have replaced “Banach space” in Theorem 2 by “normed linear space”: Indeed, no form of the axiom of choice (not even DC) is required to show that each normed space  $X$  has a completion  $\bar{X}$ , and we may identify the corresponding dual spaces  $X^* = (\bar{X})^*$  by restriction.

**Proof of Theorem 1.** Shelah has proved that there is a model of  $\text{ZF} + \text{DC}$  in which each subset of a complete separable metric space has the property of Baire (this is claimed in [39] and explicitly proved for  $\mathbb{R}$  in [36]; see also [16, 40]). Since in particular each subset of  $\mathbb{R}$  must have the Baire property in this model, it follows from [31] or [46: Theorem 13.5] (this was also remarked without proof in [38]) that the following statement holds in this model:

**(PM) $_{\omega}$**  There is no finitely additive probability measure  $\mu$  on  $\mathbb{N}$  which is defined for all subsets of  $\mathbb{N}$  and vanishes for finite sets.

Alternatively, one may use the model sketched in [32] to see that  $\text{ZF} + \text{DC} + \text{PM}_{\omega}$  is consistent. In [45] it is proved that  $\text{ZF} + \text{DC} + \text{PM}_{\omega}$  implies in particular that the dual space of the normed Köthe space  $L_{\infty}(S)$  with a  $\sigma$ -finite measure space  $S$  coincides in the canonical way with the associate space  $L_1(S)$  of integrable functions. Hence, if  $f \in (L_{\infty}/C_0)^*$  is given, and  $i : L_{\infty} \rightarrow L_{\infty}/C_0$  denotes the quotient mapping, then  $f \circ i \in L_{\infty}^*$  may be written in the form  $(f \circ i)(x) = \int x(s)y(s) ds$  with some  $y \in L_1$ . Since  $f \circ i$  vanishes on  $C_0$ , this implies  $y = 0$  (a.e.) and, consequently,  $f \circ i = 0$ . But  $i$  is onto, and so  $f = 0$ .

So far, we have only excluded the existence of non-trivial *bounded* linear functionals on  $L_{\infty}/C_0$ . But in Shelah’s model, every linear functional on a Banach space is automatically bounded, as follows from the results in [47] ■

**Proof of Theorem 2.** The result can be proved by a modification of the proof from [24]. However, for our case the proof can be simplified. We present a version which also avoids the explicit use of model theory:

Let  $X$  be a real linear space,  $p$  be a sublinear functional on  $X$ , and  $f_0$  be a linear functional defined on a subspace  $X_{f_0} \subseteq X$  such that  $f_0(x) \leq p(x)$  ( $x \in X_{f_0}$ ). Let  $F$  denote the set of all linear extensions  $f$  of  $f_0$  to some subspace  $X_f \subseteq X$  which satisfy  $f(x) \leq p(x)$  ( $x \in X_f$ ). Let  $Y = l_{\infty}(F)$  be the set of all bounded maps  $y : F \rightarrow \mathbb{R}$  endowed with the sup-norm. By  $Y_0$  we denote the subspace of all  $y \in Y$  with the following property: There are finitely many  $x_1, \dots, x_n \in X$  such that  $y(f) = 0$  for all  $f \in F$  with  $\{x_1, \dots, x_n\} \subseteq X_f$ .

One part of the classical proof of the Hahn-Banach theorem shows that for each finitely many  $x_1, \dots, x_n \in X$  there actually is some  $f \in F$  with  $\{x_1, \dots, x_n\} \subseteq X_f$  (this part of the classical proof is constructive). In particular, each  $y \in Y_0$  actually vanishes at some point. Defining  $e_0 \in Y$  by  $e_0(f) \equiv 1$ , we thus have  $\text{dist}(e_0, Y_0) = 1$ . Considering the quotient space  $Z = Y/\overline{Y_0}$  with the quotient mapping  $[\cdot] : Y \rightarrow Z$  onto the canonical equivalence class, we may conclude that  $e = [e_0]$  satisfies  $\|e\| = 1$ . Observe that  $Y_0$  and thus also  $\overline{Y_0}$  is an order ideal in  $Y$  (see, e.g., [49: Theorem 15.19]), and so  $Z$  becomes a Riesz space with the canonical order [49: §19]. The element  $e$  is a strong unit in  $Z$ .

Now we define a mapping  $G : X \rightarrow Z$  by letting  $G(x)$  the equivalence class containing a bounded (e.g. trivial) extension of the mapping  $y : f \mapsto f(x)$  (defined for all  $f \in F$  with  $x \in X_f$ ):  $G(x)$  does actually not depend on the particular choice of the extension, because if  $y_1, y_2$  are two extensions of  $y$ , we must have  $y_1(f) = y_2(f)$  for each  $f \in F$  with  $x \in X_f$ , i.e.  $y_1 - y_2 \in Y_0 \subseteq \overline{Y_0}$ . Now we may conclude that  $G$  is linear. Moreover, for  $x \in X_{f_0}$  the identity  $y(f) = f(x) = f_0(x)e_0(f)$  implies  $G(x) = [f_0(x)e_0] = f_0(x)e$ . Similarly,  $y(f) = f(x) \leq p(x)e_0(f)$  implies  $G(x) \leq p(x)e$  for all  $x \in X$ .

If we now find a positive linear functional  $L : Z \rightarrow \mathbb{R}$  with  $L(e) > 0$ , we are done, because by multiplying with a constant we may assume  $L(e) = 1$ ; then  $F_0(x) = L(G(x))$  is the required extension: Indeed,  $F_0$  is linear,  $F_0(x) = L(f_0(x)e) = f_0(x)$  ( $x \in X_{f_0}$ ), and  $F_0(x) = L(G(x)) \leq L(p(x)e) = p(x)$  ( $x \in X$ ).

To find such an  $L$ , we use that by assumption there is some non-trivial bounded linear functional  $l$  on the non-trivial normed space  $Z$  (without DC, it is not clear whether  $Z$  is a Banach space, but as remarked above, we do not need this fact). Then  $L_0(y) = l(\overline{[y]})$  defines a non-trivial bounded functional on  $Y = l_\infty(F)$  which vanishes on  $\overline{Y_0}$ . Trivially,  $L_0$  is order-bounded, and so  $|L_0|$  is defined (see, e.g., [49: §20]) and is a non-trivial positive functional which satisfies

$$|L_0|(f) = \sup \{ |L_0(g)| : |g| \leq f \} \quad (0 \leq f \in F).$$

This in particular implies that  $|L_0|$  vanishes on  $\overline{Y_0}$ , and so  $|L_0|$  induces a non-trivial positive functional  $L$  on  $Z$  by the formula  $L([y]) = |L_0|(y)$  ■

It is an interesting and still not completely solved question how strong Solovay's axiom is (in ZF + DC):

**(LM)** All subsets of  $\mathbb{R}$  are Lebesgue measurable.

Solovay has proved in his celebrated work [38] the consistency of ZF + DC + LM under the assumption that the existence of an inaccessible cardinal is consistent with ZF + AC. Later, Shelah has shown [36] (see also [33]) that conversely ZF + DC + LM implies the consistency of an inaccessible cardinal. The latter is known to be unprovable by methods formalizable in ZF + AC (see [15: Theorem 27]). In contrast, it can be proved that the following apparently similar statement is consistent with ZF + DC (we have used this fact in the proof of Theorem 1):

**(BP)** All subsets of  $\mathbb{R}$  have the Baire property.

Recall that HB is equivalent to the fact that there is a measure on each Boolean algebra [24]. It has been proved recently that ZF + HB already implies that LM fails [9]. However, the proof uses a measure on “quite a large” Boolean algebra. It is still an open problem how large this Boolean algebra must be. More precisely, it is not known whether ZF + DC + LM implies  $\text{PM}_\omega$ . Observe that, in contrast, it is known that ZF + DC + BP implies  $\text{PM}_\omega$  (we have used this fact in the proof of Theorem 1). Thus, we have the paradoxical situation that the (logically) less restrictive axiom BP implies  $\text{PM}_\omega$ , while this is not known for the (logically) more powerful axiom LM. This is even more astonishing, if one recalls that it is known that already a weakening of LM (for so-called  $\Sigma_2^1$ -sets) implies the corresponding weakening of the statement BP (for  $\Sigma_2^1$ -sets), see [34]. If we would know that LM implies  $\text{PM}_\omega$ , we could alternatively have used any model satisfying ZF + DC + LM in the proof of Theorem 1 (even if BP fails in that model which is, however, not the case in Solovay's model [38]; in the proof of Theorem 1 one then has to replace the result [47] on the automatic continuity of operators by the corresponding result [10]). However, we intend to show now that the “natural” ways to prove  $\text{PM}_\omega$  from ZF + DC + LM must fail. All following arguments take place in ZF + DC:

One of the classical ways [37] (see also [14: Problem 1.4.10] or [25]) to prove the existence of non-measurable sets on  $[0, 1]$  is to consider the family of functions  $x_n : [0, 1] \rightarrow \mathbb{R}$ ,

$$x_n(t) = [2^n t] - 2[2^{n-1}t], \tag{1}$$

(i.e.  $x_n(t)$  is the  $n$ -th number in the binary expansion of  $t$ ), and to define  $f(t) = \lim_n x_n(t)$ , where the limit is understood with respect to a (free) ultrafilter on  $\mathbb{N}$ . Then  $f$  turns out to be non-measurable (observe that by this argument one does not need the full strength of AC: The Boolean prime ideal theorem PI suffices). The same method can be used with certain periodic or almost period functions like  $x_n(t) = \sin(2\pi nt)$  if the ultrafilter is chosen appropriately [22, 41, 43]; see [5] for a characterization of those ultrafilters. Recalling that free ultrafilters may be identified in a canonical way with two-valued singular measures, it is a natural approach to use instead more general singular measures with values in  $\mathbb{R}$ . If each sequence  $y_t = (x_n(t))_n$  is bounded, this means that we put  $f(t) = L(y_t)$ , where  $L$  is an appropriately chosen singular functional on  $l_\infty$  (i.e.  $L$  vanishes on  $c_0$ ). We show now that  $f$  is measurable under natural assumptions on  $L$  and  $x_n$ .

We call a sequence  $x_n$  of functions *Cesàro-constant*, if for almost all  $t$  the limit

$$\lim_{n \rightarrow \infty} \frac{x_1(t) + \dots + x_n(t)}{n} = c$$

exists and is independent of  $t$ .

**Lemma 1** (ZF + DC). *Let  $x_n : [0, 1] \rightarrow \mathbb{R}$  be (Lebesgue) integrable and assume:*

(1) *For each  $c \in \mathbb{R}$  the following number is independent of  $n$ :*

$$M_c = \text{mes}(\{t : x_n(t) < c\}). \tag{2}$$

(2) *For each pair of indices  $k \neq n$  and numbers  $a, b \in \mathbb{R}$ , we have*

$$\text{mes}(\{t : x_k(t) < a\} \cap \{t : x_n(t) < b\}) = M_a M_b. \tag{3}$$

*Then  $x_n$  is Cesàro-constant on  $[0, 1]$  with*

$$c = \int_0^1 x_n(t) dt, \tag{4}$$

*where  $c$  is independent of  $n$ .*

**Proof.** The statement is nothing else but the strong law of large numbers formulated for the probability space  $[0, 1]$ , in the form of e.g. [4]. In fact, interpreting  $x_n$  as a random variable, condition (2) means that all  $x_n$  have the same distribution function, and (4) is the (common) expectation. Condition (3) means that the variables  $x_n$  are pairwise independent ■

The assumptions of Lemma 1 evidently hold for sequence (1) (even if we replace 2 by some other natural number). Let us show now that such a sequence  $x_n$  can not lead to a non-measurable function for a large class of singular functionals  $L$ , even if we perturbate the sequence somewhat:

**Theorem 3** (ZF + DC). *Let  $x_n : [0, 1] \rightarrow \mathbb{R}$  be integrable such that (2) and (3) hold. Assume that  $y_n : [0, 1] \rightarrow \mathbb{R}$  is Cesàro-constant (for example,  $y_n \rightarrow 0$  a.e.). If the sequence  $z_t = (x_n(t) + y_n(t))_n$  is bounded for almost all  $t$ , and  $L$  is a functional on  $l_\infty$  such that*

$$\liminf_{n \rightarrow \infty} \frac{\xi_1 + \dots + \xi_n}{n} \leq L((\xi_n)_n) \leq \limsup_{n \rightarrow \infty} \frac{\xi_1 + \dots + \xi_n}{n}, \tag{5}$$

*then the function  $f(t) = L(z_t)$  is measurable (namely a.e. constant).*

**Proof.** By Lemma 1,  $z_n = x_n + y_n$  is Cesàro-constant with some constant  $c$ . Hence, for almost all  $t$ , we have that for  $\xi_n = x_n(t) + y_n(t)$  both sides of (5) are equal to  $c$ . Thus,  $f(t) = c$  a.e. ■

A natural choice is  $x_n(t) = x(nt)$  where  $x$  is some periodic function, like  $x_n(t) = \sin(nt)$ . However, condition (3) is usually not satisfied in this case, even if  $x$  is 1-periodic and continuous. Anyway, even if  $x$  is only almost periodic, the conclusion of Theorem 3 holds. In this case, we can prove even more:

Recall that a positive linear functional  $L$  on  $l_\infty$  is called a *Banach-Mazur limit*, if  $L((\xi_n)_n) = \lim \xi_n$  whenever the limit exists and  $L((\xi_n)_n) = L((\xi_{n+1})_n)$ . Observe that the classical construction of Banach-Mazur limits (with HB) leads to functionals  $L$  which satisfy (5). We show now that for sequences  $x_n(t) = x(nt)$  with an almost periodic function  $x$  one may even choose an arbitrary Banach-Mazur limit  $L$  and still ends up with a measurable function.

Following Lorentz, we call a bounded sequence  $x = (\xi_n)_n$  *almost convergent* (to some number  $c$ ) if

$$\lim_{n \rightarrow \infty} \sup_{j \in \mathbb{N}_0} \left| \frac{\xi_{1+j} + \dots + \xi_{n+j}}{n} - c \right| = 0.$$

Using HB, Lorentz has proved [18] that a sequence  $x = (\xi_n)_n$  is almost convergent to  $c$  if and only if  $L(x) = c$  for each Banach-Mazur limit  $L$  (one can give a short proof of this equivalence by non-standard methods [27]; but be aware that the construction of non-standard models requires a more powerful form of the axiom of choice than HB). However, we will only use that direction of this equivalence which can be carried out in ZF: If  $x = (\xi_n)_n$  is almost convergent to  $c$ , then  $L(x) = c$  for each Banach-Mazur limit  $L$ .

Recall [28: §23] that a measurable almost periodic function  $x$  on  $\mathbb{R}$  is automatically continuous and bounded, and may be approximated uniformly on  $\mathbb{R}$  by trigonometric polynomials.

**Lemma 2** (ZF). *Let  $x$  be almost periodic on  $\mathbb{R}$  and continuous. Put  $x_n(t) = x(nt)$ . Then for all except countably many  $t \in \mathbb{R}$  the sequence  $x_t = (x_n(t))_n$  is almost convergent to*

$$c = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x(s) ds.$$

**Proof.** First, assume that  $x(t) = e^{i\varphi t}$  with  $\varphi \in \mathbb{R}$ . For  $\varphi = 0$  the statement is trivial, and otherwise

$$\sup_{j \in \mathbb{N}_0} \left| \frac{x_{1+j}(t) + \dots + x_{n+j}(t)}{n} \right| = \sup_{j \in \mathbb{N}_0} \left| \frac{1}{n} e^{i\varphi(j+1)t} \sum_{k=0}^{n-1} e^{i\varphi kt} \right| = \frac{1}{n} \left| \frac{e^{i\varphi nt} - 1}{e^{i\varphi t} - 1} \right|$$

tends to 0 for all  $t$  with  $e^{i\varphi t} \neq 1$  (and the exceptional set is just countable). Thus, the statement follows for all trigonometric polynomials.

For the general case, choose a sequence of trigonometric polynomials  $p_k$  which converges uniformly on  $\mathbb{R}$  to  $x$ . By what we just proved, we have for all except countably many  $t$  that

$$\lim_{n \rightarrow \infty} \sup_{j \in \mathbb{N}_0} \left| \frac{p_k((1+j)t) + \dots + p_k((n+j)t)}{n} - \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T p_k(t) dt \right| = 0 \tag{6}$$

for all  $k \in \mathbb{N}$ . Since  $p_k(mt) \rightarrow x_m(t)$  uniformly in  $m$ , we have

$$\lim_{k \rightarrow \infty} \sup_n \sup_{j \in \mathbb{N}_0} \left| \frac{p_k((1+j)t) + \dots + p_k((n+j)t)}{n} - \frac{x_{1+j}(t) + \dots + x_{n+j}(t)}{n} \right| = 0,$$

and since  $p_k \rightarrow x$  uniformly, we also have

$$\lim_{k \rightarrow \infty} \sup_{T > 0} \left| \frac{1}{T} \int_0^T p_k(t) dt - \frac{1}{T} \int_0^T x(t) dt \right| = 0.$$

Thus, we may pass to the limit  $k \rightarrow \infty$  in (6) and replace there  $p_k$  by  $x$  ■

Lemma 2 generalizes [17: Example 6.5(iii)].

**Theorem 4** (ZF). *Let  $x$  be almost periodic on  $\mathbb{R}$  and continuous. Put  $x_n(t) = x(nt)$ . By  $y_t$ , we denote the sequence  $(x_n(t))_n$ . Then for any Banach-Mazur limit  $L$  the function  $f(t) = L(y_t)$  is measurable on  $\mathbb{R}$  (namely a.e. constant).*

**Proof.** By Lemma 2, the sequence  $y_t$  is, for all except countably many  $t$ , almost convergent to a constant  $c$  which is independent of  $t$ . By the result of Lorentz cited above, this means  $f(t) = c$  ■

The same method of proof does not apply for sequence (1). In fact, if we only assume that the functions in Lemma 1 are not only pairwise but even completely independent (and exclude the trivial case that they are equally distributed), we do not have almost convergence.

**Proposition 1** (ZF + DC). *Let  $x_n : [0, 1] \rightarrow \mathbb{R}$  be Lebesgue integrable such that (2) holds. Assume, moreover:*

- (1) *For each finitely many pairwise different indices  $n_1, \dots, n_k$  and numbers  $c_1, \dots, c_k$  we have*

$$\text{mes}(\{t : x_{n_1}(t) < c_1\} \cap \dots \cap \{t : x_{n_k}(t) < c_k\}) = M_{c_1} \cdots M_{c_k}. \tag{7}$$

- (2) *The functions  $x_n$  are not a.e. constant.*

*Then for almost no  $t \in [0, 1]$  the sequence  $x_n(t)$  is almost convergent.*

**Proof.** By Lemma 1 we only have to prove that for almost all  $t$  the sequence  $x_n(t)$  is not almost convergent to (4). Since the functions  $x_n$  are not equally distributed, there is some  $c_0 < c$  such that  $M_{c_0} > 0$ . Put

$$D_{n,j} = \bigcup_{k=1}^n \{t \in [0, 1] : x_{j+n+k}(t) \geq c_0\}.$$

By (7), we have  $\text{mes } D_{n,j}^c = M_{c_0}^n > 0$ . Moreover, for fixed  $n$  the events  $D_{n,j}$  are completely independent, and so we have

$$\text{mes } \bigcap_{j=0}^J D_{n,j} = \prod_{j=0}^J \text{mes } D_{n,j} = (1 - M_{c_0}^n)^{J+1} \rightarrow 0 \quad (J \rightarrow \infty),$$

which implies that  $D = \cup_{n=1}^\infty \cap_{j=0}^\infty D_{n,j}$  is a null set. But all  $t \in D^c$  have the following property: For each  $n$  we find some  $j$  such that each of the numbers  $x_{jn+1}(t), \dots, x_{jn+n}(t)$  is less than  $c_0 < c$ . This implies that  $x_n(t)$  can not be almost convergent to  $c$  ■

The assumptions of Proposition 1 are satisfied for sequence (1). It thus remains an open problem in ZF + DC + HB whether there exists a Banach-Mazur limit  $L$  such that  $f(t) = L(y_t)$  is non-measurable where  $y_t$  is the sequence given by (1). In this connection, we remark that the results in [26] imply that the abstract averaging integral of any sequence in Proposition 1 in the space  $L_1([0, 1])$  with respect to each Banach-Mazur limit  $L$  equals the integral over the constant function (4). However, this does not answer the question, since it is not clear whether this abstract integral may be written (a.e.) as the pointwise integral with respect to  $L$ : A result like [44: Theorem 4.4.2] (see also [13: p. 68 – 70] or [6]) is not available since Fubini’s theorem may fail for the “product measure *Lebesgue measure*  $\times L$ ”.

To find a required Banach-Mazur limit one only needs that the function  $f$  is not a.e. constant:

**Proposition 2** (ZF + DC). *Let  $y_t$  denote sequence (1),  $L$  be a Banach-Mazur limit, and  $f(t) = L(y_t)$ . If  $A \subseteq [0, 1]$  is disjoint with  $\{1 - \xi : \xi \in A\}$ , then*

$$f^{-1}(A) = \{t \in [0, 1] : f(t) \in A\}$$

*is either a null set or non-measurable. In particular, if  $f$  is measurable, we must have  $f(t) = \frac{1}{2}$  a.e.*

**Proof.** The second statement follows from the first for  $A = [0, \frac{1}{2})$  and  $A = (\frac{1}{2}, 1]$ .

If  $M = f^{-1}(A)$  is measurable, then  $g(t) = \chi_M(t)$  is integrable. Denote the integral by  $c$ . Observe that  $g(t) = g(\frac{1}{2}t)$  and  $g(2^{-n} + t) = g(t)$ . Hence, the substitution rule implies

$$\int_{k2^{-n}}^{(k+1)2^{-n}} g(t) dt = 2^{-n} \int_0^1 g(2^{-n}(k + s)) ds = 2^{-n} \int_0^1 g(s) ds = 2^{-n}c.$$

We may conclude that for all dyadic numbers  $0 \leq a < b \leq 1$  the relation

$$\frac{1}{b - a} \int_a^b g(t) dt = c$$

holds. This implies that  $g(t) = c$  for almost all  $t$ : Indeed, considering appropriate nicely shrinking dyadic intervals, we find  $g(t) = c$  for any Lebesgue point  $t$  (see [35] for the terminology). Hence,  $g$  is a.e. constant, i.e.  $M$  has either measure 0 or 1 (we could also have applied the zero-one law for tail sets [8] to prove this fact; see also [2, 29]).



It remains to exclude that  $\text{mes } M = 1$ : But denoting the constant sequence  $(1)_n$  by  $e$ , we have for any non-dyadic number  $t \in [0, 1]$  the relation

$$1 - f(t) = L(e) - L(y_t) = L(e - y_t) = L(y_{1-t}) = f(1 - t).$$

Hence, the sets

$$M_0 = f^{-1}(\{\xi : 1 - \xi \in A\}) \quad \text{and} \quad \{1 - t : t \in M\}$$

differ at most by the (countably many) dyadic numbers. In particular, if  $\text{mes } M = 1$ , we also have  $\text{mes } M_0 = 1$ . But this is not possible, since the assumption on  $A$  implies that  $M \cap M_0 = \emptyset$  ■

Following Lorentz [18] (see also [1: Chapter II, §3.3]), we introduce for any sequence  $x = (\xi_n)_n$  the numbers

$$q(x) = \inf_{j_1, \dots, j_p} \limsup_{n \rightarrow \infty} \frac{1}{p} \sum_{i=1}^p \xi_{j_i+n} \tag{8}$$

$$q'(x) = \sup_{j_1, \dots, j_p} \liminf_{n \rightarrow \infty} \frac{1}{p} \sum_{i=1}^p \xi_{j_i+n}. \tag{9}$$

It is easily verified that  $q$  is sublinear. Consequently,  $q'(x) = -q(-x)$  is superlinear and  $q'(x) \leq q(x)$  (because  $0 = q(0) = q(x + (-x)) \leq q(x) + q(-x) = q(x) - q'(x)$ ).

The following results are formulated in terms of  $q'$ , but they can analogously be formulated in terms of  $q$  (by swapping the roles of 0 and 1):

**Proposition 3** (ZF). *Let  $x = (\xi_n)_n$  with  $\xi_n \in \{0, 1\}$ . Then  $q'(x) = 0$  if and only if we find for each pattern  $j_1, \dots, j_p$  infinitely many  $n$  with  $\xi_{n+j_1} = \dots = \xi_{n+j_p} = 0$ . Moreover, it is equivalent that for each  $n$  there is some  $j$  with  $\xi_{j+1} = \xi_{j+2} = \dots = \xi_{j+n} = 0$ .*

**Proof.** Denoting the sum in (9) by  $s_n$ , we have  $s_n \in \{0, p^{-1}, 2p^{-1}, \dots, 1\}$ . Hence,  $\liminf s_n = 0$  if and only if  $s_n = 0$  for infinitely many  $n$ . In view of  $\xi_n \geq 0$ , this implies the first statement. Observe that the existence of one  $n$  for each pattern  $j_1, \dots, j_p$  with  $\xi_{n+j_1} = \dots = \xi_{n+j_p} = 0$  implies the existence of infinitely many such  $n$  (and thus is equivalent with  $q'(x) = 0$ ): Indeed, putting  $m = \max\{j_1, \dots, j_p\} + 1$ , consider the patterns  $\{km + j_1, \dots, km + j_p : k = 1, \dots, k_0\}$  with  $k_0 = 1, 2, \dots$ . This implies the second equivalence: For necessity consider the particular pattern  $j_1 = 1, \dots, j_p = p$ , and for sufficiency put  $n = \max\{j_1, \dots, j_p\}$  ■

It follows from the proof of [18: Theorem 1] that a bounded sequence  $x = (\xi_n)_n$  is almost convergent to  $c$  if and only if  $q(x) = q'(x)$  ( $= c$ ). Hence, the following result can be considered as a strengthening of Proposition 1:

**Proposition 4** (ZF + DC). *Let  $y_t$  denote sequence (1). Then for almost all  $t \in [0, 1]$  we have  $q'(y_t) = 0$ .*

**Proof.** It follows from the proof of Proposition 1 that for almost all  $t \in [0, 1]$  the following is true: For each  $n$  we find some  $j$  such that  $x_{jn+1}(t) = x_{jn+2}(t) = \dots = x_{jn+n}(t) = 0$ . In view of Proposition 3, this implies  $q'(y_t) = 0$  ■

Proposition 4 makes the following statement look reasonable (let  $y_t$  denote sequence (1)).

**(LF)** There is a non-null set  $T \subseteq [0, 1]$  such that for each finitely many  $t_1, \dots, t_n \in T$  we have

$$q'(\sup\{y_{t_1}, \dots, y_{t_n}\}) = 0. \quad (10)$$

Taking into account that  $q'$  is monotone and homogeneous, it is equivalent to replace (10) by

$$q'(y_{t_1} + \dots + y_{t_n}) = 0. \quad (11)$$

Putting  $N_t = \{n : x_n(t) = 0\}$  where  $x_n$  is given by (1), LF is equivalent to the statement

**(LF)'** There is a filter  $\mathcal{F}$  on  $\mathbb{N}$  such that  $\{t \in [0, 1] : N_t \in \mathcal{F}\}$  is not a null set and such that for each  $F \in \mathcal{F}$  and each  $n \in \mathbb{N}$  there is some  $j \in \mathbb{N}$  with  $j, j+1, \dots, j+n \in F$ .

This equivalence follows from Proposition 3: If  $\mathcal{F}$  is a filter as above, one may put  $T = \{t : N_t \in \mathcal{F}\}$  in LF; conversely, if a set  $T$  is given as in LF, one may choose  $\mathcal{F}$  as the filter generated by the sets  $N_t$  ( $t \in T$ ). Recall that any filter  $\mathcal{F}$  on  $\mathbb{N}$  is either a null set or non-measurable [3: Proposition 4.1] (see also [2]). Hence, if LF holds, the set  $T$  is contained in a non-measurable subset of  $[0, 1]$ . This has some interesting consequences:

Despite of Proposition 4,  $T$  can not have measure 1. Moreover, in view of [38], it is not possible to prove LF within ZF + DC (if the existence of an inaccessible cardinal is consistent). We do not know whether it is possible to prove LF within ZF + DC + HB.

However, LF is *consistent* with ZF + DC + HB. In fact, the following stronger result follows from [42] which was brought to our attention by a referee of a preceding version of this paper.

**Theorem 5.** ZF + AC *implies* LF.

**Proof.** We apply a special case from [42], namely that the intersection of countably many free ultrafilters on  $\mathbb{N}$  is non-measurable. Let  $\mathcal{U}$  be a free ultrafilter on  $\mathbb{N}$ . For  $k \in \mathbb{N}_0$  put

$$\mathcal{F}_k = \{F \subseteq \mathbb{N} : \{j \in \mathbb{N} : j + k \in F\} \in \mathcal{U}\}.$$

One may check straightforwardly that  $\mathcal{F}_k$  is a free ultrafilter. Hence,  $\mathcal{F} = \bigcap_{k=0}^{\infty} \mathcal{F}_k$  is a non-measurable filter on  $\mathbb{N}$ . We claim that  $\mathcal{F}$  has the property required in LF'. Indeed, a set  $F \subseteq \mathbb{N}$  belongs to  $\mathcal{F}$  if and only if  $\{j \in \mathbb{N} : j + k \in F\} \in \mathcal{U}$  for  $k \in \mathbb{N}_0$ . Hence, we have for any  $F \in \mathcal{F}$  and any  $n$  that the set  $\bigcap_{k=0}^n \{j \in \mathbb{N} : j + k \in F\}$  belongs to  $\mathcal{U}$  and thus contains some element  $j \in \mathbb{N}$ . This means  $j, j+1, \dots, j+n \in F$  ■

**Theorem 6.** In ZF + DC + HB the statement LF is equivalent to the following statement: There is a Banach-Mazur limit  $L$  such that  $\{t : L(y_t) = 0\}$  is not Lebesgue measurable.

**Proof.** If  $M = \{t : L(y_t) = 0\}$  is not Lebesgue measurable, then we may choose  $T = M$ : Indeed, by [18] we must have  $q'(x) \leq L(x)$  for  $x \in l_\infty$ , in particular

$$0 \leq q'(y_{t_1} + \dots + y_{t_n}) \leq L(y_{t_1} + \dots + y_{t_n}) = 0$$

whenever  $t_1, \dots, t_n \in M$ , i.e. (11) holds.

Conversely, let LF hold. In view of Proposition 2, it suffices to prove that there exists some Banach-Mazur limit  $L$  such that  $T \subseteq \{t : L(x_t) = 0\}$ . To find such an  $L$ , let  $X_0 \subseteq l_\infty$  denote the linear hull of the sequences  $y_t$  with  $t \in T$ . We claim that  $q'(x) \leq 0$  for any  $x \in X_0$ : Let  $x = \lambda_1 y_{t_1} + \dots + \lambda_k y_{t_k}$  with  $t_m \in T$  and  $\lambda_m \in \mathbb{R}$ . In view of (10) and Proposition 3, we find for any pattern  $j_1, \dots, j_p$  infinitely many  $n$  such that

$$x_{j_1+n}(t_m) = x_{j_2+n}(t_m) = \dots = x_{j_p+n}(t_m) = 0 \quad (m = 1, \dots, k).$$

With the notation  $x = (\xi_n)_n$ , this implies

$$\liminf_{n \rightarrow \infty} \frac{1}{p} \sum_{i=1}^p \xi_{j_i+n} = \frac{1}{p} \liminf_{n \rightarrow \infty} \sum_{m=1}^k \lambda_m \sum_{i=1}^p x_{j_i+n}(t_m) \leq 0,$$

and so  $q'(x) \leq 0$ , as claimed.

Since  $x \in X_0$  implies  $-x \in X_0$ , we thus also have  $q(x) = -q'(-x) \geq 0$ . Hence, if we define  $L(x) = 0$  ( $x \in X_0$ ), the estimate

$$q'(x) \leq L(x) \leq q(x) \tag{12}$$

holds on  $X_0$ . Applying HB, we may extend  $L$  linear to  $l_\infty$  with  $L(x) \leq q(x)$ . In view of  $-q'(x) = q(-x) \geq L(-x) = -L(x)$ , (12) holds on  $l_\infty$ . Then  $L$  is a Hahn-Banach limit which vanishes on  $X_0$ : For non-negative sequences  $x$  we have  $L(x) \geq q'(x) \geq 0$ . If  $x$  is convergent to  $c$ , then  $q(x) = q'(x) = c$ , i.e.  $L(x) = c$ . Finally, if  $x = (\xi_n)_n$  and  $y = (\xi_{n+1})_n$ , then  $q(x - y) \leq 0$  and  $q(y - x) \leq 0$ , as can be seen by the choice  $j_1 = 1, \dots, j_p = p$  as  $p \rightarrow \infty$  in (8). Hence,  $L(x) - L(y) = L(x - y) \leq 0$  and  $L(y) - L(x) = L(y - x) \leq 0$ , i.e.  $L(x) = L(y)$  ■

The construction of  $L$  in Theorem 6 follows the construction in [18].

**Corollary 1.** *ZF + AC implies the existence of a Banach-Mazur limit  $L$  such that  $\{t : L(y_t) = 0\}$  is not Lebesgue measurable.*

## References

- [1] Banach, S.: *Introduction to Linear Operations*. Amsterdam et al.: North-Holland 1987. Translation of: *Théorie des Opérations Linéaires*. Warsaw: Polish Sci. Publ. 1979.
- [2] Bartoszyński, T. and H. Judah: *Measure and category – filters on  $\omega$* . In: *Set Theory of the Continuum* (eds.: H. Judah et al.). New York et al.: Springer 1992, pp. 175 – 201.
- [3] Bartoszyński, T. and H. Judah: *Set Theory. On the Structure of the Real Line*. Wellesley, Massachusetts (USA): A. K. Peters 1995.
- [4] Bauer, H.: *Probability Theory*. Berlin et al.: de Gruyter 1996.
- [5] Borde, P. and L. Haddad: *Etude de la mesurabilité de la fonction  $^\circ\cos(2\pi\omega x)$ , ou les entiers audibles* (submitted).
- [6] Burgess, D. C. J.: *Abstract moment problems with applications to the  $l^p$  and  $L^p$  spaces*. Proc. London Math. Soc. (3) 4 (1954), 107 – 128.

- [7] Buskes, G.: *The Hahn-Banach Theorem Surveyed* (Diss. Math. (Rozprawy Mat.): Nr. 327). Warszawa: Polish Sci. Publ. 1993.
- [8] Feller, W.: *An Introduction to Probability Theory and its Applications*, Vol. I. New York: John Wiley & Sons 1951.
- [9] Foreman, M. and F. Wehrung: *The Hahn-Banach theorem implies the existence of a non-Lebesgue measurable set*. Fund. Math. 138 (1991), 13 – 19.
- [10] Garnir, H. G.: *Solovay's axiom and functional analysis*. Lect. Notes Math. 399 (1974), 189 – 204.
- [11] Garnir, H. G., de Wilde, M. and J. Schmets: *Analyse Fonctionnelle. Théorie constructive des espaces linéaires à semi-normes*. Tome I: *Théorie générale*. Basel et al.: Birkhäuser 1968.
- [12] Halpern, J. D.: *The independence of the axiom of choice from the Boolean prime ideal theorem*. Fund. Math. 55 (1964), 57 – 66.
- [13] Hille, E. and R. S. Phillips: *Functional Analysis and Semi-Groups* (Amer. Math. Soc. Coll. Publ.). Providence, R. I. (USA): Amer. Math. Soc. 1957.
- [14] Jech, T. J.: *The Axiom of Choice*. Amsterdam: North-Holland Publ. Comp. 1973.
- [15] Jech, T. J.: *Set Theory*, 2nd ed. Berlin et al.: Springer 1997.
- [16] Judah, H. and S. Shelah: *Baire property and the axiom of choice*. Israel J. Math. 84 (1993), 435 – 450.
- [17] Läuchli, H.: *Auswahlaxiom in der Algebra*. Comm. Math. Helv. 37 (1962/1963), 1 – 18.
- [18] Lorentz, G. G.: *A contribution to the theory of divergent sequences*. Acta Math. 80 (1948), 167 – 190.
- [19] Loś, J. and C. Ryll-Nardzewski: *On the applications of Tychonoff's theorem in mathematical proofs*. Fund. Math. 38 (1951), 233 – 237.
- [20] Loś, J. and C. Ryll-Nardzewski: *Effectiveness of the representation theory for Boolean algebras*. Fund. Math. 41 (1954), 49 – 56.
- [21] Luxemburg, W. A. J.: *Two applications of the method of construction of ultrapowers in analysis*. Bull. Amer. Math. Soc. (N.S.) 68 (1962), 416 – 419.
- [22] Luxemburg, W. A. J.: *Addendum to "On the measurability of a function which occurs in a paper by A. C. Zaanen"*. Proc. Netherl. Acad. Sci. (A) 66 (1963), 587 – 590 (Indag. Math. 25 (1963), 587 – 590).
- [23] Luxemburg, W. A. J. (ed.): (Intern. Symp. on the) *Applications of Model Theory to Algebra, Analysis, and Probability* (1967). Toronto: Holt, Rinehart and Winston 1969.
- [24] Luxemburg, W. A. J.: *Reduced powers of the real number system and equivalents of the Hahn-Banach extension theorem*. In: (Intern. Symp. on the) *Applications of Model Theory to Algebra, Analysis, and Probability* (1967). Toronto: Holt, Rinehart and Winston 1969, pp. 123 – 137.
- [25] Luxemburg, W. A. J.: *What is nonstandard analysis?* Amer. Math. Monthly 80 (1973), no. 6, part II, 38 – 67.
- [26] Luxemburg, W. A. J.: *Asymptotic averages*. In: *Aspects of Positivity in Functional Analysis* (Tübingen 1985; eds.: R. Nagel et al.). Amsterdam - New York: North-Holland Publ. Comp. 1986, pp. 169 – 176.
- [27] Luxemburg, W. A. J.: *Nonstandard hulls, generalized limits and almost convergence*. In: *Analysis and Geometry. Trends in Research and Teaching* (eds.: B. Fuchssteiner et al.). Mannheim et al.: BI Wissenschaftsverlag 1992, pp. 19 – 45.

- [28] Maak, W.: *Fastperiodische Funktionen*, 2nd ed. Berlin et al.: Springer 1967.
- [29] Oxtoby, J. C.: *Maß und Kategorie*. Berlin et al.: Springer 1971.
- [30] Pawlikowski, J.: *The Hahn-Banach theorem implies the Banach-Tarski paradox*. Fund. Math. 138 (1991), 20 – 22.
- [31] Pincus, D.: *The strength of the Hahn-Banach theorem*. Lect. Notes Math. 369 (1973), 203 – 248.
- [32] Pincus, D. and R. M. Solovay: *Definability of measures and ultrafilters*. J. Symbolic Logic 42 (1977), 179 – 190.
- [33] Raisonier, J.: *A mathematical proof of S. Shelah's theorem on the measure problem and related results*. Israel J. Math. 48 (1984), 48 – 56.
- [34] Raisonier, J. and J. Stern: *Mesurabilité et propriété de Baire*. C.R. Acad. Sci. Paris Sér. I Math. 296 (1983), 323 – 326.
- [35] Rudin, W.: *Real and Complex Analysis*, 3rd ed. Singapore: McGraw-Hill 1987.
- [36] Shelah, S.: *Can you take Solovay's inaccessible away?* Israel J. Math. 48 (1984), 1 – 47.
- [37] Sierpiński, W.: *Fonctions additives non complètement additives et fonctions non mesurables*. Fund. Math. 30 (1938), 96 – 99.
- [38] Solovay, R. M.: *A model of set-theory in which every set of reals is Lebesgue measurable*. Ann. Math. (2) 92 (1970), 1 – 56.
- [39] Stern, J.: *Le problème de la mesure*. Astérisque 121/122 (1985), 325 – 346.
- [40] Stern, J.: *Regularity properties of definable sets of reals*. Ann. Pure Appl. Logic 29 (1985), 289 – 324.
- [41] Stroyan, K. D. and W. A. J. Luxemburg: *Introduction to the Theory of Infinitesimals*. New York et al.: Acad. Press 1976.
- [42] Talagrand, M.: *Compacts de fonctions mesurables et filtres non mesurables*. Studia Math. 67 (1980), 13 – 43.
- [43] Taylor, R. F.: *On some properties of bounded internal functions*. In: (Intern. symp. on the) Applications of model theory to algebra, analysis, and probability (1967; ed.: W. A. J. Luxemburg). Toronto: Holt, Rinehart and Winston 1969, pp. 167 – 170.
- [44] Väth, M.: *Ideal Spaces*. Lect. Notes Math. 1664 (1997).
- [45] Väth, M.: *The dual space of  $L_\infty$  is  $L_1$* . Indag. Math. 9 (1998), 619 – 625.
- [46] Wagon, S.: *The Banach-Tarski Paradox*, 2nd ed. Cambridge: Univ. Press 1986.
- [47] Wright, J. D. M.: *Functional Analysis for the Practical Man*. In: Functional Analysis: Surveys and Recent Results. Proc. Conf. on Funct. Anal., Paderborn (Germany) 1976 (eds.: K.-D. Bierstedt and B. Fuchssteiner). Amsterdam: North-Holland publ. Comp. 1977, pp. 283 – 290.
- [48] Zaanen, A. C.: *Integration*. Amsterdam: North-Holland Publ. Comp. 1967.
- [49] Zaanen, A. C.: *Introduction to Operator Theory in Riesz Spaces*. Berlin et al.: Springer 1997.
- [50] Zabreïko, P. P.: *Ideal spaces of functions* (in Russian), Part I. Vestnik Jaroslav. Univ. 8 (1974), 12 – 52.