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Nonlinear Diffusion Equations on Bounded Fractal Domains

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Abstract. We investigate nonlinear diffusion equations $\frac{\partial u}{\partial t} = \Delta u + f(u)$ with initial data and zero boundary conditions on bounded fractal domains. We show that these equations possess global solutions for suitable f and small initial data by employing the iteration scheme and the maximum principle that we establish on the bounded fractals under consideration. The Sobolev-type inequality is the starting point of this work, which holds true on a large class of bounded fractal domains and gives rise to an eigenfunction expansion of the heat kernel.

Keywords: *Diffusion equations, fractals, Laplacian, Sobolev-type inequality, heat kernel, iteration scheme, maximum principle*

AMS subject classification: 35K31, 28A58

1. Introduction

We consider the nonlinear diffusion equation

$$\frac{\partial u}{\partial t} = \Delta u + f(u) \qquad (t > 0, x \in V \backslash V_0) \tag{1.1}$$

with given initial data and zero boundary conditions

$$\begin{aligned} u|_{t=0} &= u_0(x) \qquad && (x \in V) \\ u|_{V_0} &= 0 && (t \geq 0) \end{aligned} \tag{1.2}$$

where V is a self-similar (compact) fractal domain in $\mathbb{R}^N$ $(N \geq 1)$ with boundary V_0 and Δ is a "Laplacian" defined on V in an appropriate way. The function $f : \mathbb{R} \to \mathbb{R}$ is assumed to be locally Lipschitz continuous. We suppose that the initial data u_0 lie in $L^2(V)$ and satisfy the compatibility condition $u_0|_{V_0} = 0$.

The boundary V_0 of a self-similar fractal V in $\mathbb{R}^N$ is defined as follows. Let $D \geq 2$ be an integer and $\{\psi_i\}_{i=1}^D$ the system of contractive similitudes:

$$|\psi_i(x) - \psi_i(y)| = \alpha_i |x - y| \qquad (x, y \in \mathbb{R}^N)$$

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where $0 < \alpha_i < 1$ $(i = 1, 2, \dots, D)$. Then there exists a unique non-empty compact set V in $\mathbb{R}^N$ such that $V = \cup_{i=1}^{D} \psi_i(V)$ (see, for example, [5]). The *boundary* V_0 of V is defined by $V_0 = \cup_{i,j=1\ (i\neq j)}^{D} \psi_i^{-1}(V_i \cap V_j)$ where $V_i = \psi_i(V)$ for $1 \leq i \leq D$ (see [22: p. 309]).

A major difficulty in studying equation (1.1) on bounded or unbounded fractal domains is how to define the Laplacian Δ. Recall that the linear equation (1.1) with $f \equiv 0$ was investigated in [15 - 17] on certain bounded fractals from the analytic point of view where the Laplacian is defined directly, and in [1, 2, 8] on more general (bounded or unbounded) fractals from the probabilistic point of view where the Laplacian is viewed as the infinitesimal generator of a strongly continuous semigroup. See also [4, 7, 10, 18]. Equation (1.1) with $f(u) = u^p$ $(p > 1)$ was considered on unbounded fractal domains in [6], where it is proved that non-negative global solutions with non-negative initial data exist if $p > 1 + \frac{2}{d_s}$ and the initial data are sufficiently small, whilst solutions blow up, that is become unbounded in a finite time, if $p \leq 1 + \frac{2}{d_s}$, where d_s is the spectral dimension of the fractal domain under consideration. See also [21].

In this paper we work with equation (1.1) on bounded fractal domains, which is significantly different from the case of unbounded fractals. We assume that there exists a Hilbert space of functions on the fractal domain V, denoted by $H_0^1(V)$, that satisfies a Sobolev-type inequality (see (2.1) below). This is the starting point of this work. Note that $H_0^1(V)$ belongs to the domain of the Dirichlet form W (see [19]). The eigenvalue problem (see (2.2) below) on the space $H_0^1(V)$ has therefore a sequence of eigenfunctions with corresponding positive eigenvalues. Then there exists a heat kernel $k : (0, \infty) \times V \times V \to [0, \infty)$ which may be expressed in terms of eigenfunctions and eigenvalues (see (2.6) below). Several properties of the heat kernel are derived, which imply a strongly continuous contraction semigroup on $L^2(V) = L^2(V; d\mu)$, where μ is the normalized s-Hausdorff measure on V with s the Hausdorff dimension of V, and $\mu(O) > 0$ for all open sets $O \subset V$. Such a measure μ exists for a self-similar set satisfying the open set condition (e.g., post-critically finite self-similar fractals; see [5, 16, 17]). The Laplacian Δ in equation (1.1) may also be interpreted as the infinitesimal generator of this semigroup associated with k (see Section 2).

Recently, a Sobolev-type inequality has been obtained in [14] on post-critically finite self-similar fractals having regular harmonic structures and satisfying the separation condition, including the well-known Sierpínski gasket and Vicsek snowflake in $\mathbb{R}^N$ $(N \geq 2)$. Consequently, such a Hilbert space $H_0^1(V)$ and heat kernel k exist (see the detail in [7, 18] for the case of the Sierpínski gasket). Whether or not a regular harmonic structure exists for a general post-critically finite self-similar fractal is still an active topic.

For such Laplacians, we obtain a maximum principle analogous to the classical result [10, 23] for smooth domains (see Section 3). In Section 4, we use an iteration scheme (see, for example, [26]) and the maximum principle to establish the existence of global solutions to problem (1.1) - (1.2) for suitable f and small initial data u_0.

2. Preliminaries and heat kernels

Let V be a self-similar fractal in $\mathbb{R}^N$ $(N \geq 1)$ and V_0 its boundary. Define the space

$$C_0(V) = \{f : f \text{ is continuous on } V \text{ and } f|_{V_0} = 0\}$$

with the usual supremum norm. Let $H_0^1(V)$ be a Hilbert space in $C_0(V)$ with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$. Throughout this paper we suppose that $H_0^1(V)$ is dense in $C_0(V)$ and

$$|u(x) - u(y)| \leq c_0 |x-y|^\alpha \|u\| \qquad (x, y \in V) \tag{2.1}$$

for all $u \in H_0^1(V)$, where $c_0 > 0$ and $\alpha \in (0, 1]$ are constants. This inequality is termed *Morrey-Sobolev imbedding inequality* in [23].

There is a class of fractals V when such a Hilbert space exists in a natural way. For example, let V be the Sierpínski gasket in $\mathbb{R}^N$ $(N \geq 2)$ with boundary V_0 defined as the $N+1$ corner points in V, that is $V_0 = \{p_0, p_1, \ldots, p_N\}$ for points p_i in $\mathbb{R}^N$ $(0 \leq i \leq N)$ with the property that $|p_i - p_j| = 1$ $(i \neq j)$ and V is the closure of $V_* \equiv \cup_{n=1}^{\infty} V_n$ under the Euclidean metric, where $V_n = \cup_{i=0}^{N} \psi_i(V_{n-1})$ $(n \geq 1)$ with $\psi_i(x) = \frac{1}{2}(x + p_i)$ $(0 \leq i \leq N)$. There exists a Hilbert space $H_0^1(V)$ which is dense in $C_0(V)$, and (2.1) holds with $c_0 = 2N + 3$ and $\alpha = \frac{\log \frac{N+2}{N}}{2 \log 2}$, where $\|u\|^2 = W(u, u)$ for $u \in H_0^1(V)$ with the Dirichlet form W defined by

$$W(u, v) = \lim_{n \to \infty} \left(\frac{N+3}{N+1} \right)^n \sum_{\substack{x, y \in V_n \\ |x-y| = 2^{-n}}} \big(u(x) - u(y)\big)\big(v(x) - v(y)\big)$$

for all $u, v \in H_0^1(V)$ (see [18] for $N = 2$ and [7] for $N \geq 2$). More general cases are treated in [14, 23].

Given the Hilbert space $H_0^1(V)$ and (2.1), we may solve the eigenvalue problem

$$\left.\begin{aligned} \Delta u &= -\lambda u \\ u|_{V_0} &= 0 \end{aligned}\right\}. \tag{2.2}$$

We say that a non-zero function $\psi \in H_0^1(V)$ satisfies this problem if there is a non-negative value λ such that $(\psi, v) = \lambda \int_V \psi(x) v(x)\, d\mu(x)$ for all $v \in H_0^1(V)$, where $(\cdot,\cdot)$ is the inner product of the Hilbert space $H_0^1(V)$ and μ is the normalized s-Hausdorff measure on V with s the Hausdorff dimension of V. Such a function ψ is termed *eigenfunction* of problem (2.2) with *eigenvalue* λ. Using (2.1) and the standard method [20, 27], we have that problem (2.2) has a sequence of solutions φ_n $(n \geq 1)$ in $H_0^1(V)$ with eigenvalues λ_n, and that the φ_n satisfy $\|\varphi_n\|_2 = 1$ and form a complete orthogonal basis of $H_0^1(V)$, that is

$$(\varphi_n, v) = \lambda_n \int_V \varphi_n(x) v(x)\, d\mu(x) \qquad (v \in H_0^1(V)) \tag{2.3}$$

$$(\varphi_i, \varphi_j) = \int_V \varphi_i(x) \varphi_j(x)\, d\mu(x) = 0 \qquad (i \neq j). \tag{2.4}$$

Moreover, the sequence of eigenvalues λ_n satisfies $0 < \lambda_n \uparrow \infty$ as $n \to \infty$ (see [7] for the Sierpínski gasket in $\mathbb{R}^N$ $(N \geq 2)$). Next, we suppose that Weyl's theorem holds, that is

$$c_1 \lambda^{\frac{d_s}{2}} \leq \rho(\lambda) \leq c_2 \lambda^{\frac{d_s}{2}} \tag{2.5}$$

for all λ sufficiently large, where $c_1, c_2 > 0$, $\rho(\lambda)$ is the number of the eigenvalues (with multiplicity) not greater than λ and d_s is the spectral dimension of V. This was addressed in [11] for the Sierpínski gasket in $\mathbb{R}^N$ $(N \geq 2)$ with $d_s = 2\frac{\log N}{\log(N+2)}$, in [17] for post-critically finite fractals and in [24] for variational fractals.

Define

$$k(t,x,y) = \sum_{n=1}^{\infty} \exp(-\lambda_n t)\varphi_n(x)\varphi_n(y) \qquad (t > 0;\ x, y \in V) \tag{2.6}$$

(see, for example, [10]). Then k has the properties of a heat kernel on V, in particular the semigroup property.

Proposition 2.1.

(i) *The series in* (2.6) *is uniformly convergent for all* $x, y \in V$ *and all* $t \geq \eta > 0$, *and so* $k(t,x,y)$ *is well-defined for all* $x, y \in V$ *and all* $t > 0$.

(ii) *For all* $x, y \in V$ *and* $t, s > 0$, $k(t+s,x,y) = \int_V k(t,x,z)k(s,z,y)\,d\mu(z)$.

Proof. Taking $y \in V_0$ in (2.1) and using (2.3), we have that for some $c > 0$

$$\sup_{x \in V} |\varphi_n(x)| \leq c\|\varphi_n\| = c\lambda_n^{\frac{1}{2}} \|\varphi_n\|_2 = c\lambda_n^{\frac{1}{2}}. \tag{2.7}$$

From (2.5), we see that

$$b_1 n^{\frac{2}{d_s}} \leq \lambda_n \leq b_2 n^{\frac{2}{d_s}} \qquad (n \geq 1) \tag{2.8}$$

for some $b_1, b_2 > 0$. Thus for $t \geq \eta > 0$

$$\sup_{x,y \in V} \left| \exp(-\lambda_n t)\varphi_n(x)\varphi_n(y) \right| \leq c^2 b_2 n^{\frac{2}{d_s}} \exp(-b_1 \eta n^{\frac{2}{d_s}})$$

and so

$$\sum_{n=1}^{\infty} \exp(-\lambda_n t)\varphi_n(x)\varphi_n(y)$$

is uniformly convergent on $[\eta, \infty)$ for all $x, y \in V$ since $\sum_{n=1}^{\infty} n^{\frac{2}{d_s}} \exp(-b_1 \eta n^{\frac{2}{d_s}})$ is convergent, proving statement (i).

Let $\eta > 0$. From (2.4) we see that for $t, s \geq \eta > 0$ and $x, y \in V$

$$\begin{aligned}
&\int_V k(t,x,z)k(s,z,y)\,d\mu(z) \\
&\quad = \sum_{n=1}^{\infty} \int_V \exp(-\lambda_n t)\varphi_n(x)\varphi_n(z) \left(\sum_{m=1}^{\infty} \exp(-\lambda_m s)\varphi_m(z)\varphi_m(y) \right) d\mu(z) \\
&\quad = \sum_{n=1}^{\infty} \exp(-\lambda_n(t+s))\varphi_n(x)\varphi_n(y)\|\varphi_n\|_2^2 \\
&\quad = k(t+s,x,y)
\end{aligned}$$

since $\|\varphi_n\|_2 = 1$ $(n \geq 1)$, proving statement (ii) ∎

Define the family of mappings $\{T_t\}_{t>0}$ on $L^2(V)$ by

$$T_t f(x) = \int_V k(t,x,y) f(y)\, d\mu(y) \qquad (x \in V) \tag{2.9}$$

for $f \in L^2(V)$. Clearly, each T_t is linear and symmetric, and satisfies the semigroup property $T_t T_s = T_{t+s}$ $(t, s > 0)$ by virtue of Proposition 2.1/(ii). Moreover, we have

Proposition 2.2. *Each T_t $(t > 0)$ is a contraction on $L^2(V)$, that is*

$$\|T_t f\|_2 \le \|f\|_2 \qquad (f \in L^2(V)). \tag{2.10}$$

Moreover,

$$\lim_{t\downarrow 0} \|T_t f - f\|_2 = 0 \qquad (f \in L^2(V)). \tag{2.11}$$

Proof. Note that $H_0^1(V)$ is dense in $L^2(V)$ since $H_0^1(V)$ is dense in $C_0(V)$, and so by (2.3) we see that $\{\varphi_n\}_{n\ge 1}$ is also a complete orthonormal basis of $L^2(V)$. Let $f \in L^2(V)$. By Parseval's relation, $\|f\|_2^2 = \sum_{n=1}^{\infty} a_n^2$, where $a_n = \int_V f(y)\varphi_n(y)\, d\mu(y)$. Therefore, by (2.9), (2.6) and (2.4), it follows that for $t > 0$

$$\|T_t f\|_2^2 = \sum_{n=1}^{\infty} a_n^2 \exp(-2\lambda_n t) \le \sum_{n=1}^{\infty} a_n^2 = \|f\|_2^2$$

giving (2.10). Further, for $f = \sum_{n=1}^{\infty} a_n \varphi_n \in L^2(V)$,

$$\|T_t f - f\|_2^2 = \sum_{n=1}^{\infty} a_n^2 \big(\exp(-\lambda_n t) - 1\big)^2 \to 0 \qquad (t \downarrow 0) \tag{2.12}$$

since $\sum_{n=1}^{\infty} a_n^2 (\exp(-\lambda_n t) - 1)^2$ is uniformly convergent on $t \ge 0$, giving statement (2.11) ∎

By Proposition 2.2 we see that $\{T_t\}_{t>0}$ is a strongly continuous contraction semigroup on $L^2(V)$. Thus we can define the infinitesimal generator Δ of it by

$$\Delta f = \lim_{h\downarrow 0} h^{-1}(T_h f - f) \tag{2.13}$$

where the limit is taken in the L^2-norm. Let

$$\mathcal{D}(\Delta) = \Big\{ f \in L^2(V) : \lim_{h\downarrow 0} h^{-1}(T_h f - f) \text{ exists in } L^2(V) \Big\}. \tag{2.14}$$

Then $\mathcal{D}(\Delta)$ is dense in $L^2(V)$ (see [28]).

Let Δ be given by (2.13). Then

$$\Delta \varphi_n(x) = -\lambda_n \varphi_n(x) \text{ pointwise in } V\backslash V_0 \qquad (n \ge 1) \tag{2.15}$$

where $\{\varphi_n\}$ is the sequence of eigenfunctions in (2.2). From (2.9), (2.6) it is easily seen that T_t is self-adjoint, that is for $f, g \in L^2(V)$

$$\int_V T_t f(x) g(x)\, d\mu(x) = \int_V T_t g(x) f(x)\, d\mu(x)$$

and so

$$\begin{aligned}\int_V \Delta f(x) g(x)\, d\mu(x) &= \lim_{h\downarrow 0} h^{-1} \int_V (T_h f - f) g(x)\, d\mu(x)\\ &= \lim_{h\downarrow 0} h^{-1} \int_V f(x)(T_h g - g)\, d\mu(x)\\ &= \int_V \Delta g(x) f(x)\, d\mu(x).\end{aligned}$$

Therefore, for $f, g \in L^2(V)$ and $\Delta f, \Delta g \in L^2(V)$ we get the Gauss-Green formula

$$\int_V \Delta f(x) g(x)\, d\mu(x) = \int_V \Delta g(x) f(x)\, d\mu(x). \tag{2.16}$$

Proposition 2.3. *Let k be as in* (2.6). *Then for all $x \in V$, $t_0 > 0$ and $y_0 \in V$, there exists $\frac{\partial k}{\partial t}(t_0, x, y_0)$ and*

$$\frac{\partial k}{\partial t}(t_0, x, y_0) = \Delta k(t_0, x, y_0). \tag{2.17}$$

Proof. Let $t_0 > 0$. From (2.7) - (2.8) the series

$$\sum_{n=1}^{\infty} \lambda_n \exp(-\lambda_n t_0) \varphi_n(x) \varphi_n(y_0)$$

is uniformly convergent for all $x, y_0 \in V$. Thus $\frac{\partial k}{\partial t}(t_0, x, y_0)$ exists and

$$\frac{\partial k}{\partial t}(t_0, x, y_0) = -\sum_{n=1}^{\infty} \lambda_n \exp(-\lambda_n t_0) \varphi_n(x) \varphi_n(y_0) \tag{2.18}$$

for all $x \in V$, $t_0 > 0$ and $y_0 \in V$. On the other hand, we see that for fixed t_0 and y_0, using (2.6) and Proposition 2.1/(ii),

$$\begin{aligned}\lim_{h\downarrow 0} h^{-1}\big(T_h k(t_0, x, y_0) - k(t_0, x, y_0)\big) &= \lim_{h\downarrow 0} h^{-1}\big(k(t_0 + h, x, y_0) - k(t_0, x, y_0)\big)\\ &= \frac{\partial k}{\partial t}(t_0, x, y_0)\end{aligned}$$

giving (2.17) by using (2.13) and the dominated convergence theorem ∎

Proposition 2.4. *Let k be as in* (2.6). *Then*

$$\left.\begin{aligned} k(t,x,y) &\geq 0 \\ \int_V k(t,x,y)\,d\mu(y) &\leq 1 \end{aligned}\right\} \qquad (t>0, x\in V). \tag{2.19}$$

Proof. Let $f \in L^2(V)$. We write $f(x) = \sum_{n=1}^{\infty} a_n \varphi_n(x)$. Then

$$\int_V f(x) T_t f(x)\,d\mu(x) = \sum_{n=1}^{\infty} \exp(-\lambda_n t) a_n^2 \leq \sum_{n=1}^{\infty} a_n^2 = \|f\|_2^2$$

whence

$$0 \leq \int_V f(x) T_t f(x)\,d\mu(x) \leq \|f\|_2^2 \qquad \big(t>0, f\in L^2(V)\big). \tag{2.20}$$

We claim that, for all $f \in L^2(V)$ with $f \geq 0$,

$$T_t f(x) \geq 0 \qquad (t>0). \tag{2.21}$$

To see this, suppose that this is false. Then there exist $x_0 \in V$ and $t_0 > 0$ such that $T_{t_0} f(x_0) < 0$. By the continuity of $T_t f$, we see that there is a neighborhood O of x_0 in V such that $T_{t_0} f(x) < 0$ for $x \in O$. Since $\mu(O) > 0$ we have that, taking $f(x) = 1$ for $x \in O$ and $f(x) = 0$ for $x \in V \backslash O$, $\int_V f(x) T_{t_0} f(x)\,d\mu(x) < 0$ which contradicts with (2.20). From (2.21) and the continuity of the heat kernel k we immediately get that $k(t,x,y) \geq 0$ on $(0,\infty) \times V \times V$.

From (2.20) we have that

$$\int_V f(x)(f(x) - T_t f(x))\,d\mu(x) \geq 0 \qquad (t>0)$$

for all $f \in L^2(V)$ which yields that, for all $f : V \to [0,1]$, $T_t f(x) \leq \max_{x\in V} f(x) \leq 1$ for all $t > 0$ and $x \in V$. We take $f \equiv 1$ on V to give (2.19) ∎

3. The maximum principle

We state the maximum principle on the fractal V. See [13] in the framework of Bauer harmonic spaces.

Proposition 3.1. *Let $T > 0$. Suppose that $v(t,\cdot) \in \mathcal{D}(\Delta)$ is continuous on $[0,T]$ and satisfies*

$$\left.\begin{aligned} \Delta v - av - \tfrac{\partial v}{\partial t} &\leq 0 \qquad (t>0, x\in V\backslash V_0) \\ v|_{t=0} = v_0(x) &\geq 0 \qquad (x\in V) \\ v|_{V_0} &= 0 \qquad (t\geq 0) \end{aligned}\right\} \tag{3.1}$$

where $a > 0$ and $\mathcal{D}(\Delta)$ is as in (2.14). *Then*

$$v(t,x) \geq 0 \qquad \big((t,x) \in (0,T] \times V\big). \tag{3.2}$$

Proof. Suppose $(t_0, x_0) \in (0, T] \times V$ is such that $v(t_0, x_0) < 0$. Since $v(t, x)$ is continuous on $[0, T] \times V$ and $v_0(x) \geq 0$, there must exist $(t_1, x_1) \in (0, T] \times V$ such that v reaches its negative minimum at (t_1, x_1). Note that $\frac{\partial v}{\partial t}(t_1, x_1) \leq 0$, and $\Delta v(t_1, x_1) \geq 0$ since using (2.19)

$$T_h v(t_1, x_1) - v(t_1, x_1) \geq v(t_1, x_1) \left(\int_V k(h, x_1, y)\, d\mu(y) - 1 \right) \geq 0.$$

Therefore,

$$0 \leq \Delta v(t_1, x_1) - \frac{\partial v}{\partial t}(t_1, x_1) \leq a v(t_1, x_1) < 0.$$

But this is a contradiction, proving the statement ∎

Corollary 3.2. *Let $T > 0$. Suppose that $w(t, \cdot) \in \mathcal{D}(\Delta)$ is continuous on $[0, T]$ and satisfies*

$$\left.\begin{array}{rl} \Delta w - aw - \frac{\partial w}{\partial t} \geq 0 & (t > 0, x \in V \backslash V_0) \\ w|_{t=0} = w_0(x) \leq 0 & (x \in V) \\ w|_{V_0} = 0 & (t \geq 0) \end{array}\right\} \tag{3.3}$$

where $a > 0$ and $\mathcal{D}(\Delta)$ is as in (2.14)*. Then*

$$w(t, x) \leq 0 \qquad \big((t, x) \in (0, T] \times V\big). \tag{3.4}$$

Proof. Let $v(t, x) = -w(t, x)$. Then the statement follows immediately from Proposition 3.1 ∎

4. Existence of solutions

We establish the existence of solutions to problem (1.1) - (1.2) for suitable f and small initial data by using an iteration scheme and the maximum principle. To do this, we first investigate the linear problem

$$\left.\begin{array}{rl} \frac{\partial u}{\partial t} = \Delta u & (t > 0, x \in V \backslash V_0) \\ u|_{t=0} = \phi(x) & (x \in V) \\ u|_{V_0} = 0 & (t \geq 0) \end{array}\right\}. \tag{4.1}$$

From (2.16), (2.17) and (2.11) we see that this problem has a unique solution

$$u(t, x) = \int_V k(t, x, y) \phi(y)\, d\mu(y) \tag{4.2}$$

if $\phi \in L^2(V)$. The following proposition states the continuity of solutions to problem (4.1). The results on regular sets were addressed in [3].

Proposition 4.1. *Let $u = u(t,x)$ be the solution of the linear problem* (4.1)*. If the initial data $\phi \in C_0(V)$, then u is continuous on $[0,\infty) \times V$.*

Proof. Since u is the solution of problem (4.1), we see that

$$u(t,x) = \int_V k(t,x,y)\phi(y)\,d\mu(y) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n t)\varphi_n(x) \quad (t>0, x\in V) \tag{4.3}$$

where $a_n = \int_v \varphi_n(y)\phi(y)\,d\mu(y)$. It is easily seen that u is continuous in $(0,\infty)\times V$ since $\sum_{n=1}^{\infty} a_n \exp(-\lambda_n t)\varphi_n(x)$ is uniformly convergent for all $x \in V$ and $t \geq \eta > 0$.

It remains to prove that u is continuous at $\{0\}\times V$. To see this, we first assume that $\phi \in H_0^1(V)$. We write $\phi(x) = \sum_{n=1}^{\infty} a_n\varphi_n(x)$. From (2.3), $\|\varphi_n\|^2 = \lambda_n$ where $\|\cdot\|$ is the norm of $H_0^1(V)$. Thus

$$\|u(t,\cdot)-\phi(\cdot)\|^2 = \left\| \sum_{n=1}^{\infty} a_n\big(\exp(-\lambda_n t)-1\big)\varphi_n(\cdot)\right\|^2 = \sum_{n=1}^{\infty} a_n^2\big(\exp(-\lambda_n t)-1\big)^2\lambda_n$$

for all $t>0$ which implies $\lim_{t\downarrow 0}\|u(t,\cdot)-\phi(\cdot)\| = 0$ since $\sum_{n=1}^{\infty} a_n^2\big(\exp(-\lambda_n t)-1\big)^2\lambda_n$ is uniformly convergent in $t \geq 0$ by noting that

$$\sum_{n=1}^{\infty} a_n^2\big(\exp(-\lambda_n t)-1\big)^2\lambda_n \leq 4\sum_{n=1}^{\infty} a_n^2\lambda_n = 4\|\phi\|^2 \qquad (t\geq 0).$$

Therefore, it follows from (2.1) that for $\phi \in H_0^1(V)$

$$\lim_{t\downarrow 0}\|u(t,\cdot)-\phi(\cdot)\|_\infty \leq c\lim_{t\downarrow 0}\|u(t,\cdot)-\phi(\cdot)\| = 0 \tag{4.4}$$

where $c>0$. For $\phi\in C_0(V)$, there is a sequence $\{\psi_n\}$ in $H_0^1(V)$ such that

$$\|\psi_n - \phi\|_\infty \to 0 \qquad (n\to\infty) \tag{4.5}$$

since $H_0^1(V)$ is dense in $C_0(V)$. On the other hand, setting

$$u_n(t,x) = \int_V k(t,x,y)\psi_n(y)\,d\mu(y)$$

we have that for $t>0$, using (2.19),

$$\big\|u(t,\cdot)-u_n(t,\cdot)\big\|_\infty = \sup_{x\in V}\left|\int_V k(t,x,y)\big(\psi_n(y)-\phi(y)\big)d\mu(y)\right| \leq \|\psi_n-\phi\|_\infty.$$

Hence, we see that for $\phi\in C_0(V)$, using (4.4) and (4.5),

$$\begin{aligned}
&\lim_{t\downarrow 0}\|u(t,\cdot)-\phi(\cdot)\|_\infty\\
&\quad\leq \lim_{t\downarrow 0}\Big[\|u(t,\cdot)-u_n(t,\cdot)\|_\infty + \|u_n(t,\cdot)-\psi_n(\cdot)\|_\infty + \|\psi_n-\phi\|_\infty\Big]\\
&\quad\leq \lim_{t\downarrow 0}\Big[\|u_n(t,\cdot)-\psi_n(\cdot)\|_\infty + 2\|\psi_n-\phi\|_\infty\Big]\\
&\quad= 2\|\psi_n-\phi\|_\infty\\
&\quad\to 0
\end{aligned}$$

as $n\to\infty$ giving the continuity of u at $\{0\}\times V$ ∎

Corollary 4.2. *Suppose that $h = h(t, x)$ is continuous on $[0, \infty) \times V$. Let $v(t, \cdot) \in \mathcal{D}(\Delta)$ be the solution of the linear diffusion problem*

$$\left.\begin{array}{lr} \frac{\partial v}{\partial t} + av = \Delta v + h(t, x) & (t > 0, x \in V \backslash V_0) \\ v|_{t=0} = v_0(x) & (x \in V) \\ v|_{V_0} = 0 & (t \geq 0) \end{array}\right\}$$

where a is a constant and $\mathcal{D}(\Delta)$ is as in (2.14). Then v is continuous on $[0, \infty) \times V$ if the initial data $v_0 \in C_0(V)$.

Proof. Let $w(t, x) = v(t, x)\exp(at)$. Then w satisfies

$$\left.\begin{array}{lr} \frac{\partial w}{\partial t} = \Delta w + \exp{(at)} h(t, x) & (t > 0, x \in V \backslash V_0) \\ w|_{t=0} = v_0(x) & (x \in V) \\ w|_{V_0} = 0 & (t \geq 0) \end{array}\right\}.$$

Therefore,

$$w(t, x) = u(t, x) + \int_0^t d\tau \int_V k(t - \tau, x, y)\exp(a\tau)h(\tau, y)\, d\mu(y) \tag{4.6}$$

where u is the solution of problem (4.1) with the initial data v_0. From Proposition 4.1, u is continuous on $[0, \infty) \times V$ since $v_0 \in C_0(V)$. The second term on the right-hand side of (4.6) is also continuous on $[0, \infty) \times V$ since h is continuous on $[0, \infty) \times V$ ∎

We require the concepts of upper and lower solutions. Let $T > 0$ and $\Gamma_T = (0, T] \times V$. A function $u_1 : \Gamma_T \to \mathbb{R}$ is an *upper* solution of problem (1.1) - (1.2) on Γ_T if $u_1(t, \cdot) \in \mathcal{D}(\Delta)$ for $t \in (0, T]$ and satisfies

$$\left.\begin{array}{rl} \Delta u_1 + f(u_1) - \frac{\partial u_1}{\partial t} \leq 0 & (\text{in } \Gamma_T) \\ u_1|_{t=0} \geq u_0(x) & (x \in V) \\ u_1|_{V_0} = 0 & (t \geq 0) \end{array}\right\}. \tag{4.7}$$

Analogously, a function $v_1 : \Gamma_T \to \mathbb{R}$ is a *lower* solution of problem (1.1) - (1.2) on Γ_T if $v_1(t, \cdot) \in \mathcal{D}(\Delta)$ for $t \in (0, T]$ and satisfies

$$\left.\begin{array}{rl} \Delta v_1 + f(v_1) - \frac{\partial v_1}{\partial t} \geq 0 & (\text{in } \Gamma_T) \\ v_1|_{t=0} \leq u_0(x) & (x \in V) \\ v_1|_{V_0} = 0 & (t \geq 0) \end{array}\right\}. \tag{4.8}$$

As before, Δ is the generator of the semigroup $\{T_t\}_{t>0}$ associated with the heat kernel k.

Given upper and lower solutions u_1 and v_1 in Γ_T with $v_1 \leq u_1$, we choose $M_0 > 0$ so large that $M_0 > L_f$, where L_f is the Lipschitz constant of f, that is $|f(w_2) - f(w_1)| \leq$

$L_f|w_2 - w_1|$ for $w_1, w_2 : \Gamma_T \to \mathbb{R}$ such that $\min_{\Gamma_T} v_1 \le w_1$ and $w_2 \le \max_{\Gamma_T} u_1$. Let $z_1 : \Gamma_T \to \mathbb{R}$ be continuous and $v_1 \le z_1 \le u_1$. We define z_2 by

$$\left.\begin{array}{rll} \Delta z_2 - M_0 z_2 - \frac{\partial z_2}{\partial t} &= -(f(z_1) + M_0 z_1) & (\text{in } \Gamma_T) \\ z_2|_{t=0} &= u_0(x) & (x \in V) \\ z_2|_{V_0} &= 0 & (t \ge 0) \end{array}\right\}. \tag{4.9}$$

From Corollary 4.2, the solution z_2 of problem (4.9) is continuous on $[0,T] \times V$ if $u_0 \in C_0(V)$. Using Propositon 3.1 and (4.7), (4.9) we see that $z_2 \le u_1$ in Γ_T. Similarly, we have $v_1 \le z_2$ by using Corollary 3.2 and (4.8), (4.9).

Let $\mathcal{F}$ be a mapping given by $z_2 = \mathcal{F} z_1$, where z_2 is the solution of problem (4.9) corresponding to z_1. Let $\Omega = \{z : \Gamma_T \to \mathbb{R} |\, v_1 \le z \le u_1\}$. Then $\mathcal{F}$ is a mapping from Ω to Ω.

Proposition 4.3. *$\mathcal{F}$ is a monotone mapping in the sense of Collatz, that is*

$$\mathcal{F}u \le \mathcal{F}v \qquad \text{if } u \le v \tag{4.10}$$

for $\min v_1 \le u, v \le \max u_1$.

Proof. Let $u \le v$ for $\min v_1 \le u, v \le \max u_1$. Then

$$\left.\begin{array}{rll} \Delta \mathcal{F}u - M_0 \mathcal{F}u - \frac{\partial \mathcal{F}u}{\partial t} &= -(f(u) + M_0 u) & (\text{in } \Gamma_T) \\ \mathcal{F}u|_{t=0} &= u_0(x) & (x \in V) \\ \mathcal{F}u|_{V_0} &= 0 & (t \ge 0) \end{array}\right\}$$

and

$$\left.\begin{array}{rll} \Delta \mathcal{F}v - M_0 \mathcal{F}v - \frac{\partial \mathcal{F}v}{\partial t} &= -(f(v) + M_0 v) & (\text{in } \Gamma_T) \\ \mathcal{F}v|_{t=0} &= u_0(x) & (x \in V) \\ \mathcal{F}v|_{V_0} &= 0 & (t \ge 0) \end{array}\right\}.$$

Therefore, setting $w = \mathcal{F}v - \mathcal{F}u$,

$$\left.\begin{array}{rll} \Delta w - M_0 w - \frac{\partial w}{\partial t} &= -\big(f(v) - f(u) + M_0(v-u)\big) & (\text{in } \Gamma_T) \\ w|_{t=0} &= 0 & (x \in V) \\ w|_{V_0} &= 0 & (t \ge 0) \end{array}\right\}.$$

Since $u \le v$ and $M_0 > L_f$, we see that $f(v) - f(u) + M_0(v-u) \ge 0$. Thus by Proposition 3.1 we have $w \ge 0$, giving the statement ∎

We now obtain a solution to problem (1.1) - (1.2) by an iteration procedure.

Lemma 4.4. *If $u_0 \in C_0(V)$ and there are upper and lower solutions u_1 and v_1 of problem* (1.1) – (1.2) *satisfying* (4.7) *and* (4.8), *respectively, then there is a function $u \in L^\infty(\Gamma_T)$ satisfying*

$$u(t,x) = \int_V k(t,x,y) u_0(y)\, d\mu(y) + \int_0^t d\tau \int_V k(t-\tau,x,y) f(u(\tau,y))\, d\mu(y) \tag{4.11}$$

with the property that $v_1 \le u \le u_1$ *in* Γ_T*, where* $T > 0$.

Proof. Inductively, we define $u_n : \Gamma_T \to \mathbb{R}$ by

$$u_{n+1} = \mathcal{F}u_n \qquad (n \ge 1)$$

where u_1 is the upper solution of problem (1.1) - (1.2). Since $\mathcal{F}$ is monotone and $u_2 \le u_1$, we see that

$$u_{n+1} = \mathcal{F}u_n \le \mathcal{F}u_{n-1} = u_n \qquad (n \ge 2),$$

that is the sequence $\{u_n\}$ is decreasing in n for all $(t,x) \in \Gamma_T$. On the other hand, we define

$$v_{n+1} = \mathcal{F}v_n \qquad (n \ge 1)$$

where v_1 is a lower solution of problem (1.1) - (1.2). It follows by Corollary 3.2 that $v_2 \ge v_1$. Thus the sequence $\{v_n\}$ is increasing in n for all $(t,x) \in \Gamma_T$. Moreover, $v_n \le u_n$ for all $n \ge 1$ since $v_1 \le u_1$, and $v_n = \mathcal{F}v_{n-1} \le \mathcal{F}u_{n-1} = u_n$ if $v_{n-1} \le u_{n-1}$. Thus

$$v_1 \le u_n \le u_1 \qquad \text{in } \Gamma_T \text{ for } n \ge 1. \tag{4.12}$$

Therefore, there exists $u : \Gamma_T \to \mathbb{R}$ with the property $v_1 \le u \le u_1$ such that

$$\lim_{n\to\infty} u_n(t,x) = u(t,x) \qquad \text{pointwise in } \Gamma_T. \tag{4.13}$$

We have

$$\begin{aligned} u_{n+1}(t,x) &= \mathcal{F}u_n(t,x) \\ &= \int_V k(t,x,y)u_0(y)\,d\mu(y) \\ &\quad + \int_0^t d\tau \int_V k(t-\tau,x,y)\Big[f(u_n(\tau,y)) + M_0\big(u_n(\tau,y) - u_{n+1}(\tau,y)\big)\Big]d\mu(y) \end{aligned}$$

giving the statement by letting $n \to \infty$ and using the dominated convergence theorem ∎

Proposition 4.5. *Let* $u = u(t,x)$ *be bounded and satisfying* (4.11)*. Suppose that* $f \in C_1(\mathbb{R})$ *and* $u_0 = u_0(x)$ *is such that*

$$\frac{\partial}{\partial t}T_t u_0 \qquad \text{exists and is bounded for all } t > 0 \text{ and } x \in V \tag{4.14}$$

where $T_t u_0 = \int_V k(t,x,y)u_0(y)\,d\mu(y)$*. Then* u *satisfies equation* (1.1) *pointwise, where* Δ *is the generator of the semigroup* $\{T_t\}_{t>0}$ *associated with the heat kernel* $k = k(t,x,y)$*.*

Proof. Set $u_0(t,x) = T_t u_0(x)$. Since u satisfies (4.11) we have that for $\delta > 0$

$$\begin{aligned} u(t+\delta,x) - u(t,x) &= u_0(t+\delta,x) - u_0(t,x) \\ &\quad + \int_0^{t+\delta} d\tau \int_V k(\tau,x,y) f\big(u(t+\delta-\tau,y)\big)d\mu(y) \\ &\quad - \int_0^t d\tau \int_V k(\tau,x,y) f\big(u(t-\tau,y)\big)d\mu(y) \\ &= u_0(t+\delta,x) - u_0(t,x) \\ &\quad + \int_t^{t+\delta} d\tau \int_V k(\tau,x,y) f\big(u(t+\delta-\tau,y)\big)d\mu(y) \\ &\quad + \int_0^t d\tau \int_V k(\tau,x,y)\Big[f\big(u(t+\delta-\tau,y)\big) - f\big(u(t-\tau,y)\big)\Big]d\mu(y). \end{aligned}$$

Letting

$$g(t) = \sup_{x \in V} |u(t+\delta, x) - u(t,x)| \qquad (t > 0)$$

we see that, using (2.19) and (4.14),

$$g(t) \le M_1 \left(\delta + \int_0^t g(t-\tau)\, d\tau \right) \qquad (t > 0)$$

since f is Lipschitz and u is bounded, where M_1 is a constant. Applying Gronwall's inequality, it follows that

$$g(t) \le M_1 \delta \exp(M_1 t) \qquad (t > 0) \tag{4.15}$$

which implies $u(t,x)$ is uniformly Lipschitz on $t \in (0,T]$ for all $x \in V$ and all $T > 0$, and so $\frac{\partial u}{\partial t}$ exists for almost every $t > 0$ and all $x \in V$. Thus the second term on the right-hand side of (4.11) is differentiable with respect to $t > 0$ and its derivative equals

$$\int_V k(t,x,y) f(u_0(y))\, d\mu(y) + \int_0^t d\tau \int_V k(\tau, x, y) \frac{\partial f(u)}{\partial u}(t-\tau, y) \frac{\partial u}{\partial t}(t-\tau, y)\, d\mu(y)$$

for all $x \in V$ and $t > 0$. It is not hard to verify that Δu exists for all $t > 0$ and all $x \in V$ since $\frac{\partial u}{\partial t}$ exists for all $t > 0$ and all $x \in V$, and

$$\Delta u(t,x) = \frac{\partial u}{\partial t}(t,x) - f(u(t,x))$$

for all $t > 0$ and all $x \in V$ (see [6]) ∎

Note that if $u_0(x) = \int_V w_0(y) k(\delta, x, y_0)\, d\mu(y)$ where $\delta > 0$ and $w_0 \in L^1(V)$, then u_0 satisfies (4.14). Another example when (4.14) holds is that $u_0 = \sum_{n=1}^{\infty} a_n \varphi_n \in L^2(V)$ with $\sum_{n=1}^{\infty} |a_n| \lambda_n^{\frac{3}{2}} < \infty$.

Theorem 4.6. *Suppose that $|f(r)| \le \lambda_1 |r|$ for $|r| \le b$, for some $b > 0$, and that $u_0 \in C_0(V)$ satisfies $|u_0(x)| \le M\varphi_1(x)$ in V where λ_1 is the smallest eigenvalue of* (2.2) *with eigenfunction φ_1 and M so small that $\max \varphi_1 \le \frac{b}{M}$. Then for any $T > 0$ there exists $u \in L^\infty((0,T) \times V)$ satisfying* (4.11)*. Moreover, if $f \in C_1$ and the initial data u_0 satisfies* (4.11)*, then $u = u(t,x)$ satisfies equation* (1.1) *pointwise for all $t > 0$ and all $x \in V \backslash V_0$.*

Proof. The proof here is motivated by [9]. Note that the eigenfunction φ_1 in (2.2) can be taken to be non-negative on V. Let $u_1(t,x) = M\varphi_1(x)$. Then

$$\left.\begin{array}{ll} \frac{\partial u_1}{\partial t} - \lambda_1 u_1 = \Delta u_1 & (t > 0, x \in V \backslash V_0) \\ u_1|_{t=0} = M\varphi_1(x) & (x \in V) \\ u_1|_{V_0} = 0 & (t \ge 0) \end{array}\right\} .$$

It is not hard to verify that u_1 is an upper solution. Similarly, $v_1(t,x) = -M\varphi_1(x)$ is a lower solution. The result follows immediately from Lemma 4.4 and Proposition 4.5 ∎

For a specific example, let $f(r) = r|r|^{p-1}$ $(p > 1)$. Then problem (1.1) - (1.2) has a global solution if the initial data are sufficiently small. I mention in passing here that a partial existence result in Theorem 4.6 might be obtained from the perturbation theory on Bauer harmonic spaces (see [12, 25]).

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