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Floquet Boundary Value Problems for Differential Inclusions: a Bound Sets Approach

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Abstract. A technique is developed for the solvability of the Floquet boundary value problem associated to a differential inclusion. It is based on the usage of a not necessarily C^1-class of Liapunov-like bounding functions. Certain viability arguments are applied for this aim. Some illustrating examples are supplied.

Keywords: *Floquet problems, bound sets, differential inclusions, viability arguments, existence results*

AMS subject classification: 34A60, 34B15

1. Introduction

Given a compact real interval $J = [a, b]$, a Carathéodory map $F : J \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ with non-empty, compact convex values and a subset S of $AC(J, \mathbb{R}^N)$, we look for solutions of the boundary value problem

$$\left.\begin{array}{l} x' \in F(t,x) \text{ for a.a. } t \in J \\ x \in S \end{array}\right\}. \qquad (P)$$

We recall that F is said to be a *Carathéodory multi-function*, when $F(t, \cdot)$ is upper semi-continuous for a.a. $t \in J$ and $F(\cdot, x)$ is measurable for each $x \in \mathbb{R}^N$.

As usual, by a *solution* $x(t)$ of problem (P) we mean an absolutely continuous function satisfying (P) for almost all $t \in J$.

In the case of a single-valued map F, i.e. for differential equations, the problem was extensively investigated (see, e.g., [11, 12, 16] and the references there) and results were

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obtained by means of various topological and analytical methods. On the contrary, less results are known up to now for the multi-valued case (see, e.g., [2, 3, 6, 10]).

In this paper, we investigate problem (P) by means of the Schauder linearization device and we treat the associated set-valued operator in the infinite dimensional solution space by a recent continuation principle obtained in [3]. The following theorem (i.e. Theorem 1) is an appropriate modification of [3: Theorem 2.33] which precisely refers to the technique employed in this work.

Theorem 1 [2: Theorem 1]. *Let us consider boundary value problem (P), where $F : J \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$, $J = [a, b]$ is a Carathéodory multi-function and $S \subset AC(J, \mathbb{R}^N)$. Let $G : J \times \mathbb{R}^N \times \mathbb{R}^N \times [0, 1] \rightsquigarrow \mathbb{R}^N$ be a Carathéodory map such that*

$$G(t, c, c, 1) \subset F(t, c) \quad \text{for all } (t, c) \in J \times \mathbb{R}^N.$$

Assume the following:

(i) *There exist a bounded retract Q of $C(J, \mathbb{R}^N)$ such that $Q \backslash \partial Q$ is non-empty (open) and a closed bounded subset S_1 of S such that the associated problem*

$$\left.\begin{aligned} & x' \in G(t, x, q(t), \lambda) \quad \text{for a.a. } t \in J \\ & x \in S_1 \end{aligned}\right\} \qquad (P_{q,\lambda})$$

is solvable with R_δ-sets of solutions, for each $(q, \lambda) \in Q \times [0, 1]$.

(ii) *There exists an integrable function $\alpha : J \to \mathbb{R}$ such that*

$$|G(t, x(t), q(t), \lambda)| \le \alpha(t) \qquad \text{a.e. in } J$$

for any $(x, q, \lambda) \in \Gamma_T$, where T denotes the map which assigns to any $(q, \lambda) \in Q \times [0, 1]$ the set of solutions of problem $(P_{q,\lambda})$ and Γ_T its graph.

(iii) $T(Q \times \{0\}) \subset Q$.

(iv) *The map T has no fixed points on the boundary ∂Q of Q, for every $(q, \lambda) \in Q \times [0, 1]$.*

Then problem (P) *admits a solution.*

Remark 1. In Theorem 1, the assumption that Q is a retract of $C(J, \mathbb{R}^N)$ is not necessary provided that problem $(P_{q,0})$ has only one solution $x_0(t) \notin \partial Q$ which does not depend on the choice of q in the closure $\overline{Q}$ of Q, i.e. provided that $T(\overline{Q} \times \{0\}) = \{x_0(t)\} \subseteq Q \setminus \partial Q$. Moreover, if $Q \subset C(J, \mathbb{R}^N)$ is convex, and subsequently a retract of $C(J, \mathbb{R}^N)$, then $Q \backslash \partial Q$ can be empty. For more details, see [3].

Notice, in particular, that such an approach requires the introduction of a suitable subset $Q \subset C([a, b], \mathbb{R}^N)$ of candidate solutions as well as the verification of the transversality condition (iv) in Theorem 1, for each associated problem $(P_{q,\lambda})$.

In our opinion, a quite natural way to construct the set Q is the following one

$$Q = \Big\{ q \in C([a, b], \mathbb{R}^N) : \ q(t) \in \overline{K}(t) \ \text{ for all } t \in [a, b] \Big\}$$

where $\{K(t)\}_{t \in J}$ denotes a one-parametric family of non-empty and open subsets of $\mathbb{R}^N$ and $\overline{K}(t)$ their closures. Throughout the paper, we shall always assume that $\{K(t)\}_{t \in J}$ is also uniformly bounded, i.e. $\|x\| < R$, for each $t \in J$ and $x \in K(t)$, where R is a positive constant.

Definition 1. We say that $\{K(t)\}_{t\in J}$ is a *bound set* for the boundary value problem (P) if there is no solution $x(t)$ of (P) such that $x(t) \in \overline{K}(t)$ for all $t \in J$ and $x(\tau) \in \partial K(\tau)$ for some $\tau \in J$.

In the single-valued case, bound sets, jointly with a family of Liapunov-like functions usually called bounding functions, were extensively used also in recent researches in order to investigate boundary value problems. We refer in particular to [12, 20, 22, 23] for a continuous right-hand side, and to the recent results by J. Mawhin and J. R. Ward Jr. [21] for the Carathéodory case. However, either the guiding and the bounding functions introduced there (see, e.g., [12, 15, 16, 20, 21, 23]) or the solutions of the given problems (see [22]) belong to the C^1-class. On the other hand, in [1, 2, 10, 17] only locally Lipschitzian in x Liapunov-like functions were employed for investigating differential inclusions, where solutions are absolutely continuous.

In Sections 2 and 3, we develop a detailed theory for the multi-valued case in a more advanced level. Since the bound sets approach is strictly related to the special boundary condition S, from now on we shall mainly refer to the Floquet boundary value problem

$$\left.\begin{aligned} x' \in F(t,x) \quad &\text{for a.a. } t \in [a,b] \\ x(b) = Mx(a) \quad & \end{aligned}\right\} \tag{1}$$

where M denotes an $N \times N$ non-singular matrix. More precisely, in Section 2 we discuss the existence of bound sets in the case of a globally upper semi-continuous right-hand side F. We also apply the obtained results in order to prove the transversality condition (iv) in Theorem 1. Examples of bound sets in the case of an autonomous boundary value problem can be found in Section 3. Section 4 gives a sufficient condition on the sets $K(t)$ assuring that Q is a retract of the space $C([a,b],\mathbb{R}^N)$. Our main result is Theorem 5 in Section 5; it gives an existence result for the Floquet boundary value problem associated to a differential inclusion. According to the fixed-point method used to prove it, it can also be seen as a viability result for the same problem. The paper concludes with Section 6 which contains an application of Theorem 5 to an anti-periodic problem.

Given a point $x \in \mathbb{R}^N$ and a constant $r > 0$, throughout the paper we shall denote by B_x^r the closed ball centered in x and having the radius r, and simply by B the unit closed ball. If X is an arbitrary metric space and $A \subseteq X$, we shall respectively denote by $\operatorname{int} A$, $\overline{A}$ and ∂A the interior, the closure and the boundary of A. Moreover, the following notation will be used for a bound set $\{K(t)\}_{t\in J}$:

$$\Gamma_{\partial K} = \big\{(t,x) :\, t \in J \text{ and } x \in \partial K(t)\big\}$$
$$\mathcal{K} = \big\{(t,x) :\, t \in J \text{ and } x \in \overline{K}(t)\big\}.$$

Finally, given a compact real interval $J = [a,b]$, $C(J,\mathbb{R}^N)$ will be the Banach space of continuous functions $x :\, J \to \mathbb{R}^N$ endowed with the usual sup norm.

2. Bound sets for the Floquet boundary value problem

We are now interested to introduce a family of Liapunov-like functions $V_{(\tau,\xi)}$, usually called *bounding functions*, satisfying suitable transversality conditions which assure that $\{K(t)\}_{t\in J}$ is a bound set for boundary value problem (1).

Throughout this section, we assume the multi-function F globally upper semi-continuous in $J\times\mathbb{R}^N$, and we divide our investigation in two steps. In the first one (see Theorem 2) we take into account only the interior points of the interval $J=[a,b]$. On the other hand, it is rather natural to expect that a particular boundary condition S should be fixed in order to treat the possible extremal points of J. As we mentioned in the introduction, this paper is mainly devoted to the Floquet problem (1). In the second step (see Theorem 3) we consider also the extremal points a and b and we give sufficient conditions for a collection $V_{(\tau,\xi)}$ of continuous functions to be a family of bounding functions for problem (1).

Given a point $(\tau,\xi)\in\Gamma_{\partial K}$, we shall consider a continuous function $V_{(\tau,\xi)}: J\times\mathbb{R}^N\to\mathbb{R}$ such that

(H1) $V_{(\tau,\xi)}(\tau,\xi)=0$

and

(H2) $V_{(\tau,\rho)}/k\leq 0$ in a neighbourhood of (τ,ξ).

Theorem 2. *Let $\{K(t)\}_{t\in J}$, $J=[a,b]$ be a non-empty, open and uniformly bounded family of subsets of $\mathbb{R}^N$. Assume that, for every $(\tau,\xi)\in\Gamma_{\partial K}$, there is a continuous function $V_{(\tau,\xi)}$ satisfying conditions* (H1) *and* (H2). *Moreover, suppose that, for all $\tau\in\operatorname{int}J$, $\xi\in\partial K(\tau)$ and $w_1,w_2\in F(\tau,\xi)$,*

$$\textbf{(H3)}\ 0\notin\left[\liminf_{\substack{v\to w_1\\ h\to 0^+}}\frac{V_{(\tau,\xi)}(\tau+h,\xi+hv)}{h},\limsup_{\substack{v\to w_2\\ h\to 0^-}}\frac{V_{(\tau,\xi)}(\tau+h,\xi+hv)}{h}\right].$$

Then all possible solutions $x(t)$ of problem (1) *satisfying $x(t)\in\overline{K}(t)$ for every $t\in J$ are such that $x(t)\in K(t)$ for every $t\in\operatorname{int}J$.*

Proof. Assume by contradiction the existence of a solution $x(t)$ of problem (1) and the existence of a point $\tau\in\operatorname{int}J$ satisfying $x(t)\in\overline{K}(t)$ for all $t\in J$ and $x(\tau)\in\partial K(\tau)$. Let I be a compact interval such that $\tau\in I\subseteq J$. Since $x(I)$ is compact and F is upper semi-continuous, then $F(t,x(t))$ is bounded on I, and consequently x is a Lipschitz function on I. Let us denote by Ω the set of limit points of

$$\frac{x(\tau+h)-x(\tau)}{h}\qquad\text{when } h\to 0^+.$$

According to the Lipschitzianity of x in a neighbourhood of the point τ and since we are in the Euclidean space $\mathbb{R}^N$, we get $\Omega\neq\emptyset$.

Taking $w_1\in\Omega$, there exists a sequence $\{h_n\}_n$ of positive numbers such that $h_n\to 0^+$ and

$$\frac{x(\tau+h_n)-x(\tau)}{h_n}\to w_1\qquad\text{when } n\to+\infty. \tag{2}$$

As a consequence of the regularity assumption on F in the point $(\tau, x(\tau))$ we prove now that $w_1 \in F(\tau, x(\tau))$. In fact, given $\varepsilon > 0$ it is possible to find $\sigma > 0$ such that, whenever $t \in J$ with $|t-\tau| < \sigma$ and $\|x - x(\tau)\| < \sigma$, then

$$F(t,x) \subset F(\tau, x(\tau)) + \varepsilon B = \cup_{w \in F(\tau,x(\tau))} B_w^{\varepsilon}.$$

By the continuity of x one can then find $0 < \eta \leq \sigma$ such that $|t-\tau| < \eta$ and $t \in J$ imply $F(t, x(t)) \subset F(\tau, x(\tau)) + \varepsilon B$. Recalling that F is convex valued, $F(\tau, x(\tau)) + \varepsilon B$ is a convex subset of $\mathbb{R}^N$. Therefore, for each n sufficiently big, we have

$$\frac{x(\tau + h_n) - x(\tau)}{h_n} = \frac{1}{h_n} \int_{\tau}^{\tau + h_n} x'(s)\, ds \in F(\tau, x(\tau)) + \varepsilon B.$$

Finally, since the set $F(\tau, x(\tau))$ is compact, it follows $w_1 \in F(\tau, x(\tau))$.

As a consequence of property (2) there exists a sequence $\{\Delta_n\}_n$ such that $\Delta_n \to 0$ as $n \to +\infty$ and

$$x(\tau + h_n) = x(\tau) + h_n[w_1 + \Delta_n] \qquad \text{for all } n \in \mathbb{N}.$$

Since, for all n, $x(\tau + h_n) \in \overline{K}(\tau + h_n)$, according to condition (H2) we obtain for n sufficiently large that

$$0 \geq \frac{V_{(\tau,x(\tau))}(\tau + h_n, x(\tau + h_n))}{h_n} = \frac{V_{(\tau,x(\tau))}(\tau + h_n, x(\tau) + h_n[w_1 + \Delta_n])}{h_n}.$$

Therefore, recalling that $\Delta_n \to 0$ when $n \to +\infty$ we have

$$\begin{aligned} 0 &\geq \liminf_{n \to +\infty} \frac{V_{(\tau,x(\tau))}(\tau + h_n, x(\tau) + h_n[w_1 + \Delta_n])}{h_n} \\ &\geq \liminf_{\substack{v \to w_1 \\ h \to 0^+}} \frac{V_{(\tau,x(\tau))}(\tau + h, x(\tau) + hv)}{h}. \end{aligned} \tag{3}$$

By a similar reasoning one can show the existence of a vector $w_2 \in F(\tau, x(\tau))$ and of a sequence $\{k_n\}_n$ such that $k_n \to 0^-$ and

$$\frac{x(\tau + k_n) - x(\tau)}{k_n} \to w_2 \qquad \text{when } n \to +\infty. \tag{4}$$

This yields

$$\limsup_{\substack{v \to w_2 \\ h \to 0^-}} \frac{V_{(\tau,x(\tau))}(\tau + h, x(\tau) + hv)}{h} \geq 0. \tag{5}$$

Notice that inequalities (3) and (5) are in contradiction with condition (H3). Hence the result is proven ∎

Remark 2. The technique we employed in order to prove the existence of vectors $w_i \in F(\tau, x(\tau))$ $(i = 1, 2)$ satisfying respectively (2) and (4) was previously used by Haddad in his seminal paper [14] on the viability theory for differential inclusions. More precisely, let $T_{\mathcal{K}}(\tau, x(\tau))$ be the Bouligand contingent cone of $\mathcal{K}$ (see [14]) in its point $(\tau, x(\tau))$. With our previous reasoning, we actually proved that

$$w_i \in F(\tau, x(\tau)) \cap T_{\mathcal{K}}(\tau, x(\tau)) \qquad (i = 1, 2)$$

and this is the necessary condition (see again [14]) for a viability problem in a finite-dimensional space. We refer to [18] for some recent viability results in an infinite dimensional Banach space.

Remark 3. Assume that $V_{(\tau,\xi)}(t, x)$ is locally Lipschitzian in x, uniformly with respect to t, in the point (τ, ξ), i.e. that there exists a constant $L_{(\tau,\xi)} > 0$ such that

$$|V_{(\tau,\xi)}(t, x) - V_{(\tau,\xi)}(t, y)| \leq L_{(\tau,\xi)} \|x - y\|$$

for all (t, x) and (t, y) in a neighbourhood of (τ, ξ). For such a function we can define in a standard manner the *upper* and *lower right Dini derivatives* at (τ, ξ) calculated in $(1, w)$ by

$$D^+ V_{(\tau,\xi)}(\tau, \xi)(1, w) = \limsup_{h \to 0^+} \frac{[V_{(\tau,\xi)}(\tau + h, \xi + hw) - V_{(\tau,\xi)}(\tau, \xi)]}{h}$$
$$D_+ V_{(\tau,\xi)}(\tau, \xi)(1, w) = \liminf_{h \to 0^+} \frac{[V_{(\tau,\xi)}(\tau + h, \xi + hw) - V_{(\tau,\xi)}(\tau, \xi)]}{h},$$

respectively, as well as the *upper* and *lower left Dini derivatives*

$$D^- V_{(\tau,\xi)}(\tau, \xi)(1, w)$$
$$D_- V_{(\tau,\xi)}(\tau, \xi)(1, w),$$

respectively, simply replacing $h \to 0^+$ by $h \to 0^-$ in the previous definitions. According to the Lipschitzianity assumption on V, all these four quantities are real numbers. Moreover, for h small enough,

$$V_{(\tau,\xi)}(\tau + h, \xi + hv) \leq V_{(\tau,\xi)}(\tau + h, \xi + hw) + L_{(\tau,\xi)} |h| \, \|v - w\|.$$

Therefore, assuming that $x(t)$ is a solution of problem (1) such that $x(\tau) = \xi$, we can reformulate inequalities (3) as

$$\begin{aligned} 0 &\geq \liminf_{n \to +\infty} \frac{V_{(\tau,x(\tau))}(\tau + h_n, x(\tau) + h_n[w_1 + \Delta_n])}{h_n} \\ &\geq \liminf_{n \to +\infty} \left[\frac{V_{(\tau,x(\tau))}(\tau + h_n, x(\tau) + h_n w_1)}{h_n} - L_{(\tau,x(\tau))} \|\Delta_n\| \right] \\ &\geq \liminf_{h \to 0^+} \frac{V_{(\tau,x(\tau))}(\tau + h, x(\tau) + h w_1)}{h} \\ &= D_+ V_{(\tau,x(\tau))}(\tau, x(\tau))(1, w_1) \end{aligned}$$

because of condition (H1). By a similar reasoning we can replace (5) by

$$D^- V_{(\tau,x(\tau))}(\tau, x(\tau))(1, w_2) \le 0.$$

Therefore, when $V_{(\tau,\xi)}$ is locally Lipschitzian in x, a contradiction immediately follows by

$$0 \notin \left[D_+ V_{(\tau,\xi)}(\tau,\xi)(1,w_1), D^- V_{(\tau,\xi)}(\tau,\xi)(1,w_2)\right] \tag{6}$$

for all $w_1, w_2 \in F(\tau,\xi)$. On the other hand, one has

$$\begin{aligned}\liminf_{\substack{v\to w\\ h\to 0^+}} &\frac{V_{(\tau,\xi)}(\tau+h, \xi+hv)}{h}\\ &\le \liminf_{h\to 0^+}\left[\frac{V_{(\tau,\xi)}(\tau+h,\xi+hw)}{h} + L_{(\tau,\xi)}\|v-w\|\right]\\ &= D_+ V_{(\tau,\xi)}(\tau,\xi)(1,w).\end{aligned}$$

In an analogous way one can prove that

$$\limsup_{\substack{v\to w\\ h\to 0^-}} \frac{V_{(\tau,\xi)}(\tau+h,\xi+hw)}{h} \ge D^- V_{(\tau,\xi)}(\tau,\xi)(1,w).$$

Hence, for every $w_1, w_2 \in F(\tau,\xi)$,

$$\begin{aligned}&\left[D_+ V_{(\tau,\xi)}(\tau,\xi)(1,w_1),\ D^- V_{(\tau,\xi)}(\tau,\xi)(1,w_2)\right]\\ &\qquad\subset \left[\liminf_{\substack{v\to w_1\\ h\to 0^+}} \frac{V_{(\tau,\xi)}(\tau+h,\xi+hv)}{h},\ \limsup_{\substack{v\to w_2\\ h\to 0^-}} \frac{V_{(\tau,\xi)}(\tau+h,\xi+hv)}{h}\right].\end{aligned}$$

Consequently, when $V_{(\tau,\xi)}(t,x)$ is locally Lipschitzian in x, (6) is a more proper assumption than (H3), because the regularity allows us to get a contradiction by means of a weaker condition than the one required in the general case.

Moreover, in [22] it was shown that, when F is single-valued, assumption (6) can be replaced by the even more general one

$$0 \notin \left[D^+ V_{(\tau,\xi)}(\tau,\xi)(1,F(\tau,\xi)),\ D_- V_{(\tau,\xi)}(\tau,\xi)(1,F(\tau,\xi))\right].$$

Finally, when the function $V_{(\tau,\xi)}$ is of class C^1,

$$\lim_{h\to 0} \frac{V_{(\tau,\xi)}(\tau+h,\xi+hw)}{h} = \big(\nabla V_{(\tau,\xi)}(\tau,\xi),(1,w)\big),$$

so condition (H3) becomes

$$0 \notin \Big[\big(\nabla V_{(\tau,\xi)}(\tau,\xi),(1,w_1)\big),\ \big(\nabla V_{(\tau,\xi)}(\tau,\xi),(1,w_2)\big)\Big]$$

for all $w_1, w_2 \in F(\tau,\xi)$. Since the set $F(\tau,\xi)$ is convex, this is equivalent to require

$$\big(\nabla V_{(\tau,\xi)}(\tau,\xi),(1,w)\big) \ne 0 \qquad \text{for all } w \in F(\tau,\xi).$$

We are now able to give an existence theorem of a bound set for Floquet problem (1). In order to study the extremal points a and b we need the invariance condition

$$M\partial K(a) = \big\{M\xi : \xi \in \partial K(a)\big\} = \partial K(b). \tag{7}$$

We point out that when the bound set is autonomous, i.e. when $K(t) \equiv K$, this is equivalent to the invariance of its boundary with respect to the subgroup of $GL^N(\mathbb{R})$ generated by M, which is a usual assumption in this setting.

Theorem 3. *Let $\{K(t)\}_{t\in[a,b]}$ be a non-empty, open and uniformly bounded family of subsets of $\mathbb{R}^N$. Assume that, for every $(\tau,\xi)\in\Gamma_{\partial K}$, there exists a continuous function $V_{(\tau,\xi)}$ satisfying conditions* (H1) *and* (H2)*, but also condition* (H3) *when $\tau\in(a,b)$. Suppose, furthermore, invariance condition* (7)*. Finally, assume that, for any $\xi\in\partial K(a), w_a\in F(a,\xi)$ and $w_b\in F(b,M\xi)$,*

$$\textbf{(H4)}\quad 0\notin\left[\liminf_{\substack{v\to w_a\\ h\to 0^+}}\frac{V_{(a,\xi)}(a+h,\xi+hv)}{h},\ \limsup_{\substack{v\to w_b\\ h\to 0^-}}\frac{V_{(b,M\xi)}(b+h,M\xi+hv)}{h}\right].$$

Then $\{K(t)\}_{t\in[a,b]}$ is a bound set for problem (1).

Proof. According to Theorem 2 we only need to show that, whenever problem (1) has a solution $x(t)$ such that $x(t)\in\overline{K}(t)$ for every $t\in[a,b]$, then $x(a)\in K(a)$ and $x(b)\in K(b)$. Suppose by contradiction that this is false. Therefore, according to invariance condition (7), both $x(a)\in\partial K(a)$ and $x(b)\in\partial K(b)$. Following the same reasoning as in the proof of Theorem 2, there exist $w_a\in F(a,x(a))$ such that

$$\liminf_{\substack{v\to w_a\\ h\to 0^+}}\frac{V_{(a,x(a))}(a+h,x(a)+hv)}{h}\le 0$$

and $w_b\in F(b,x(b))$ such that

$$\limsup_{\substack{v\to w_b\\ h\to 0^-}}\frac{V_{(b,x(b))}(b+h,x(b)+hv)}{h}\ge 0.$$

But $x(b)=Mx(a)$ by the boundary condition and the contradiction follows from condition (H4) ∎

Remark 4. Take $V_{(\tau,\xi)}(t,x)$ locally Lipschitzian in x, uniformly with respect to t in the point (τ,ξ). Reasoning as in Remark 3, it is possible to show that condition (H4) can be replaced by

$$0\notin\left[D_+V_{(a,\xi)}(a,\xi)(1,w_a),\ D^-V_{(b,M\xi)}(b,M\xi)(1,w_b)\right]$$

for all $w_a\in F(a,\xi)$ and $w_b\in F(b,M\xi)$. Finally, in the C^1-case it becomes

$$0\notin\left[\big(\nabla V_{(a,\xi)}(a,\xi),(1,w_a)\big),\ \big(\nabla V_{(b,M\xi)}(b,M\xi),(1,w_b)\big)\right].$$

Remark 5. In the theory of bound sets a collection of continuous functions $V_{(\tau,\xi)}$ satisfying assumptions (H1) - (H4) is usually said to be a *family of bounding functions* for problem (1).

Existence results for Floquet problems, obtained by means of Theorem 1, require in particular the knowledge of a subset $Q\subset C([a,b],\mathbb{R}^N)$ having no solution of the associated quasi-linearized problems on its boundary. A quite natural way to construct such a subset Q is

$$Q=\left\{q\in C([a,b],\mathbb{R}^N):\ q(t)\in\overline{K}(t)\ \text{ for all } t\in[a,b]\right\}\tag{8}$$

where $\{K(t)\}_{t\in[a,b]}$ denotes a family of non-empty, open uniformly bounded subsets of $\mathbb{R}^N$.

The following theorem gives sufficient conditions on such a subset Q in order that no solution of problem (1) belongs to its boundary. Since it makes use of a family of bounding functions, it is an application of previous investigations.

Theorem 4. *Let* $\{K(t)\}_{t\in[a,b]}$ *be a family of non-empty, open and uniformly bounded subsets of* $\mathbb{R}^N$ *and define* $Q \subset C([a,b],\mathbb{R}^N)$ *as in* (8)*. Let* $\Gamma_{\partial K}$ *be closed in* $\mathbb{R}^{N+1}$*, and for each* $(\tau,\xi) \in \Gamma_{\partial K}$ *assume the existence of a continuous function* $V_{(\tau,\xi)} : [a,b]\times\mathbb{R}^N \to \mathbb{R}$ *satisfying conditions* (H1) - (H4)*. Then Floquet problem* (1) *has no solution on* ∂Q*.*

Proof. First of all we show that every function x in ∂Q admits at least a point $\tau = \tau_x \in [a,b]$ such that $x(\tau) \in \partial K(\tau)$. For this suppose that $x(t) \in K(t)$ for every $t \in [a,b]$ and consider the function

$$d : [a,b] \to \mathbb{R}^+, \quad t \to \mathrm{dist}\{x(t), \partial K(t)\}.$$

Given $t_0 \in [a,b]$, take a sequence $\{t_n\}_n$ converging to t_0 and such that $d(t_n)$ converges to a real number l. Let us prove that $d(t_0) \le l$.

By the boundedness of $\overline{K}(t)$ the compactness of $\partial K(t)$ for each t follows. Hence, for every n there exists $y_n \in \partial K(t_n)$ such that $d(t_n) = \|x(t_n) - y_n\|$. Moreover, since $\overline{K}(t)$ is uniformly bounded, then $\{y_n\}_n$ has a subsequence, again denoted for the sake of simplicity by $\{y_n\}_n$, which converges to a point $y_0 \in \mathbb{R}^N$. Notice that $(t_n, y_n) \in \Gamma_{\partial K}$ for each n; the closure of $\Gamma_{\partial K}$ then implies $y_0 \in \partial K(t_0)$. Therefore, by the definition of the function d we have

$$l = \|x(t_0) - y_0\| \ge d(t_0).$$

We have so obtained that d is a lower semi-continuous function on $[a,b]$. Hence $d([a,b])$ has a minimum d_0, and since we assumed $x(t) \in K(t)$ for all t, d_0 must be positive. Therefore, $B^{d_0}_{x(t)} \subset K(t)$ for each $t \in [a,b]$, implying $y \in Q$ for any function $y \in C([a,b],\mathbb{R}^N)$ having $\|x - y\| < d_0$. Thus $x \in \mathrm{int}\, Q$.

Let us consider now a function x in ∂Q. According to its definition, Q is closed in the Banach space $C([a,b],\mathbb{R}^N)$. Hence $x \in Q$ and so $x(t) \in \overline{K}(t)$ for each $t \in [a,b]$. As a consequence of the previous reasoning, there exists $\tau = \tau_x \in [a,b]$ such that $x(\tau) \in \partial K(\tau)$. By Theorem 3 the family $\{K(t)\}_{t\in[a,b]}$ is a bound set for problem (1). Thus $x(t)$ can not be a solution of problem (1) and the result is proven ∎

3. Examples of bound sets for autonomous Floquet problems

The present example deals with sufficient conditions for the existence of bound sets for the autonomous Floquet boundary value problem

$$\left.\begin{aligned} x' \in F(x) \quad &\text{for a.a. } t \in [a,b] \\ x(b) = Mx(a) & \end{aligned}\right\} \tag{9}$$

where $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ is an upper semi-continuous multi-function with non-empty, compact and convex values. Since the problem is autonomous, in our opinion, it is natural to look for a bound set K constant in time. Therefore, the family of bounding functions will be also taken independent of time. We shall study separately the case of a convex set K when a family of C^1-bounding functions arises naturally, and the opposite one where we have to look for a less regular bounding function.

More precisely, for each $\xi \in \partial K$ we shall consider a continuous function $V_\xi : \mathbb{R}^N \to \mathbb{R}$ such that

(h1) $V_\xi(\xi) = 0$

and

(h2) $V_\xi/_{\overline{K}} \leq 0$ in a neighbourhood of ξ.

Hence, Theorem 3 can be reformulated as follows.

Proposition 1. *Let K be a non-empty, open and bounded subset of $\mathbb{R}^N$ having an invariant boundary with respect to the subgroup of $GL^N(\mathbb{R})$ generated by M. Assume that, for every $\xi \in \partial K$, there exists a continuous function V_ξ satisfying conditions* (h1) - (h2)*. Moreover, suppose that, for all $\xi \in \partial K$ and $w_1, w_2 \in F(\xi)$,*

(h3) $0 \notin \left[\liminf\limits_{\substack{v\to w_1\\ h\to 0^+}} \dfrac{V_\xi(\xi+hv)}{h},\ \limsup\limits_{\substack{v\to w_2\\ h\to 0^-}} \dfrac{V_\xi(\xi+hv)}{h} \right].$

Finally, assume that, for any $\xi \in \partial K, w_a \in F(\xi)$ and $w_b \in F(M\xi)$,

(h4) $0 \notin \left[\liminf\limits_{\substack{v\to w_a\\ h\to 0^+}} \dfrac{V_\xi(\xi+hv)}{h},\ \limsup\limits_{\substack{v\to w_b\\ h\to 0^-}} \dfrac{V_{M\xi}(M\xi+hv)}{h} \right].$

Then K is a bound set for problem (9)*.*

We first consider the case in which K is convex. Geometrically this means that for every $\xi \in \partial K$ there exist a vector n_ξ not necessarily unique and a neighbourhood U_ξ of ξ such that

$$\big(n_\xi, (x-\xi)\big) \leq 0 \qquad \text{for all } x \in U_\xi \cap \overline{K} \tag{10}$$

(for this purpose see [19: p. 156]). Let V_ξ be the C^1-function defined by

$$V_\xi : \mathbb{R}^N \to \mathbb{R}, \qquad x \to \big(n_\xi, (x-\xi)\big).$$

It follows immediately by (10) that V_ξ satisfies conditions (h1) and (h2). Moreover, we have $\nabla V_\xi(\xi) = n_\xi$. Hence, recalling the discussion in Remarks 3 and 4 for a C^1-bounding function, conditions (h3) and (h4) are respectively equivalent to

(h3) $(n_\xi, w) \neq 0$ for all $w \in F(\xi)$

and

(h4) $0 \notin \big[(n_\xi, w_a), (n_{M\xi}, w_b)\big]$ for all $w_a \in F(\xi)$ and $w_b \in F(M\xi)$.

Consider now the case when K is not locally convex in some ξ of its boundary. Then as shown in [22: Example 4.2], for differential equations in $\mathbb{R}^2$ in general it is not possible for a C^1-function V_ξ to satisfy at the same time all conditions (h1) - (h3). On the other hand, take the Lipschitzian function (for this purpose see [8: Proposition 2.4.1]) $V_\xi(x) = \operatorname{dist}(x, \overline{K})$ which trivially verifies both conditons (h1) and (h2). Again, by Remark 3, condition (h3) is equivalent to

$$0 \notin \left[\liminf_{h\to 0^+} \frac{\operatorname{dist}(\xi+hw_1, \overline{K})}{h}, \limsup_{h\to 0^-} \frac{\operatorname{dist}(\xi+hw_2, \overline{K})}{h} \right] \tag{11}$$

for all $w_1, w_2 \in F(\xi)$. The non-negativity of the distance function implies that the previous condition is satisfied if and only if at least one between the left and the right extremes of the interval is different from zero, since they are always respectively non-negative and non-positive.

It is easy to prove the equivalence

$$\limsup_{h\to 0^-} \frac{\operatorname{dist}(\xi + hw_2, \overline{K})}{h} \neq 0 \quad \Longleftrightarrow \quad \liminf_{h\to 0^+} \frac{\operatorname{dist}(\xi - hw_2, \overline{K})}{h} \neq 0.$$

Therefore, recalling the definition of the Bouligand contingent cone $T_{\overline{K}}(\xi)$ of $\overline{K}$ in ξ (see, e.g., [14]), (11) is equivalent to

$$w_1 \notin T_{\overline{K}}(\xi) \quad \text{or} \quad (-w_2) \notin T_{\overline{K}}(\xi) \qquad \text{for all } w_1, w_2 \in F(\xi).$$

If $w \notin T_{\overline{K}}(\xi)$ for all $w \in F(\xi)$, then (11) holds. On the contrary, if there exists $w_1 \in F(\xi) \cap T_{\overline{K}}(\xi)$, (11) is verified if and only if $(-w) \notin T_{\overline{K}}(\xi)$ for all $w \in F(\xi)$. Hence, we get that condition (h3) is equivalent to

(h3) $F(\xi) \cap T_{\overline{K}}(\xi) = \emptyset$ or $(-F(\xi)) \cap T_{\overline{K}}(\xi) = \emptyset$.

Finally, if K is not locally convex also in $M\xi$, we can define $V_{M\xi}$ identically equal to V_ξ and, by a reasoning similar to the previous one, taking account of Remark 4 one can show that condition (h4) is equivalent to

(h4) $F(\xi) \cap T_{\overline{K}}(\xi) = \emptyset$ or $(-F(M\xi)) \cap T_{\overline{K}}(M\xi) = \emptyset$.

4. Retracts of the space of continuous functions

Given a metric space X, we recall here for completeness the well-known definition of a retract of X.

Definition 2. A subset Q of a metric space X is said to be a *retract* of X if there exists a continuous function $\phi: X \to Q$ such that $\phi(q) = q$ for every $q \in Q$.

In this section we are interested in studying retracts of the space $C([a,b], \mathbb{R}^N)$. We consider, in particular, subsets Q of $C([a,b], \mathbb{R}^N)$ of the type given by (8), and we give sufficient conditions on the sets $K(t)$ in order that Q is such a retract.

Proposition 2. *Let $\{\overline{K}(t)\}_{t\in[a,b]}$ be a family of non-empty subsets of $\mathbb{R}^N$. Assume that the graph $\mathcal{K}$ of the map $t \rightsquigarrow \overline{K}(t)$ is a retract of $[a,b] \times \mathbb{R}^N$ and admits a retraction $\phi : [a,b] \times \mathbb{R}^N \to \mathcal{K}$ which is the identity on its first component, i.e. that $\phi(t,x) = (t, \tilde{\phi}(t,x))$ for all $(t,x) \in [a,b] \times \mathbb{R}^N$ where $\tilde{\phi} : [a,b] \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous function. Then*

$$Q = \left\{ q \in C([a,b], \mathbb{R}^N) : \, q(t) \in \overline{K}(t) \ \text{ for all } t \in [a,b] \right\}$$

is a retract of the space $C([a,b], \mathbb{R}^N)$.

Proof. Let us consider the function

$$\phi' : C([a,b],\mathbb{R}^N) \to Q, \qquad q \to \phi'(q) : [a,b] \to \mathbb{R}^N, \qquad t \to \tilde{\phi}(t, q(t)).$$

From the definition of Q and the properties of ϕ it immediately follows that ϕ' is well defined and that $\phi'(q) = q$ for every $q \in Q$. Take now $\{q_n\}_n$ converging to $q \in C([a,b],\mathbb{R}^N)$. Then there exists a compact subset $T \subset \mathbb{R}^N$ such that $q(t)$ and $q_n(t)$ belong to T for every t and every n. The continuity of $\tilde{\phi}$ implies its uniform continuity in $[a,b] \times T$. Hence, by the convergence of q_n to q we get the convergence of $\phi'(q_n)$ to $\phi'(q)$, and also the continuity of ϕ' is proved ∎

Remark 6. The above Q is an example of a so-called *absolute retract space* such that $\operatorname{ind} T(\cdot,0)|_Q = 1$ (see, e.g., [13: pp. 274 – 275] for the definition of an index in a metric ANR-space). It is well-known that a retract of a convex subset of any metric space is also an absolute retract space and that a retract of an open subset of a Banach space is a so-called *absolute neighbourhood retract* (ANR) space. Notice that in Theorem 1 one may replace the conditon that Q be a retract of $C(J,\mathbb{R}^N)$ by the assumption that Q be an ANR and $\operatorname{ind} T(\cdot,0)|_Q \neq 0$. It is therefore possible to assume alternatively that Q is either a retract of any convex subset of $C([a,b],\mathbb{R}^N)$ or of any open subset of $C([a,b],\mathbb{R}^N)$ jointly with $\operatorname{ind} T(Q \times \{0\}) \neq 0$. For more details concerning the theory of retracts see, e.g., [7].

Remark 7. If $\overline{K}(t)$ is convex (which trivially implies the same property for the set Q), then the associated bounding functions can be taken smooth in x, as pointed out both in previous section and in [12: p. 43] (cf. also the main converse theorem in [9: Theorem 1.2]).

5. An existence result for the Floquet boundary value problem

We consider now the Floquet boundary value problem

$$\left.\begin{aligned} x' + A(t)x \in F(t,x) \quad &\text{for a.a. } t \in [a,b] \\ x(b) = Mx(a) \quad & \end{aligned}\right\} \tag{12}$$

where $A : [a,b] \to \mathbb{R}^N \times \mathbb{R}^N$ is a continuous $N \times N$ matrix, $F : [a,b] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ is a globally upper semi-continuous multi-function with non-empty, compact and convex values and M is a regular $N \times N$ matrix. Under appropriate conditions on A and F and with the fixed-point technique described in Theorem 1, we prove its solvability. Notice, in particular, that we define the set Q of candidate solutions (see condition d)) as in (8) and use the bounding functions approach developed in Section 2 in order to show the transversality condition (iv) of Theorem 1.

Theorem 5. *Let us consider the Floquet boundary value problem* (12)*. Assume the following:*

a) *The associated homogeneous problem of* (12) *has only the trivial solution.*

b) *There exists a globally upper semi-continuous multi-function* $G : [a,b] \times \mathbb{R}^N \times \mathbb{R}^N \times [0,1] \rightsquigarrow \mathbb{R}^N$ *with non-empty, compact and convex values such that* $G(t,c,c,1) \subset F(t,c)$ *for all* $(t,c) \in [a,b] \times \mathbb{R}^N$.

c) *There exists a family* $\{K(t)\}_{t\in[a,b]}$ *of non-empty, open and uniformly bounded subsets of* $\mathbb{R}^N$ *such that the graph* $\mathcal{K}$ *of the map* $t \rightsquigarrow \overline{K}(t)$ *is a retract of* $[a,b] \times \mathbb{R}^N$ *with a retraction* $\phi : [a,b] \times \mathbb{R}^N \to \mathcal{K}$ *which is the identity on its first component, i.e.* $\phi(t,x) = (t, \tilde{\phi}(t,x))$. *Moreover, assume that the graph of its boundary* $\Gamma_{\partial K} = \{(t,x) : t \in [a,b]$ *and* $x \in \partial K(t)\}$ *is closed.*

d) $G(t,\cdot,q,\lambda)$ *is Lipschitzian with a sufficiently small Lipschitz constant for each* $t \in [a,b]$ *and* $(q,\lambda) \in Q \times [0,1]$ *where*

$$Q = \Big\{ q \in C([a,b],\mathbb{R}^N) : q(t) \in \overline{K}(t) \text{ for all } t \in [a,b] \Big\}.$$

e) *There exists a Lebesgue integrable function* $\alpha : [a,b] \to \mathbb{R}$ *such that*

$$|G(t,x(t),q(t),\lambda)| \le \alpha(t) \qquad \text{for a.a. } t \in [a,b] \text{ and all } (x,q,\lambda) \in \Gamma_T$$

where T *is the multi-function which assigns to any* $(q,\lambda) \in Q \times [0,1]$ *the set of solutions of the problem*

$$\left.\begin{aligned} x' + A(t)x \in G(t,x,q(t),\lambda) \quad &\text{for a.a. } t \in [a,b] \\ x(b) = Mx(a) \quad & \end{aligned}\right\}$$

and Γ_T *is its graph.*

f) $T(Q \times \{0\}) \subset Q$ *and* ∂Q *is fixed-point free, i.e.* $\{q \in Q : q \in T(q,0)\} \cap \partial Q = \emptyset$.

g) *For every* $(\tau,\xi) \in \Gamma_{\partial K}$ *there exists a continuous function* $V_{(\tau,\xi)} : [a,b] \times \mathbb{R}^N \to \mathbb{R}$ *satisfying conditions* (H1) *and* (H2).

h) *For every* $\tau \in (a,b), \xi \in \partial K(\tau), (q,\lambda) \in Q \times (0,1]$ *and* $w_1, w_2 \in G(\tau,\xi,q(\tau),\lambda) - A(\tau)\xi$ *one has*

$$0 \notin \left[\liminf_{\substack{v\to w_1 \\ h\to 0^+}} \frac{V_{(\tau,\xi)}(\tau+h,\xi+hv)}{h}, \limsup_{\substack{v\to w_2 \\ h\to 0^-}} \frac{V_{(\tau,\xi)}(\tau+h,\xi+hv)}{h} \right].$$

i) $M\partial K(a) = \{M\xi : \xi \in \partial K(a)\} = \partial K(b)$.

j) *For any* $\xi \in \partial K(a), (q,\lambda) \in Q \times (0,1], w_a \in G(a,\xi,q(a),\lambda) - A(a)\xi$ *and* $w_b \in G(b,M\xi,q(b),\lambda) - A(b)M\xi$ *there holds*

$$0 \notin \left[\liminf_{\substack{v\to w_a \\ h\to 0^+}} \frac{V_{(a,\xi)}(a+h,\xi+hv)}{h}, \limsup_{\substack{v\to w_b \\ h\to 0^-}} \frac{V_{(b,M\xi)}(b+h,M\xi+hv)}{h} \right].$$

Then problem (12) *admits a solution.*

Proof. Let us prove that all the assumptions of Theorem 1 are satisfied. Define the set Q as in d). The continuity of A and the global upper semi-continuity of G are sufficient to get the global upper semi-continuity of $G(t,x,q(t),\lambda) - A(t)x$ for every fixed $(q,\lambda) \in Q \times [0,1]$ (see, e.g., [4: pp. 9 – 10] or [5: p. 41]). Therefore, also thanks to conditions a) and d) and the continuity of A, we are able to apply [6: Theorem 4] (see also [2: Proposition 1]) to each quasi-linearized associated problem

$$\left.\begin{aligned} x' \in G(t,x,q(t),\lambda) - A(t)x \text{ for a.a. } t \in [a,b] \\ x(b) = Mx(a) \qquad\qquad\qquad \end{aligned}\right\}$$

and to assure its solvability with an R_δ set of solutions, i.e. that the set of solutions is, in particular, non-empty, compact and connected (hence lying in some B_0^R when R is sufficiently big). Moreover, it follows from the proof of the main result in [6] that the ball B_0^R can be taken the same for all $q \in Q$.

Taking

$$S_1 = B_0^R \cap \{x \in AC([a,b],\mathbb{R}^N) : x(b) = Mx(a)\}$$

the boundedness of B_0^R implies the same property for S_1. Moreover, according to assumption c) and Proposition 2, Q is a retract of $C([a,b],\mathbb{R}^N)$. Since $Q \backslash \partial Q$ is non-empty, condition i) of Theorem 1 holds.

Assumptions g) - j) guarantee that $\{K(t)\}_{t\in[a,b]}$ is a bound set for each problem

$$\left.\begin{aligned} x' \in G(t,x,q(t),\lambda) - A(t)x \;\text{ for a.a. } t \in [a,b] \\ x(b) = Mx(a) \qquad\qquad\qquad \end{aligned}\right\}.$$

Therefore, since by assumption c) $\Gamma_{\partial K}$ is closed, according to Theorem 4, for $\lambda \in (0,1]$ we have $T(Q \times [0,1]) \cap \partial Q) = \emptyset$. This implies condition (iv) of Theorem 1 (for $\lambda = 0$ it follows from assumption f)) and the proof is complete ∎

Remark 8. Because of the method used to solve problem (12) we obtained in particular solutions belonging to the set Q. Consequently, previous theorem gives an existence result for the Floquet viability problem

$$\left.\begin{aligned} x' + A(t) \in F(t,x) \;\text{ for a.a. } t \in [a,b] \\ x(a) = Mx(b) \qquad\qquad \\ x(t) \in K(t) \;\text{ for all } t \in [a,b] \end{aligned}\right\}.$$

Remark 9. In view of Remark 1, Theorem 5 can be reformulated in the sense that, instead of assumptions c) and f), we can assume respectively assumptions c′) and f′) or c″) and f″) as follows:

c′) $\{K(t)\}_{t\in[a,b]}$ is a suitable (i.e. with respect to conditions d) – j)) family of non-empty, open and uniformly bounded subsets of $\mathbb{R}^N$ having $\Gamma_{\partial K}$ closed in $\mathbb{R}^{N+1}$.

f′) $T(Q \times \{0\}) = \{q_0\} \subseteq Q \backslash \partial Q$.

Or

c″) $\{K(t)\}_{t\in[a,b]}$ is a suitable family of non-empty, open uniformly bounded and convex subsets of $\mathbb{R}^N$ with $\Gamma_{\partial K}$ closed in $\mathbb{R}^{N+1}$.

f″) $T(Q \times \{0\}) \subset Q$ and ∂Q is fixed-point free.

6. An application for the anti-periodic problem

Consider the anti-periodic problem

$$\left.\begin{aligned} x' &\in F_1(t,x) + F_2(t,x) \\ x(a) &= -x(b) \end{aligned}\right\} \tag{13}$$

where $x = (x_1, \dots, x_N)$, $F = F_1 + F_2 = (f_{11}, \dots, f_{1N}) + (f_{21}, \dots, f_{2N})$, $F_1, F_2 : [a,b] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ are globally upper semi-continuous multi-functions which are bounded in $t \in [a,b]$ for every $x \in \mathbb{R}^N$ and linearly bounded in $x \in \mathbb{R}^N$ for every $t \in [a,b]$. Furthermore, assume that there exist constants $R_i > 0$ $(i = 1, \dots, N)$ such that the following conditions (14) - (16) are satisfied:

$$\big|f_{1i}(t, x(\pm R_i))\big| > \max_{\substack{t\in[a,b]\\ x\in\overline{K}}} |f_{2i}(t,x)| \qquad \big(i = 1, \dots, N;\ \ t \in (a,b)\big) \tag{14}$$

where $x(\pm R_i) = \big(x_1, \dots, x_{i-1}, \pm R_i, x_{i+1}, \dots, x_N\big)$, $|x_j| \leq R_j$ and $K = \big\{x \in \mathbb{R}^N : |x_i| < R_i\ (i = 1, \dots, N)\big\}$.

$$\big[f_{1i}(a, x(\pm R_i)) + f_{2i}(a,y)\big]\,\big[f_{1i}(b, -x(\pm R_i)) + f_{2i}(b,z)\big] < 0\ (i = 1, \dots, N) \tag{15}$$

where $x, y, z \in \overline{K}$.

$$F_1(t,\cdot) \text{ is Lipschitzian with a sufficiently small constant } L \tag{16}$$

for every $t \in [a,b]$; in the single-valued case, when $F \in C([a,b] \times \mathbb{R}^N, \mathbb{R}^N)$, it is enough to take $L \leq \frac{\pi}{b-a}$ (cf. [2]).

In order to apply Theorem 5 for the solvability of problem (11), let us still consider the enlarged family of problems

$$\left.\begin{aligned} x' &\in \lambda F_1(t,x) + \lambda F_2(t,q(t)) \quad (\lambda \in [0,1]) \\ x(a) &= -x(b) \end{aligned}\right\} \tag{13_q}$$

where

$$q \in Q = \Big\{\widetilde{q} \in C([a,b], \mathbb{R}^N) : \widetilde{q}(t) \in \overline{K} \ \text{ for all } t \in [a,b]\Big\}.$$

Observe that if $\xi \in \partial K$, then

$$\xi = \xi(\pm R_i) = \big(\xi_1, \dots, \xi_{i-1}, \pm R_i, \xi_{i+1}, \dots, \xi_N\big)$$

for some i and $|\xi_j| \leq R_j$ for all $j \neq i$. Therefore, let us define for problem (13_q) the bounding functions as (cf. [12: p. 78]) $V_\xi(x) = \pm x_i - R_i$ $(i = 1, \dots, N)$ where $\xi = \xi(\pm R_i) \in \partial K$.

One can easily check that K is a bound set for problems (13_q) provided (14) - (16) hold. Indeed, making also use of the discussions both in Remarks 3 and 4 we have the following:

ad. g) $V_\xi(\xi) = 0$ and $V_\xi(x) \le 0$ for $x \in \overline{K}$.

ad. h) $\big(\nabla V_\xi(\xi), \big(\lambda F_1(t,\xi) + \lambda F_2(t,q(t))\big)\big) = \pm\lambda\big[f_{1i}(t,\xi) + f_{2i}(t,q(t))\big] \neq 0$ for all $t \in (a,b)$, where $\lambda \in (0,1]$, according to (16).

ad. j) $\big(\nabla V_\xi(\xi), \big(\lambda F_1(a,\xi)+\lambda F_2(a,q(a))\big)\big)\cdot\big(\nabla V_{-\xi}(-\xi)\big(\lambda F_1(b,-\xi)+\lambda F_2(b,q(b))\big)\big) = -\lambda^2\big[f_{1i}(a,\xi) + f_{2i}(a,q(a))\big]\cdot\big[f_{1i}(b,-\xi) + f_{2i}(b,q(b))\big] > 0$ where $\lambda \in (0,1]$, according to (17).

The other conditions in Theorem 5 are satisfied as follows:

ad. a) and ad. f) The associated homogeneous problem ($\lambda = 0$)

$$\left.\begin{aligned} x' &= 0 \\ x(a) &= -x(b) \end{aligned}\right\} \tag{13_0}$$

has only the trivial solution $x(t) = T(q,\{0\}) \equiv 0$ by which $T(Q\times\{0\}) \equiv 0 \in Q$.

ad. b) $G(t,x,q,\lambda) = \lambda F_1(t,x) + \lambda F_2(t,q)$; hence $F_1(t,c) + F_2(t,c) = F(t,c)$.

ad. c) Since K is convex, it is an absolute retract, and so a retract of $\mathbb{R}^N$.

ad. d) It follows immediately from (16).

ad. e) It follows by the hypothesis on the growth restrictions on F_1 and F_2.

ad. i) $\partial K(a) = -\partial K(b)$, $\partial K = \{\xi \in \mathbb{R}^N : (\xi_1, \xi_{i-1}, \pm R_i, \xi_{i+1}, \dots, \xi_N),\ |\xi_j| \le R_j\}$.

Hence, applying Theorem 5, anti-periodic problem (13) admits a solution provided (14) - (16) take place jointly with the above growth restrictions on F_1 and F_2.

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