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Quadratic Forms and Nonlinear Non-Resonant Singular Second Order Boundary Value Problems of Limit Circle Type

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Abstract. New existence results are presented for non-resonant second order singular boundary value problems

$$\left.\begin{aligned} &\frac{1}{p(t)}(p(t)y'(t))' + \tau(t)\,y(t) = \lambda\,f(t,y(t)) \quad \text{a.e. on } [0,1] \\ &\lim_{t\to 0^+} p(t)y'(t) = y(1) = 0 \end{aligned}\right\}$$

where one of the endpoints is regular and the other may be singular or of limit circle type.

Keywords: *Singular and non-resonant problems, points of limit circle type, existence criteria for solutions*

AMS subject classification: 34B15

1. Introduction

In this paper we develop an existence theory for

$$\frac{1}{p(t)}(p(t)y'(t))' + \tau(t)y(t) = \lambda\,f(t,y(t)) \qquad \text{a.e. on } [0,1]$$

which makes use of the relationship between the asymptotic behavior of the non-linearity $\frac{f(t,y)}{y}$ and the spectrum of the differential operator. In particular, we examine the non-resonant second order singular boundary value problem

$$\left.\begin{aligned} &\frac{1}{p(t)}(p(t)y'(t))' + \tau(t)\,y(t) = \lambda\,f(t,y(t)) \quad \text{a.e. on } [0,1] \\ &\lim_{t\to 0^+} p(t)y'(t) = y(1) = 0 \end{aligned}\right\}. \qquad (P_\lambda)$$

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Throughout $p \in C[0,1] \cap C^1(0,1)$ together with $p > 0$ on $(0,1)$, τ is measurable with $\tau > 0$ a.e. on $[0,1]$ and $\int_0^1 p(x)\tau(x)\,dx < \infty$, and $\lambda \in \mathbb{R}$ is some parameter. We do not assume $\int_0^1 \frac{ds}{p(s)} < \infty$ but rather $\int_0^1 \frac{1}{p(s)} \left(\int_0^s p(x)\tau(x)\,dx\right)^{\frac{1}{2}} ds < \infty$. As a result for the eigenvalue problem

$$\left.\begin{array}{l} Lu = \lambda u \quad \text{a.e. on } [0,1] \\ \lim_{t\to 0^+} p(t)u'(t) = u(1) = 0 \end{array}\right\} \tag{1.1}$$

where $Lu = -\frac{1}{pq}(pu')'$, one of the endpoints, $t = 1$, will be regular and the other, $t = 0$, may be singular or of limit circle type [6, 7]. For nonlinear non-resonant problems of limit circle type only a handful of papers have appeared in the literature (see [1, 3, 6]). All other papers, to our knowledge, concerning nonlinear non-resonant problems discuss the case when $t = 0$ and $t = 1$ are regular points (see [2, 4, 5, 7] and the references therein). In [6], Fonda and Mawhin presented a technique for discussing non-resonant problems (i.e. (1.1) with $p \equiv 1$) based on quadratic forms. We will use part of this technique in this paper. However, as we will see, many extra steps will be needed to discuss non-resonant problems when one of the endpoints is of limit circle type.

For notational purposes let w be a weight function. By $L^2_w[0,1]$ we mean the space of functions u such that $\int_0^1 w(t)|u(t)|^2 dt < \infty$ (also, if $u \in L^2_w[0,1]$, we define $\|u\|_w = \left(\int_0^1 w(t)|u(t)|^2 dt\right)^{\frac{1}{2}}$). Let $AC[0,1]$ be the space of functions which are absolutely continuous on $[0,1]$.

The following well known existence principle [6, 7] (which is a special case of the Leray-Schauder continuation theorem), due to O'Regan, will be needed in Section 2.

Theorem 1.1. *Suppose the following conditions are satisfied:*

(i) $p \in C[0,1] \cap C^1(0,1)$ *with* $p > 0$ *on* $(0,1)$.

(ii) $\tau \in L^1_p[0,1]$ *with* $\tau > 0$ *a.e. on* $[0,1]$.

(iii) $\int_0^1 \frac{1}{p(s)} \left(\int_0^s p(x)\tau(x)\,dx\right)^{1/2} ds < \infty$.

(iv) $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ *is a Carathéodory function, i.e.*

(i) $t \mapsto f(t,y)$ *is measurable for all* $y \in \mathbb{R}$

(ii) $y \mapsto f(t,y)$ *is continuous for a.e.* $t \in [0,1]$.

(v) $\frac{f(t,y(t))}{\tau(t)} \in L^2_{p\tau}[0,1]$ *whenever* $y \in L^2_{p\tau}[0,1]$.

In addition, assume that problem (P_0) *has only the trivial solution. Further, suppose there is a constant* M_0*, independent of* λ*, with*

$$\|y\|_{p\tau} = \left(\int_0^1 p(t)\tau(t)|y(t)|^2 dt\right)^{\frac{1}{2}} \neq M_0$$

for any solution y *(here* $y \in L^2_{p\tau}[0,1]$ *with* $y \in C(0,1] \cap C^1(0,1)$ *and* $py' \in AC[0,1]$*) to problem* (P_λ)*, for each* $\lambda \in (0,1)$*. Then problem* (P_1) *has at least one solution.*

Finally, we remark that problems of type (P_λ) occur in many applications in the physical sciences, for example in radially symmetric nonlinear diffusion in the n-dimensional sphere we have $p(t) = t^{n-1}$; these problems involve a homogeneous Neumann condition at zero, i.e. $\lim_{t\to 0^+} t^{n-1}u'(t) = 0$. Another example is the Poisson-Boltzmann equation

$$\left.\begin{array}{ll} y'' + \frac{\alpha}{t}\, y' = f(t,y) & (0 < t < 1) \\ y'(0^+) = y(1) = 0 & (\alpha \geq 1) \end{array}\right\} \tag{1.2}$$

which occurs in the theory of thermal explosions and in the theory of electrohydrodynamics. The results related to problem (1.2) in the literature [1, 3] usually consider the situation when inf $\frac{\partial f}{\partial y}$ and sup $\frac{\partial f}{\partial y}$ are bounded and satisfy a "non-resonant" condition. In this paper we improve the above existence result (in fact, in our theory the existence of $\frac{\partial f}{\partial y}$ is not assumed).

We also note that the results in [6] are a special case of Theorems 2.1 and 2.2 in this paper (see the special example after the proof of Theorem 2.1).

2. Non-resonance type problems

In this section we present two existence results for singular boundary value problem (P_1). Conditions (i) - (v) of Theorem 1.1 will be assumed throughout this section. Notice condition (iii) implies (see [7]) $\int_0^1 p(x)\tau(x)\big(\int_x^1 \frac{ds}{p(s)}\big)^2 dx < \infty$.

Our first result establishes existence if a certain integral inequality is satisfied.

Theorem 2.1. *Suppose conditions* (i) - (v) *of Theorem* 1.1 *hold and suppose problem* (P_0) *has only the trivial solution. In addition, assume* f *has the decomposition*

$$f(t,u) = g_1(t,u)\, u + g_2(t,u) + h(t,u)$$

where $g_1, g_2, h : [0,1]\times\mathbb{R} \to \mathbb{R}$ *are Carathéodory functions and the following conditions are satisfied:*

- **(i)** $ug_2(t,u) \geq 0$ *for a.e.* $t \in [0,1]$ *and* $u \in \mathbb{R}$.
- **(ii)** $\exists \tau_1 \in C[0,1]$ *with* $\tau_1(t)\tau(t) \leq g_1(t,u) \leq 0$ *for a.e.* $t \in [0,1]$ *and* $u \in \mathbb{R}$.
- **(iii)** $|h(t,u)| \leq \phi_1(t) + \phi_2(t)|u|^\gamma$ *for a.e.* $t \in [0,1]$, *with* $0 \leq \gamma < 1$.
- **(iv)** $\int_0^1 p(t)\phi_1(t)\big(\int_t^1 \frac{ds}{p(s)}\big)^{1/2} dt < \infty$ *and* $\int_0^1 p(t)\phi_2(t)\big(\int_t^1 \frac{ds}{p(s)}\big)^{(\gamma+1)/2} dt < \infty$.
- **(v)** $\int_0^1 \big[p(u')^2 - (\tau - \tau_1\tau)pu^2\big] dt > 0$ *for any* $0 \neq u \in K^\star$

where

$$K^\star = \left\{ w : [0,1] \to \mathbb{R} \,\middle|\, \begin{array}{l} w \in L^2_{p\tau}[0,1] \text{ with } w \in C(0,1] \\ w' \in L^2_p[0,1] \text{ and } w(1) = 0 \end{array} \right\}.$$

Then problem (P_1) *has a solution* $y \in L^2_{p\tau}[0,1]$ *with* $y \in C(0,1] \cap C^1(0,1)$ *and* $py' \in AC[0,1]$.

Proof. We first show that there exists $\varepsilon > 0$ with

$$\int_0^1 \left[p(y')^2 - (\tau - \tau_1\tau)py^2\right]dt \geq \varepsilon\left(\|y\|_{p\tau}^2 + \|y'\|_p^2\right) \tag{2.1}$$

for any $y \in K^\star$. If this is not the case, then there exists a sequence $\{y_n\} \subset K^\star$ with

$$\|y_n\|_{p\tau}^2 + \|y_n'\|_p^2 = 1 \tag{2.2}$$

$$\int_0^1 \left[p(y_n')^2 - (\tau - \tau_1\tau)py_n^2\right]dt \to 0 \quad \text{as } n \to \infty. \tag{2.3}$$

The Riesz compactness criteria together with a standard result in functional analysis (if E is a reflexive Banach space, then any norm bounded sequence in E has a weakly convergent subsequence) implies that there is a subsequence S of integers with

$$y_n \to y \text{ in } L_{p\tau}^2[0,1] \qquad \text{and} \qquad y_n' \rightharpoonup y' \text{ in } L_p^2[0,1] \tag{2.4}$$

as $n \to \infty$ in S where $\rightharpoonup$ denotes weak convergence.

Note $\{y_n\}$ is bounded in $L_{p\tau}^2[0,1]$ (see (2.2)) and, for $r > 0$, Hölder's inequality yields

$$\begin{aligned}\int_0^1 p(t)\tau(t)|y_n(t+r) - y_n(t)|^2 dt &= \int_0^1 p\tau \left|\int_t^{t+r} y_n'(s)ds\right|^2 dt \\ &\leq \|y_n'\|_p^2 \int_0^1 p\tau \int_t^{t+r} \frac{ds}{p(s)} dt \\ &\leq \int_0^1 p\tau \int_t^1 \frac{ds}{p(s)} dt - \int_0^1 p\tau \int_{t+r}^1 \frac{ds}{p(s)} dt \\ &\to 0 \quad \text{as } r \to 0^+\end{aligned}$$

by the Lebesgue dominated convergence theorem and assumption (iii) of Theorem 1.1. Thus $\{y_n\}$ is relatively compact in $L_{p\tau}^2[0,1]$.

Next, a standard result in functional analysis [7] yields

$$\int_0^1 p[y']^2 dt \leq \liminf \int_0^1 p[y_n']^2 dt. \tag{2.5}$$

Now (2.3) - (2.5) and the fact that $\liminf[s_n + t_n] \geq \liminf s_n + \liminf t_n$ for sequences $\{s_n\}$ and $\{t_n\}$ imply

$$\int_0^1 \left[p(y')^2 - (\tau - \tau_1\tau)py^2\right]dt \leq 0 \tag{2.6}$$

since

$$\liminf \int_0^1 (\tau - \tau_1\tau)py_n^2 dt = \int_0^1 (\tau - \tau_1\tau)py^2 dt.$$

Note $y(1) = 0$ since in fact $y_n \to y$ in $C[\varepsilon, 1]$ $(\varepsilon > 0)$ by the Arzela-Ascoli theorem. By assumption (v) we have $y \equiv 0$. However,

$$\begin{aligned}\|y_n\|_{p\tau}^2 + \|y_n'\|_p^2 &= \int_0^1 p\tau y_n^2 dt + \int_0^1 (\tau - \tau_1\tau) p y_n^2 dt \\ &\quad + \int_0^1 \left[p(y_n')^2 - (\tau - \tau_1\tau) p y_n^2\right] dt \\ &\to 0 \quad \text{as } n \to \infty \text{ in } S\end{aligned}$$

which is impossible. Thus (2.1) holds for some $\varepsilon > 0$.

Let y be a solution to problem (P_λ) for some $0 < \lambda < 1$. Note, in particular, $y \in K^\star$. Multiply the differential equation by y and integrate from 0 to 1 to obtain

$$\int_0^1 \left[p(y')^2 - \tau p y^2\right] dt = -\lambda \int_0^1 p y^2 g_1(t, y)\, dt - \lambda \int_0^1 p y g_2(t, y)\, dt - \lambda \int_0^1 p y h(t, y)\, dt$$

and so (use assumptions (i) - (ii))

$$\int_0^1 \left[p(y')^2 - (\tau - \tau_1\tau) p y^2\right] dt \le \int_0^1 p|y h(t, y)|\, dt.$$

This together with assumption (iii) and (2.1) imply that there exists $\varepsilon > 0$ (fix it) with

$$\varepsilon\left(\|y\|_{p\tau}^2 + \|y'\|_p^2\right) \le \int_0^1 p\phi_1 |y|\, dt + \int_0^1 p\phi_2 |y|^{\gamma+1} dt.$$

Since $y(1) = 0$, we have from Hölder's inequality

$$|y(t)| = \left|\int_t^1 y'(s)\, ds\right| \le \|y'\|_p \left(\int_t^1 \frac{ds}{p(s)}\right)^{\frac{1}{2}}$$

for $t \in (0, 1)$, and so

$$\varepsilon\left(\|y\|_{p\tau}^2 + \|y'\|_p^2\right) \le K_0 \|y'\|_p + K_1 \|y'\|_p^{\gamma+1} \tag{2.7}$$

where

$$K_0 = \int_0^1 p(t)\phi_1(t) \left(\int_t^1 \frac{ds}{p(s)}\right)^{\frac{1}{2}} dt \quad \text{and} \quad K_1 = \int_0^1 p(t)\phi_2(t) \left(\int_t^1 \frac{ds}{p(s)}\right)^{\frac{\gamma+1}{2}} dt.$$

Now (2.7) guarantees that there is a constant $M > 0$, independent of λ, with $\|y'\|_p \le M$. This together with (2.7) guarantees the existence of a constant $M_0 > 0$, independent of λ, with $\|y\|_{p\tau} \le M_0$. The result now follows from Theorem 1.1 ∎

We now discuss briefly assumption (v) of Theorem 2.1. Inequalities of this type play a major role in the literature of calculus of variation. We illustrate the ideas involved with a simple example. Consider the problem

$$\left.\begin{aligned} &\frac{1}{p}(py')' + \mu q y = f(t,y) \ \text{ a.e. on } [0,1] \\ &\lim_{t\to 0^+} p(t)y'(t) = y(1) = 0 \end{aligned}\right\} \tag{2.8}$$

with $q \in L^1_p[0,1]$, $q > 0$ a.e. on $[0,1]$, and

$$\mu(1-\tau_1(t)) < \lambda_0 \ \text{ for } t \in [0,1], \tag{2.9}$$

λ_0 being the first eigenvalue of problem (1.1) with $Lu = -\frac{1}{pq}(pu')'$. Let also assumptions (i), (iii) - (v) of Theorem 1.1 and assumptions (i) - (iv) of Theorem 2.1 hold, with $\tau(t) = \mu q(t))$. Recall (see [7: Chapter 11], limit circle case) that L has a countable number of real eigenvalues $\lambda_i > 0$ (arranged so that $\lambda_0 < \lambda_1 < \lambda_2 < \ldots$) with corresponding (orthonormal) eigenfunctions ψ_i. The set $\{\psi_i\}$ form a basis of $L^2_{pq}[0,1]$, and so for any $u \in K^\star$ we have

$$u(t) = \sum_{i=0}^{\infty} \eta_i \psi_i(t), \qquad \eta_i = \langle u, \psi_i \rangle_{pq}$$

where $\langle u, v\rangle_{pq} = \int_0^1 pqu\bar{v}dt$.

We claim that problem (2.8) has at least one solution. This follows immediately from Theorem 2.1 once we show its condition (v) is satisfied. First notice from (2.9) (note $\tau_1 \in C[0,1]$) that there exists $\delta > 0$ with $\mu(1-\tau_1(t)) \le \lambda_0 - \delta$ for $t \in [0,1]$. Now for $u \in K^\star$ we have

$$\begin{aligned} \int_0^1 \left[p(u')^2 - (\tau - \tau_1\tau)pu^2\right]dt &\ge \int_0^1 \left[p(u')^2 - (\lambda_0 - \delta)pqu^2\right]dt \\ &= \sum_{i=0}^{\infty} \eta_i^2 \left[\lambda_i - (\lambda_0 - \delta)\right] \int_0^1 pq\psi_i^2 dt \end{aligned}$$

since $(p\psi_i')' + \lambda_i pq\psi_i = 0$ a.e. on $[0,1]$ and $\lim_{t\to 0^+} p(t)\psi_i(t) = \psi_i(1) = 0$. Consequently,

$$\int_0^1 \left[p(u')^2 - (\tau - \tau_1\tau)pu^2\right]dt \ge \delta \sum_{i=0}^{\infty} \eta_i^2 \int_0^1 pq\psi_i^2 dt = \delta \int_0^1 pq|u|^2 dt > 0$$

for $u \neq 0$. Thus condition (v) of Theorem 2.1 holds, so our claim is established.

For the remainder of this paper let

$$E = \Big\{ y \in L^2_{p\tau}[0,1] : \, y' \in L^2_p[0,1] \text{ and } y(1) = 0 \Big\}.$$

For $u, v \in E$ we define

$$\langle u, v\rangle = \int_0^1 p\tau u\bar{v}dt + \int_0^1 pu'\overline{v'}dt.$$

We show E is complete. Let $\{y_n\}$ be a Cauchy sequence in E. Then there exist functions $y \in L^2_{p\tau}[0,1]$ and $u \in L^2_p[0,1]$ with $y_n \to y$ in $L^2_{p\tau}[0,1]$ and $y_n' \to u$ in $L^2_p[0,1]$ as $n \to \infty$. Let

$$v(t) = -\int_t^1 u(s)\,ds.$$

Note $v(1) = 0$. Also, notice since $y_n \in E$ (so $y_n(1) = 0$) that

$$\begin{aligned}
&\int_0^1 p(t)\tau(t)|y_n(t) - v(t)|^2 dt \\
&\quad = \int_0^1 p(t)\tau(t)\left|\int_t^1 (y_n - v)'(s)ds\right|^2 dt \\
&\quad \le \left(\int_0^1 p(t)\tau(t)\int_t^1 \frac{ds}{p(s)}dt\right)\left(\int_0^1 p(s)|(y_n - v)'(s)|^2 ds\right) \\
&\quad = \left(\int_0^1 p(t)\tau(t)\int_t^1 \frac{ds}{p(s)}dt\right)\left(\int_0^1 p(s)|y_n'(s) - u(s)|^2 ds\right)
\end{aligned}$$

and the right-hand side goes to zero as $n \to \infty$. Thus $y_n \to v$ in $L^2_{p\tau}[0,1]$ as $n \to \infty$, and so $y = v$ a.e. on $[0,1]$. As a result, $y_n \to v$ in E, so E is complete. [In fact, in the following theorem, we could let E be the space of functions $y \in L^2_{p\tau}[0,1]$ with $y' \in L^2_p[0,1]$.]

Theorem 2.2. *Suppose conditions* (i) - (v) *of Theorem* 1.1 *hold and assume problem* (P_0) *has only the trivial solution. In addition, assume f has the decomposition*

$$f(t,u) = g(t,u)u + h(t,u)$$

where $g, h : [0,1] \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying conditions (iii) - (iv) *of Theorem 2.1. Also, suppose the following conditions are satisfied:*

- **(i)** *There exist $0 \le -\tau_1, \tau_2 \in C[0,1]$ with $\tau_1(t)\tau(t) \le g_1(t,u) \le \tau_2(t)\tau(t)$ for a.e. $t \in [0,1]$ and $u \in \mathbb{R}$.*
- **(ii)** *$E = \Omega \oplus \Gamma$ where $\Omega \subseteq K^\star$ is finite-dimensional and for every $0 \ne y = u + v \in K^\star$ with $u \in \Omega, v \in \Gamma$ we have $R(y) > 0$*

where

$$R(y) = \int_0^1 \left[p(v')^2 - (\tau - \tau\tau_1)pv^2\right]dt - \int_0^1 \left[p(u')^2 - (\tau - \tau\tau_2)pu^2\right]dt.$$

Then problem (P_1) has at least one solution.

Remark 2.1. The set $K^\star$ in condition (ii) here is as defined in condition (v) of Theorem 2.1. In (ii) we have $y = u + v$ with $u \in \Omega$ and $v \in \Gamma$, so $\int_0^1 p\tau uv\,dt + \int_0^1 pu'v'dt = 0$.

Proof of Theorem 2.2. We first show that there exists $\varepsilon > 0$ with

$$R(y) \ge \varepsilon\left(\|y\|^2_{p\tau} + \|y'\|^2_p\right) \tag{2.10}$$

for any $y \in K^\star$; here $y = u + v$ with $u \in \Omega$ and $v \in \Gamma$. If this is false, then there exists a sequence $\{y_n\} \subset K^\star$ with $\|y_n\|^2_{p\tau} + \|y_n'\|^2_p = 1$ and

$$R(y_n) \to 0 \qquad \text{as } n \to \infty. \tag{2.11}$$

Note $y_n = u_n + v_n$ with $u_n \in \Omega$ and $v_n \in \Gamma$. Now there is a subsequence S of integers with

$$y_n \to y \text{ in } L^2_{p\tau}[0,1] \qquad \text{and} \qquad y_n' \rightharpoonup y' \text{ in } L^2_p[0,1] \tag{2.12}$$

as $n \to \infty$ in S. Also, since strong and weak convergence are the same in finite-dimensional spaces we have

$$u_n' \to u' \qquad \text{in } L^2_p[0,1] \text{ as } n \to \infty \text{ in } S. \tag{2.13}$$

We also have

$$\int_0^1 p[v']^2 dt \le \liminf \int_0^1 p[v_n']^2 dt. \tag{2.14}$$

Now (2.11) - (2.14) imply that $R(y) \le 0$. From assumption (ii) we have $y \equiv 0$. Finally (note $E = \Omega \oplus \Gamma$, so $\int_0^1 p\tau u_n v_n dt + \int_0^1 p u_n' v_n' dt = 0$),

$$\begin{aligned}\|y_n\|^2_{p\tau} + \|y_n'\|^2_p &= R(y_n) + \int_0^1 p\tau[v_n^2 + u_n^2]\, dt + 2\int_0^1 p[u_n']^2 dt \\ &\quad + \int_0^1 \big([\tau - \tau_1\tau]pv_n^2 - [\tau - \tau_2\tau]pu_n^2\big)dt \\ &\to 0 \quad \text{as } n \to \infty \text{ in } S\end{aligned}$$

which is impossible. Thus (2.10) holds for some $\varepsilon > 0$.

Let y $(= u + v)$ be a solution to problem (P_λ) for some $0 < \lambda < 1$. Then

$$-\int_0^1 (v-u)[(py')' + p\tau y]dt = -\lambda \int_0^1 p(v-u)yg(t,y)\, dt - \lambda \int_0^1 p(v-u)h(t,y)\, dt$$

and so integration by parts yield

$$\begin{aligned}\int_0^1 \big[p(v')^2 + pv^2(-\tau + \lambda g(t,y))\big]\, dt - \int_0^1 \big[p(u')^2 + pu^2(-\tau + \lambda g(t,y))\big]\, dt \\ \le \int_0^1 p|v-u||h(t,y)|\, dt.\end{aligned} \tag{2.15}$$

Now

$$\begin{aligned}pv^2\big[-\tau + \lambda g(t,y)\big] &= pv^2\big[-(\tau - \tau_1\tau) + \lambda g(t,y) - \tau_1\tau\big] \\ &\ge pv^2\big[-(\tau - \tau_1\tau) + (\lambda - 1)\tau_1\tau\big] \\ &\ge -p(\tau - \tau_1\tau)v^2 \quad \text{a.e. on } [0,1].\end{aligned}$$

Similarly,

$$pu^2\big[-\tau + \lambda g(t,y)\big] \le -p(\tau - \tau_2\tau)u^2 \qquad \text{a.e. on } [0,1].$$

Putting these into (2.15) yields

$$R(y) \le \int_0^1 p|v-u||h(t,y)|\,dt.$$

This together with (2.10) implies that there is an $\varepsilon > 0$ with

$$\varepsilon\big(\|y\|_{p\tau}^2 + \|y'\|_p^2\big) \le \int_0^1 p|v-u|\,|h(t,y)|\,dt. \tag{2.16}$$

Next, notice that for $t \in (0,1)$ we have

$$|v(1)-u(1)| \le |v(t)-u(t)| + \int_t^1 |(v-u)'(s)|\,ds$$

and so for $t \in (0,1)$

$$|v(1)-u(1)| \le |v(t)-u(t)| + \|v'-u'\|_p \left(\int_t^1 \frac{ds}{p(s)}\right)^{\frac{1}{2}}. \tag{2.17}$$

Note also that

$$\|v-u\|_{p\tau}^2 + \|v'-u'\|_p^2 = \|y\|_{p\tau}^2 + \|y'\|_p^2 \tag{2.18}$$

and this together with (2.17) yields for $t \in (0,1)$

$$|v(1)-u(1)| \le |v(t)-u(t)| + \big(\|y\|_{p\tau}^2 + \|y'\|_p^2\big)^{\frac{1}{2}} \left(\int_t^1 \frac{ds}{p(s)}\right)^{\frac{1}{2}}.$$

Multiply this by $\sqrt{p(t)\tau(t)}$ and integrate from 0 to 1 (using Hölder's inequality) to obtain

$$\begin{aligned} &|v(1)-u(1)| \int_0^1 \sqrt{p(t)\tau(t)}dt \\ &\qquad \le \|v-u\|_{p\tau} + \big(\|y\|_{p\tau}^2 + \|y'\|_p^2\big)^{\frac{1}{2}} \left(\int_0^1 p\tau \int_t^1 \frac{ds}{p(s)} dt\right)^{\frac{1}{2}}. \end{aligned}$$

This together with (2.18) yields

$$|v(1)-u(1)| \le K_2\big(\|y\|_{p\tau}^2 + \|y'\|_p^2\big)^{\frac{1}{2}} \tag{2.19}$$

where

$$K_2 = \frac{1 + \big(\int_0^1 p(t)\tau(t) \int_t^1 \frac{ds}{p(s)} dt\big)^{\frac{1}{2}}}{\int_0^1 \sqrt{p(t)\tau(t)}dt}.$$

Also, for $t \in (0,1)$ we have

$$|v(t)-u(t)| \le |v(1)-u(1)| + \|v'-u'\|_p \left(\int_t^1 \frac{ds}{p(s)}\right)^{\frac{1}{2}}$$

and so (use (2.18) and (2.19)) for $t \in (0,1)$

$$|v(t) - u(t)| \leq \left(\|y\|_{p\tau}^2 + \|y'\|_p^2\right)^{\frac{1}{2}} \left\{ K_2 + \left(\int_t^1 \frac{ds}{p(s)} \right)^{\frac{1}{2}} \right\}. \tag{2.20}$$

In addition, since $y(1) = 0$ we have $|y(t)| \leq \int_t^1 |y'(s)|\, ds$ for $t \in (0,1)$ and so

$$|y(t)| \leq \left(\|y\|_{p\tau}^2 + \|y'\|_p^2\right)^{\frac{1}{2}} \left(\int_t^1 \frac{ds}{p(s)} \right)^{\frac{1}{2}} \tag{2.21}$$

for $t \in (0,1)$. Put condition (iii) of Theorem 2.1 into (2.16) to obtain

$$\begin{aligned} &\varepsilon\left(\|y\|_{p\tau}^2 + \|y'\|_p^2\right) \\ &\quad \leq \int_0^1 p(t)|v(t) - u(t)|\phi_1(t)\, dt + \int_0^1 p(t)|v(t) - u(t)|\, |y(t)|^\gamma \phi_2(t)\, dt. \end{aligned}$$

This together with (2.20) - (2.21) gives

$$\begin{aligned} \varepsilon\left(\|y\|_{p\tau}^2 + \|y'\|_p^2\right) &\leq \left(\|y\|_{p\tau}^2 + \|y'\|_p^2\right)^{\frac{1}{2}} \left[K_2 \int_0^1 p(t)\phi_1(t)\, dt + K_0 \right] \\ &\quad + \left(\|y\|_{p\tau}^2 + \|y'\|_p^2\right)^{\frac{\gamma+1}{2}} \left[K_2 \int_0^1 p(t)\phi_2(t) \left(\int_t^1 \frac{ds}{p(s)} \right)^{\frac{\gamma}{2}} dt + K_1 \right] \end{aligned}$$

where

$$K_0 = \int_0^1 p(t)\phi_1(t) \left(\int_t^1 \frac{ds}{p(s)} \right)^{\frac{1}{2}} dt \quad \text{and} \quad K_1 = \int_0^1 p(t)\phi_2(t) \left(\int_t^1 \frac{ds}{p(s)} \right)^{\frac{\gamma+1}{2}} dt.$$

Now since $0 \leq \gamma < 1$, there exists a constant $M > 0$, independent of λ, with $\|y\|_{p\tau}^2 + \|y'\|_p \leq M$. The result now follows from Theorem 1.1 ∎

References

[1] Chawla, M. M. and P. N. Shivakumar: *On the existence of solutions of a class of singular nonlinear two point boundary value problems.* J. Comp. Appl. Math. 19 (1987), 379 – 388.

[2] Dunninger, D. R. and J. C. Kurtz: *A priori bounds and existence of positive solutions for singular nonlinear point boundary value problems.* SIAM J. Math. Anal. 17 (1986), 595 – 609.

[3] El Gebeily, M. A., Boumenir, A. and A. B. M. Elgindi: *Existence and uniqueness of solutions of a class of two-point singular nonlinear boundary value problems.* J. Comp. Appl. Math. 46 (1993), 345 – 355.

[4] Fonda, A. and J. Mawhin: *Quadratic forms, weighted eigenfunctions and boundary value problems for nonlinear second order ordinary differential equations.* Proc. Royal Soc. Edinburgh 112A (1989), 145 – 153.

[5] Mawhin, J. and W. Omano: *Two point boundary value problems for nonlinear perturbations of some singular linear differential equations at resonance.* Comm. Math. Univ. Carolinae 30 (1989), 537 – 550.

[6] O'Regan, D.: *Nonresonant nonlinear singular problems in the limit circle case.* J. Math. Anal. Appl. 197 (1996), 708 – 725.

[7] O'Regan, D.: *Existence Theory for Nonlinear Ordinary Differential Equations.* Dordrecht: Kluwer Acad. Publ. 1997.

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