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Lipschitz Continuity of Polyhedral Skorokhod Maps

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Abstract. We show that a special stability condition of the associated system of oblique projections (the so-called ℓ-paracontractivity) guarantees that the corresponding polyhedral Skorokhod problem in a Hilbert space X is solvable in the space of absolutely continuous functions with values in X. If moreover the oblique projections are transversal, the solution exists and is unique for each continuous input and the Skorokhod map is Lipschitz continuous in both spaces $C([0,T];X)$ and $W^{1,1}(0,T;X)$. Also, an explicit upper bound for the Lipschitz constant is derived.

Keywords: *Polyhedral Skorokhod problem, oblique reflections, Lipschitz continuity*

AMS subject classification: 34A60, 47A30, 52B70, 90C25

0. Introduction

A class of models called *Skorokhod problems* is widely used in areas such as elastoplasticity, queueing theory, iterative optimization methods, mathematical economics (see references in [2, 4]). Here we consider a particular case of *polyhedral* Skorokhod problems which can be described as follows.

A characteristic polyhedral set Z is given in a Hilbert space X. For a given input function $u(t)$ defined in a time interval $[0,T]$ with values in X we look for an output $x(t)$ with values in Z such that the derivative $\dot{u}(t)-\dot{x}(t)$ (in an appropriate sense) belongs to a given *reflection cone* $\mathcal{R}(x(t))$ at the point $x(t)$. If the reflection rules, for each input u in a suitable function space and for each initial condition $x_0 \in Z$, determines a unique output x, then the mapping $\mathcal{S} : [x_0, u] \mapsto \xi := u - x$ is called the *Skorokhod map*. Its analytical properties for different classes of inputs and in different metrics on the space of inputs and outputs play a crucial role in applications. In particular, the Lipschitz continuity of $\mathcal{S}$ in the metric of uniform convergence has been studied during the last

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20 years [2 - 4, 7, 10]. This is, partially, due to the fact that this property allows one to consider the operator $\mathcal{S}$ in the set of all continuous inputs $u(\cdot)$ which is more natural for the investigation of stability with respect to small perturbations.

The case when the reflection cone $\mathcal{R}(z)$ coincides with the outward normal cone to Z at each point $z \in Z$, like for instance in stress-strain laws of elastoplasticity, where the normality rule follows from the maximal dissipation condition, constitutes the important class of *polyhedral Skorokhod problems with normal reflection*. The corresponding Skorokhod map then represents the constitutive operator of the Prager linear kinematic hardening model and is called *multi-dimensional play operator*. Its Lipschitz continuity with respect to the sup-norm was first proved in [10] (see also [7] where this theorem is reproduced), then (by a different method) in [3, 4]. Recently, in [8], a recurrent upper bound for the Lipschitz constant has been found.

The general situation of *oblique reflection* arises in various models of human activity, and below in Section 8 we show a typical example from queuing theory. Sufficient conditions for the Lipschitz continuity were formulated in [3, 4] in terms of existence of a special convex set $B \in X$ with $0 \in \operatorname{Int} B$. Conditions of existence of solution can also be found in [3, 4]; however, they are different from the sufficient conditions of Lipschitz continuity and require additional assumptions on the reflection directions.

In all applications, the question of Lipschitz continuity of the input-output operator is substantial for the stability of numerical computations. An explicit knowledge of the Lipschitz constant is useful in particular for estimating the discretization error and the efficiency of the algorithm.

The analysis of the Skorokhod problem in this paper is based on the concept of ℓ-paracontractivity introduced in [6]. This is a special stability property of the *associated projection system* of linear operators of oblique projection on hyperplanes parallel to the faces of Z along the reflection directions (see Section 3). We first prove that ℓ-paracontractivity alone is sufficient for the existence of an absolutely continuous output $x(t)$ for every absolutely continuous input $u(t)$ and every initial condition. If, in addition, the associated projection system is *transversal*, that is, no reflection direction at a point z is orthogonal to all normal directions at z, then the Skorokhod map is of Lipschitz type in the space $W^{1,1}(0,T;X)$ as well as in the space $C([0,T];X)$ of continuous functions. If moreover Z has non-empty interior, then, for every continuous function u, the function $\xi = \mathcal{S}[x_0, u]$ has bounded variation.

An important property of ℓ-paracontracting sets of oblique projections is their *robustness* with respect to small shifts of reflection vectors for fixed normal directions. This property implies the Lipschitz continuity of Skorokhod problems under the transversality constraint whenever the reflection vectors are close to normal ones. On the other hand, it does not yield an explicit upper bound for the Lipschitz constant of a deviated Skorokhod problem. We obtain independently such an upper bound by a modified method of Lyapunov functions (cf. [8]).

The paper is organized as follows. In Section 1 we state the Skorokhod problem in the space of continuous functions. Section 2 is devoted to a survey of basic properties of oblique projections. In Section 3 we prove that the ℓ-paracontractivity ensures the existence of a solution for each initial condition. In Section 4 we establish a Lipschitz-type estimate for the sup-norm. Section 5 contains the main result which consists in proving that ℓ-paracontractivity and transversality imply the Lipschitz continuity of the Skorokhod map in both spaces $W^{1,1}(0,T;X)$ and $C([0,T];X)$. In Section 6 we derive an estimate for the total variation of the output, and an upper bound for the Lipschitz constant is derived in Section 7. We conclude the paper with an example from queuing theory in Section 8.

1. The Skorokhod problem

Let X be a Hilbert space endowed with a scalar product $\langle\cdot,\cdot\rangle$ and with the norm $|\cdot| = \langle\cdot,\cdot\rangle^{\frac{1}{2}}$. We consider a polyhedral set $Z \subset X$ defined in terms of a system $n_1,\dots,n_p$ of unit outward normal vectors as the intersection of half-spaces $H_j \ \ (j = 1,\dots,p)$ by the formula

$$Z = \bigcap_{j\in J} H_j, \quad H_j = \big\{z \in X : \ \langle z, n_j\rangle \le \beta_j \text{ for } j \in J\big\}, \quad J = \{1,\dots,p\} \tag{1.1}$$

where $\beta_j \ge 0$ are given real numbers. We associate with Z a system $\{r_1,\dots,r_p\}$ of unit vectors called *reflection vectors*. For $z \in Z$ we denote by

$$\tilde{J}(z) = \big\{j \in J : \ \langle z, n_j\rangle = \beta_j\big\} \tag{1.2}$$

the set of indices corresponding to 'active' constraints at the point z. The set-valued mapping $\tilde{J} : Z \to 2^J$ is upper semicontinuous in the sense that for all $z \in Z$ there exists $\varepsilon > 0$ such that

$$|z' - z| < \varepsilon \quad \Longrightarrow \quad \tilde{J}(z') \subset \tilde{J}(z). \tag{1.3}$$

Indeed, it suffices to put

$$\varepsilon = \min\big\{\beta_j - \langle z, n_j\rangle : \ j \in J \setminus \tilde{J}(z)\big\}.$$

For a function $w : [0,T] \to Z$ and any subset $A \subset [0,T]$ we put

$$\tilde{J}_A(w) = \bigcup_{t\in A} \tilde{J}(w(t)).$$

For any subset $J' \subset J$ we denote by $\mathcal{C}(J')$ the convex cone generated by vectors r_j with indices from J', that is

$$\mathcal{C}(J') = \left\{ y = \sum_{j\in J'} \alpha_j r_j \,\middle|\, \alpha_j \ge 0 \text{ for } j \in J' \right\}$$

and for $z \in Z$ we call the set

$$\mathcal{R}(z) = \mathcal{C}(\tilde{J}(z)) \tag{1.4}$$

the *reflection cone* at the point z. Similarly, for a function $w : [0,T] \to Z$ and any set $A \subset [0,T]$ we define

$$\mathcal{R}_A(w) = \mathcal{C}(\tilde{J}_A(w)). \tag{1.5}$$

As an immediate consequence of (1.3), we see that for every $w \in C([0,T];Z)$ and every compact set $A \subset [0,T]$ there exists $\varepsilon > 0$ such that for each $\tilde{w} \in C([0,T];Z)$ the implication

$$|w - \tilde{w}|_A < \varepsilon \quad \Longrightarrow \quad \mathcal{R}_A(\tilde{w}) \subset \mathcal{R}_A(w) \tag{1.6}$$

holds where for $v \in C([0,T];X)$ we put $|v|_A = \max_{t\in A} |v(t)|$.

We state the Skorokhod problem in the framework of continuous functions as follows:

Definition 1.1. Let $u \in C([0,T];X)$ be a given function. A pair of functions $\xi, x \in C([0,T];X)$ is said to be a solution to the *Skorokhod problem* with characteristic Z given by (1.1) and with reflection vectors $r_1, \ldots, r_p$, if

$$\left.\begin{array}{ll} x(t) + \xi(t) = u(t) & \text{for every } t \in [0,T] \\ x(t) \in Z & \text{for every } t \in [0,T] \\ \xi(t_2) - \xi(t_1) \in \mathcal{R}_{[t_1,t_2]}(x) & \text{for every } 0 \le t_1 < t_2 \le T \end{array}\right\}. \tag{1.7}$$

The alternative formulation given in [3, 4] includes also discontinuous inputs and outputs. The restriction to continuous functions enables us to make the geometrical ideas more clear and the proofs more transparent. Due to (1.3), we see that whenever the derivatives $\dot{u}(t), \dot{x}(t), \dot{\xi}(t)$ exist for some t, the third condition in (1.7) yields

$$\dot{\xi}(t) \in \mathcal{R}(x(t)). \tag{1.8}$$

In other words, the vector $\dot{u}(t)$ is decomposed into a tangential component $\dot{x}(t)$ and a reflection component $\dot{\xi}(t)$. The problem has been studied in detail in the case of *normal reflection*, that is, $n_j = r_j$ for every $j \in J$, and a survey of results can be found in [2]. In fact, the Skorokhod problem can then be stated as an evolution variational inequality in a Hilbert space which makes it accessible to classical analytical methods. Here, we are particularly interested in the case of *oblique reflection*, where no a priori assumption is made on the relationship between n_j and r_j.

We immediately see, however, that a necessary condition for the well-posedness of the Skorokhod problem reads

$$\langle r_j, n_j \rangle > 0 \tag{1.9}$$

whenever the j-th constraint is non-degenerate, that is, if there exists $x_j \in Z$ such that $\tilde{J}(x_j) = \{j\}$. Indeed, if $\langle r_j, n_j \rangle \le 0$, then taking $x(0) = x_j$ and $\dot{u}(t) \equiv n_j$ in $[0,T]$, we conclude from the convexity of Z and from (1.6) that $\langle x(t) - x(0), n_j \rangle \le 0$ and $\langle \xi(t) - \xi(0), n_j \rangle \le 0$ for small $t > 0$, which is a contradiction.

Put

$$Y = \operatorname{span}\{n_1, \ldots, n_p;\, r_1, \ldots, r_p\}$$

and let $Y^\perp$ be the orthogonal complement of Y in X. For every functions $u, x, \xi \in C([0,T];X)$ satisfying (1.7) and an arbitrary $w \in C(0,T;Y^\perp)$, the functions $\tilde{u} = u + w, \tilde{x} = x + w, \tilde{\xi} = \xi$ also satisfy (1.7). We can therefore restrict our considerations to the (finite-dimensional) space Y instead of X.

This motivates the following hypothesis which is assumed to be valid in all what follows:

Hypothesis 1.2. $X = \operatorname{span}\{n_1, \ldots, n_p;\, r_1, \ldots, r_p\}$ and (1.9) holds for every $j \in J$.

If the solution to the Skorokhod problem with a given initial condition $x(0) = x_0 \in Z$ is unique, we define the *Skorokhod map* $\mathcal{S} : Z \times C([0,T];X) \to C([0,T];X)$ by the formula

$$\mathcal{S}[x_0, u] = \xi. \tag{1.10}$$

By construction, the mapping $\mathcal{S}$ is causal and rate-independent, hence it belongs to the class of *hysteresis operators*.

2. Oblique projections

For $j \in J$, let Q_j be the projection onto $\operatorname{span}\{r_j\}$ orthogonal to n_j, that is,

$$Q_j x = \frac{\langle x, n_j\rangle}{\langle r_j, n_j\rangle}\, r_j \qquad \text{for } x \in X. \tag{2.1}$$

The family $\mathcal{Q}$ of complementary projections $\{(I - Q_j)|\, j \in J\}$, where $I : X \to X$ is the identity mapping, is called the *associated projection system* of the Skorokhod problem.

Let us introduce the following basic definition (cf. [6]).

Definition 2.1. Let Hypothesis 1.2 hold. The system $\mathcal{Q}$ is said to be *ℓ-paracontracting* if there exists a norm in X denoted by $\|\cdot\|$ such that for every $x \in X$ and every $j \in J$ we have

$$\|x\| \geq \|(I - Q_j)x\| + |Q_j x|. \tag{2.2}$$

In the case of the Skorokhod problem with normal reflection, such a norm can be constructed explicitly (see [1, 2, 9]).

The following result shows that the ℓ-paracontracting property is robust with respect to small shifts of the reflection vectors. In particular, it remains valid if the reflection directions are sufficiently close to the normal ones.

Lemma 2.2. *Let the system $\mathcal{Q}$ be ℓ-paracontracting and let $\{r'_1, \ldots, r'_p\}$ be a set of unit vectors such that for every $j \in J$ we have*

$$\langle n_j, r'_j\rangle > 0, \qquad \left|\frac{r_j}{\langle n_j, r_j\rangle} - \frac{r'_j}{\langle n_j, r'_j\rangle}\right| + \left\|\frac{r_j}{\langle n_j, r_j\rangle} - \frac{r'_j}{\langle n_j, r'_j\rangle}\right\| < \frac{1}{\langle n_j, r'_j\rangle}.$$

Then the system $\mathcal{Q}'$ of projections $I - Q'_j$, where the vectors r_j are replaced with r'_j, is also an ℓ-paracontracting system.

Proof. Put

$$\delta = \max_{j\in J}\left\{\langle n_j, r'_j\rangle\left(\left|\frac{r_j}{\langle n_j, r_j\rangle} - \frac{r'_j}{\langle n_j, r'_j\rangle}\right| + \left\|\frac{r_j}{\langle n_j, r_j\rangle} - \frac{r'_j}{\langle n_j, r'_j\rangle}\right\|\right)\right\} < 1.$$

For every $j \in J$ and $x \in X$ we then have

$$\begin{aligned}
\|(I - Q'_j)x\| &\leq \|(I - Q_j)x\| + \|(Q'_j - Q_j)x\| \\
&\leq \|x\| - |Q'_j x| + |(Q'_j - Q_j)x| + \|(Q'_j - Q_j)x\| \\
&\leq \|x\| - |Q'_j x| + |\langle x, n_j\rangle|\left(\left|\frac{r_j}{\langle n_j, r_j\rangle} - \frac{r'_j}{\langle n_j, r'_j\rangle}\right| + \left\|\frac{r_j}{\langle n_j, r_j\rangle} - \frac{r'_j}{\langle n_j, r'_j\rangle}\right\|\right) \\
&\leq \|x\| - |Q'_j x| + \delta\left|\frac{\langle x, n_j\rangle}{\langle n_j, r'_j\rangle}\right| \\
&= \|x\| - (1 - \delta)|Q'_j x|.
\end{aligned}$$

Dividing this inequality by $1 - \delta$, we see that the assertion holds with respect to the norm $\|\cdot\|' = \frac{1}{1-\delta}\|\cdot\|$ ∎

We have the following easy consequence of Definition 2.1.

Lemma 2.3. *If $\mathcal{Q}$ is ℓ-paracontracting, then*

$$\|x\| \geq \|(I-\gamma Q_j)x\| + \gamma|Q_j x| \tag{2.3}$$

for every $j \in J$, $x \in X$ and $0 \leq \gamma \leq 1$.

Proof. Multiplying (2.2) by γ and using the triangle inequality, we get

$$\gamma\|x\| \geq \|\gamma(I-Q_j)x\| + \gamma|Q_j x| \geq \|(I-\gamma Q_j)x\| - (1-\gamma)\|x\| + \gamma|Q_j x|$$

and (2.3) follows easily ∎

Let us define nonlinear operators of oblique projection onto half-spaces H_j $(j \in J)$ as

$$\pi_j(x) = \begin{cases} x & \text{if } \langle x, n_j\rangle \leq \beta_j \\ (I-Q_j)x + \beta_j Q_j n_j & \text{if } \langle x, n_j\rangle > \beta_j. \end{cases} \tag{2.4}$$

We will need the following two properties of the operators π_j.

Proposition 2.4. *Let $\mathcal{Q}$ be ℓ-paracontracting. Then for each $j \in J$ the following inequalities hold:*

(i) $|\pi_j(x) - x| \leq \|x - z\| - \|\pi_j(x) - z\|$ *for all $x \in X$ and all $z \in H_j$.*

(ii) $\big\|\pi_j(x_1) - \pi_j(x_2)\big\| \leq \|x_1 - x_2\| - \big|(x_1 - \pi_j(x_1)) - (x_2 - \pi_j(x_2))\big|$ *for all $x_1, x_2 \in X$.*

Proof. **(i)** Let us denote $v = x - z$ and $w = \pi_j(x) - z$. Then

$$w = (I-\gamma Q_j)v \qquad \text{where } \gamma = \begin{cases} 0 & \text{if } \langle x, n_j\rangle \leq \beta_j \\ \frac{\langle x, n_j\rangle - \beta_j}{\langle x, n_j\rangle - \langle z, n_j\rangle} & \text{if } \langle x, n_j\rangle > \beta_j. \end{cases}$$

We have $0 \leq \gamma \leq 1$ because $\langle z, n_j\rangle \leq \beta_j$. Hence the assertion follows from Lemma 2.3.

(ii) If $\langle x_i, n_j\rangle \leq \beta_j$ for one or both of x_1 and x_2, it suffices to use assertion (i). Otherwise $\pi_j(x_1) - \pi_j(x_2) = (I - Q_j)(x_1 - x_2)$ and the statement follows directly from (2.2) ∎

We further define a mapping $\pi : X \to Z$ called *quasiprojection* such that for every $x \in X$ close to a point $z \in Z$ the difference $x - \pi(x)$ lies in the reflection cone of z (a precise formulation will be given in Proposition 2.6 below).

We take a specific sequence $\{j_k\}_{k\geq 0}$ of indices from J, namely

$$j_k = k\,[\mathrm{mod}\, p] + 1 \qquad (k \in \mathbb{N}_0) \tag{2.5}$$

and for a given $x \in X$ we define recursively the sequence

$$\left.\begin{aligned} y_0 &= x \\ y_{k+1} &= \pi_{j_k}(y_k) \quad (k \in \mathbb{N}_0) \end{aligned}\right\}. \tag{2.6}$$

By construction, $y_{k+1} \in H_{j_k}$ for every $k \in \mathbb{N}_0$. Moreover, from Proposition 2.4 we get

$$\sum_{k=0}^{\infty} |y_{k+1} - y_k| \le \|x - z\| \qquad (z \in Z).$$

Hence the sequence $\{y_k\}$ is convergent and we define the *quasiprojection operator* $\pi : X \to X$ by

$$\pi(x) = \lim_{k\to\infty} y_k \qquad \text{for } x \in X. \tag{2.7}$$

From the construction $\pi(x) \in Z$ follows.

We now list further properties of π.

Proposition 2.5. *Let $\mathcal{Q}$ be ℓ-paracontracting. Then for every $x \in X$ we have*

(i) $\|\pi(x) - z\| \le \|x - z\| - |x - \pi(x)|$ *for all* $z \in Z$

(ii) $\|x - \pi(x)\| \le 2 \min_{z\in Z} \|x - z\|$

(iii) $\|\pi(x_1) - \pi(x_2)\| \le \|x_1 - x_2\| - \big|(x_1 - \pi(x_1)) - (x_2 - \pi(x_2))\big|$ *for all* $x_1, x_2 \in X$.

Proof. **(i)** Let $\{y_k\}$ be the sequence (2.6). By Proposition 2.4/(i),

$$|y_{k+1} - y_k| \le \|y_k - z\| - \|y_{k+1} - z\| \tag{2.8}$$

for every k. Summing up over $k \in \mathbb{N}_0$ we obtain the assertion.

(ii) Let $z^* \in Z$ be such that $\|x - z^*\| = \min_{z\in Z} \|x - z\|$. From (2.8) we obtain $\|y_k - z^*\| \le \|y_0 - z^*\|$, hence

$$\|x - y_k\| \le \|x - z^*\| + \|y_k - z^*\| \le 2\|x - z^*\|$$

and assertion (ii) follows.

(iii) Let $\{y_k^{(i)}\}$ for $i = 1, 2$ be sequences (2.6) with initial conditions $y_0^{(i)} = x_i$. By Proposition 2.4/(ii), for all k we have

$$\|y_{k+1}^{(1)} - y_{k+1}^{(2)}\| \le \|y_k^{(1)} - y_k^{(2)}\| - |(y_k^{(1)} - y_{k+1}^{(1)}) - (y_k^{(2)} - y_{k+1}^{(2)})|$$

and, analogously to assertion (i), a summation argument completes the proof ∎

The following property of π plays a substantial role in our argument.

Proposition 2.6. *Let $\mathcal{Q}$ be ℓ-paracontracting. Let $z \in Z$ be given and let $\varepsilon > 0$ be such that the implication*

$$\|x - z\| < \varepsilon \quad \Longrightarrow \quad \langle x, n_j \rangle < \beta_j \ \ \forall\, j \in J \setminus \tilde{J}(z)$$

holds for every $x \in X$. Then

$$x - \pi(x) \in \mathcal{R}(z) \qquad \forall\, x \in X \text{ with } \|x - z\| < \varepsilon \tag{2.9}$$

where $\mathcal{R}(z)$ is the reflection cone defined by (1.4).

Proof. Let $\{y_k\}$ be sequence (2.6). By (2.9), $\|y_k - z\| \le \|x - z\| < \varepsilon$ for every k, hence

$$\langle y_k, n_j\rangle < \beta_j \qquad \forall\, j \in J \setminus \tilde{J}(z). \tag{2.10}$$

On the other hand,

$$x - \pi(x) = \sum_{k=0}^{\infty}(y_k - y_{k+1}) = \sum_{k\in K} Q_{j_k}(y_k - \beta_{j_k} n_{j_k}) = \sum_{k\in K} \frac{\langle y_k, n_{j_k}\rangle - \beta_{j_k}}{\langle r_{j_k}, n_{j_k}\rangle}\, r_{j_k} \tag{2.11}$$

where $K = \{k : \langle y_k, n_{j_k}\rangle > \beta_{j_k}\}$. Therefore, by (2.10), $j_k \in \tilde{J}(z)$ for every $k \in K$, and from (2.11) we conclude that there exist coefficients $\alpha_j \ge 0$ such that

$$x - \pi(x) = \sum_{j\in\tilde{J}(z)} \alpha_j r_j \tag{2.12}$$

which we wanted to prove ∎

3. Skorokhod problem in $W^{1,1}(0,T;X)$

We first solve the Skorokhod problem for absolutely continuous input functions u. Keeping the notation from Section 2, we construct a solution by time-discrete approximation.

With any given input sequence (finite or infinite) $\{u_0, u_1, \ldots\}$ and initial condition $x_0 \in Z$ we associate output sequences $\{x_0, x_1, \ldots\}$ and $\{\xi_0, \xi_1, \ldots\}$ by the recurrent formulas

$$\left.\begin{aligned} x_{i+1} &= \pi(x_i + u_{i+1} - u_i) \\ \xi_i &= u_i - x_i \end{aligned}\right\} \qquad (i \in \mathbb{N}_0) \tag{3.1}$$

where π is the quasiprojection operator (2.7). For every $i \ge 1$ we have in particular $x_i \in Z$ and

$$\xi_i - \xi_{i-1} = \big(x_{i-1} + u_i - u_{i-1}\big) - \pi\big(x_{i-1} + u_i - u_{i-1}\big), \tag{3.2}$$

hence Proposition 2.5 yields

$$|\xi_i - \xi_{i-1}| \le \|x_{i-1} + u_i - u_{i-1} - z\| - \|x_i - z\| \qquad (z \in Z). \tag{3.3}$$

Let two input sequences $\{u_i^{(j)}\}$ $(j = 1,2)$ be given. We denote by $\{x_i^{(j)}\}, \{\xi_i^{(j)}\}$ the corresponding output sequences and by $\{\bar{u}_i\}, \{\bar{x}_i\}, \{\bar{\xi}_i\}$ the differences $\bar{u}_i = u_i^{(2)} - u_i^{(1)}, \bar{x}_i = x_i^{(2)} - x_i^{(1)}, \bar{\xi}_i = \xi_i^{(2)} - \xi_i^{(1)}$. From Proposition 2.5/(iii) we then obtain

$$|\bar{\xi}_i - \bar{\xi}_{i-1}| \le \|\bar{x}_{i-1} + \bar{u}_i - \bar{u}_{i-1}\| - \|\bar{x}_i\|. \tag{3.4}$$

The existence result can be stated as follows.

Theorem 3.1. *Let $\mathcal{Q}$ be an ℓ-paracontracting system, and let $u \in W^{1,1}(0,T;X)$ and $x_0 \in Z$ be given. Then there exist functions $x, \xi \in W^{1,1}(0,T;X)$ satisfying the conditions of Definition* 1.1 *and $x(0) = x_0$.*

Proof. For a given $n \in \mathbb{N}$, we divide the interval $[0,T]$ into an equidistant partition

$$0 = t_0^{(n)} < t_1^{(n)} < \ldots < t_n^{(n)} = T, \quad t_i^{(n)} = \frac{i}{n}T \qquad (i = 0, \ldots, n)$$

and put, keeping n fixed for the moment,

$$u_i = u(t_i^{(n)}) \qquad (i = 0, \ldots, n). \tag{3.5}$$

Let an initial condition x_0 be given. We define x_i for $i = 1, \ldots, n$ by formula (3.1), and for $t \in [t_{i-1}^{(n)}, t_i^{(n)})$ we put

$$\left.\begin{aligned} u^{(n)}(t) &= u_{i-1} + \tfrac{n}{T}(t - t_{i-1}^{(n)})(u_i - u_{i-1}) \\ x^{(n)}(t) &= x_{i-1} + \tfrac{n}{T}(t - t_{i-1}^{(n)})(x_i - x_{i-1}) \end{aligned}\right\}. \tag{3.6}$$

As a consequence of (3.3) where we put $z = x_{i-1}$, for every $i = 1, \ldots, n$ the inequality

$$\|x_i - x_{i-1}\| \;\le\; \|u_i - u_{i-1}\| \tag{3.7}$$

holds. The sequence $\{x^{(n)}\}$ is thus equibounded in $C([0,T];X)$ and $\{\dot{x}^{(n)}\}$ is equi-integrable in $L^1(0,T;X)$, $x^{(n)}(t) \in Z$ for every $t \in [0,T]$. There exists therefore $x \in W^{1,1}(0,T;X)$ such that $x(t) \in Z$ for every $t \in [0,T]$ and $x(0) = x_0$, and a subsequence of $\{x^{(n)}\}$ (still indexed by (n)) such that $x^{(n)} \to x$ uniformly in $C([0,T];X)$ and $\dot{x}^{(n)} \to \dot{x}$ in $L^1(0,T;X)$ weakly as $n \to \infty$.

It remains to prove that the function $\xi(t) = u(t) - x(t)$ satisfies for a.e. $t \in (0,T)$ the condition

$$\dot{\xi}(t) \in \mathcal{R}(x(t)). \tag{3.8}$$

Let $t \in (0,T)$ be a Lebesgue point of both u and x, and let $\varepsilon > 0$ be chosen according to (1.3) in such a way that the implication

$$\|x(t) - \hat{x}\| < \varepsilon \quad \Longrightarrow \quad \langle \hat{x}, n_j \rangle < \beta_j \;\; \forall\, j \in J \setminus \tilde{J}(x(t)) \tag{3.9}$$

holds for every $\hat{x} \in X$. We fix $n_0 \in \mathbb{N}$ and $\delta > 0$ such that

$$\max_{\tau \in [0,T]} \|x^{(n)}(\tau) - x(\tau)\| < \tfrac{1}{3}\varepsilon \qquad \text{for } n \ge n_0 \tag{3.10}$$

$$\|x(t) - x(\tau)\| < \tfrac{1}{3}\varepsilon \qquad \text{for } \tau \in (t - \delta, t + \delta) \tag{3.11}$$

$$\|u(\sigma) - u(\tau)\| < \tfrac{1}{3}\varepsilon \qquad \text{for } \sigma, \tau \in (t - \delta, t + \delta). \tag{3.12}$$

Let now $n \ge n_0$ and $i \in \{1, \ldots, n\}$ be such that $t_{i-1}^{(n)}, t_i^{(n)} \in (t - \delta, t + \delta)$, and for $\tau \in (t - \delta, t + \delta)$ put $\xi^{(n)}(\tau) = u^{(n)}(\tau) - x^{(n)}(\tau)$. Then

$$\xi^{(n)}(t_i^{(n)}) - \xi^{(n)}(t_{i-1}^{(n)}) = \big(x_{i-1} + u_i - u_{i-1}\big) - \pi\big(x_{i-1} + u_i - u_{i-1}\big).$$

According to (3.10) - (3.12), the point $\hat{x} = x_{i-1} + u_i - u_{i-1}$ satisfies the inequality

$$\|\hat{x} - x(t)\| \ \leq \ \|x^{(n)}(t_{i-1}^{(n)}) - x(t)\| + \|u(t_i^{(n)}) - u(t_{i-1}^{(n)})\| < \varepsilon$$

and from (3.9) and Proposition 2.6

$$\xi^{(n)}(t_i^{(n)}) - \xi^{(n)}(t_{i-1}^{(n)}) \in \mathcal{R}(x(t))$$

follows. Since the functions $\xi^{(n)}$ are piecewise linear, for large n we have

$$\xi^{(n)}(t_2) - \xi^{(n)}(t_1) \in \mathcal{R}(x(t))$$

for every $t - \delta < t_1 \leq t \leq t_2 < t + \delta$, and passing to the limit we obtain (3.8). The proof is complete ∎

Remark 3.2. If $u_1, u_2 \in W^{1,1}(0,T;X)$ are two input functions, then from (3.4) it follows for the piecewise linear approximations that for $t \in (t_{i-1}^{(n)}, t_i^{(n)})$ we have

$$\big|\dot{\xi}_2^{(n)}(t) - \dot{\xi}_1^{(n)}(t)\big| + \frac{n}{T}\big(\bar{x}_i^{(n)} - \bar{x}_{i-1}^{(n)}\big) \leq \big\|\dot{u}_2^{(n)}(t) - \dot{u}_1^{(n)}(t)\big\| \tag{3.13}$$

where $\bar{x}_i^{(n)} = \big\|x_2^{(n)}(t_i^{(n)}) - x_1^{(n)}(t_i^{(n)})\big\|$.

Let $0 < a < b < T$ be arbitrarily chosen. For n sufficiently large we find indices $1 < j < k < n$ such that $t_{j-2}^{(n)} < a \leq t_{j-1}^{(n)}$ and $t_k^{(n)} \leq b < t_{k+1}^{(n)}$. Integrating (3.13) we obtain

$$\begin{aligned} \int_a^b \big|\dot{\xi}_2^{(n)}(t) - \dot{\xi}_1^{(n)}(t)\big| dt + \big(c_k \bar{x}_{k+1}^{(n)} + (1-c_k)\bar{x}_k^{(n)}\big) - \big(d_j \bar{x}_{j-2}^{(n)} + (1-d_j)\bar{x}_{j-1}^{(n)}\big) \\ \leq \int_a^b \big\|\dot{u}_2^{(n)}(t) - \dot{u}_1^{(n)}(t)\big\| dt \end{aligned} \tag{3.14}$$

where $c_k = \frac{(b - t_k^{(n)})n}{T}$ and $d_j = \frac{(t_{j-1}^{(n)} - a)n}{T}$. The sequences $\{u_1^{(n)}\}, \{u_2^{(n)}\}$ converge strongly in $W^{1,1}(0,T;X)$ and $\{\dot{\xi}_1^{(n)}\}, \{\dot{\xi}_2^{(n)}\}$ converge weakly in $L^1(0,T;X)$. Passing to the limit as $n \to \infty$ in (3.14) we thus obtain

$$\begin{aligned} \int_a^b \big|\dot{\xi}_2(t) - \dot{\xi}_1(t)\big|\, dt &\leq \liminf_{n\to\infty} \int_a^b \big|\dot{\xi}_2^{(n)}(t) - \dot{\xi}_1^{(n)}(t)\big|\, dt \\ &\leq \|x_2(a) - x_1(a)\| - \|x_2(b) - x_1(b)\| + \int_a^b \|\dot{u}_2(t) - \dot{u}_1(t)\|\, dt. \end{aligned} \tag{3.15}$$

Since a and b have been arbitrary, we can write the above inequality in differential form

$$|\dot{\xi}_2(t) - \dot{\xi}_1(t)| + \frac{d}{dt}\|x_2(t) - x_1(t)\| \leq \|\dot{u}_2(t) - \dot{u}_1(t)\| \quad \text{a.e.} \tag{3.16}$$

which is the same as in the normal reflection case (see [1]).

We cannot conclude for the moment that the solution to the Skorokhod problem is unique in $W^{1,1}(0,T;X)$ (see Example 3.3 below); we only made sure that solutions which can be constructed as discrete limits are unique. The uniqueness and Lipschitz continuity in $W^{1,1}(0,T;X)$ will be obtained under an additional assumption below in Theorem 5.8.

Example 3.3. Let $\{e_1, e_2\}$ be an orthonormal basis in $X = \mathbb{R}^2$. We consider the set $Z = \{x \in X : \langle x, e_1\rangle = 0\}$. This corresponds to the choice $n_1 = -n_2 = e_1$ and $\beta_1 = \beta_2 = 0$ in (1.1). We choose the reflection vectors $r_1 = \frac{e_2+e_1}{\sqrt{2}}$ and $r_2 = \frac{e_2-e_1}{\sqrt{2}}$. Then the system $\mathcal{Q}$ is ℓ-paracontracting with the norm

$$\|x\| = (1+\sqrt{2})|\langle x, e_1\rangle| + |\langle x, e_2\rangle|.$$

For the input function $u(t) \equiv 0$, all functions of the form $\xi(t) = \lambda(t)e_2$ and $x(t) = -\lambda(t)e_2$ with a non-decreasing function λ such that $\lambda(0) = 0$ are solutions of the Skorokhod problem (1.7) with initial condition $x(0) = 0$. However, the time discretization method converges to the trivial solution $\xi = x \equiv 0$.

4. Uniqueness and Lipschitz continuity in $C([0,T];X)$

Sufficient conditions for Lipschitz continuity of the Skorokhod map with respect to the norm $|\cdot|_{[0,T]}$ of uniform convergence were given in [3, 4] in terms of existence of a special bounded set $B \subset X$ (condition $(\mathcal{B})$ in Theorem 4.1 below, with the additional requirement $0 \in \mathrm{Int}(B)$). We now study this problem in more detail and summarize our results in Theorem 4.9 at the end of this section.

We first derive some geometrical properties of the associated projection system.

Theorem 4.1. *Let Hypothesis* 1.2 *hold, let* Q_j $(j \in J)$ *be the projections defined by* (2.1) *and let* $B \subset X$ *be a closed convex set with* $0 \in B$. *Then the following two conditions are equivalent:*

$(\mathcal{A})$ *For all* $x \in B$ *and all* $j \in J$, $w := (I - Q_j)x \pm Q_j n_j \in B$ *where* I *is the identity operator.*

$(\mathcal{B})$ *For all* $x \in B$, *all* $y \in \mathcal{N}_B(x)$ *and all* $j \in J$, $|\langle x, n_j\rangle| < 1$ *implies* $\langle y, r_j\rangle = 0$ *where* $\mathcal{N}_B(x)$ *denotes the outward normal cone to* B *at the point* x.

Notation 4.2. In the sequel, by a $\mathcal{Q}$*-invariant set* we understand any convex closed set B containing the origin and satisfying condition $(\mathcal{A})$.

Proof of Theorem 4.1.

$(\mathcal{A}) \Rightarrow (\mathcal{B})$: By definition, for every $x \in B$ and every $y \in \mathcal{N}_B(x)$, $\langle y, x - w\rangle \geq 0$ for all $w \in B$. Assuming condition $(\mathcal{A})$, we may put $w = (I - Q_j)x \pm Q_j n_j$ and obtain

$$0 \leq \langle y, Q_j(x \mp n_j)\rangle = \langle y, r_j\rangle \frac{\langle x, n_j\rangle \mp 1}{\langle n_j, r_j\rangle}.$$

If $|\langle x, n_j\rangle| < 1$ for some $j \in J$, the above inequality immediately yields $\langle y, r_j\rangle = 0$ and $(\mathcal{B})$ follows.

$(\mathcal{B}) \Rightarrow (\mathcal{A})$: Let $x \in B$ and $j \in J$ be given and let A be the rectangle $A = [0,1] \times [-1,1]$. For $(\alpha, \beta) \in A$ put

$$x_{\alpha,\beta} = \alpha(I - Q_j)x + \beta Q_j n_j.$$

Let $G = \{(\alpha, \beta) \in A : x_{\alpha,\beta} \in B\}$ be the set of 'good' indices. The set G is obviously non-empty (since $(0,0) \in G$) and closed (since B is closed). The proof will be complete if we check that $G = A$.

With the convex closed set B we can associate the *projection pair* (P_B, Q_B) defined as follows. For a given $x \in X$, define $w = Q_B x$ and $y = P_B x = x - Q_B x$ by the formula

$$w \in B, \qquad |y| = \min\big\{|x - z| : z \in B\big\}. \tag{4.1}$$

As a consequence of the definition, the point $y = P_B x$ belongs to the outward normal cone $\mathcal{N}_B(w)$. Let $(\bar\alpha, \bar\beta) \in G$ be given such that $0 \le \bar\alpha < 1$ and $-1 < \bar\beta < 1$. We choose arbitrary $(\alpha, \beta) \in A$ such that

$$|\beta| + |\alpha - \bar\alpha|\,|(I - Q_j)x| + \frac{|\beta - \bar\beta|}{\langle r_j, n_j\rangle} < 1 \tag{4.2}$$

and put $w_{\alpha,\beta} = Q_B x_{\alpha,\beta}$ and $y_{\alpha,\beta} = P_B x_{\alpha,\beta}$. Then

$$\begin{aligned} |\langle w_{\alpha,\beta}, n_j\rangle| &\le |\langle x_{\alpha,\beta}, n_j\rangle| + |\langle y_{\alpha,\beta}, n_j\rangle| \\ &\le |\beta| + |y_{\alpha,\beta}| \\ &\le |\beta| + |x_{\alpha,\beta} - x_{\bar\alpha,\bar\beta}| \\ &\le |\beta| + |\alpha - \bar\alpha||(I - Q_j)x| + \frac{|\beta - \bar\beta|}{\langle r_j, n_j\rangle}. \end{aligned}$$

From (4.2) $|\langle w_{\alpha,\beta}, n_j\rangle| < 1$ follows, and Condition $(\mathcal{B})$ yields

$$\langle y_{\alpha,\beta}, r_j\rangle = 0. \tag{4.3}$$

On the other hand, by definition of the outward normal cone, $\langle y_{\alpha,\beta}, w_{\alpha,\beta} - w\rangle \ge 0$ for all $w \in B$. We can choose in particular $w = \alpha x$, and from (4.3) we obtain

$$0 \le \langle y_{\alpha,\beta}, w_{\alpha,\beta} - \alpha x\rangle = \langle y_{\alpha,\beta}, w_{\alpha,\beta} - x_{\alpha,\beta}\rangle = -|y_{\alpha,\beta}|^2.$$

We conclude that $x_{\alpha,\beta} \in B$, hence the set G is relatively open in A. Therefore $G = A$, and Theorem 4.1 is proved ∎

We now give some useful consequences of Theorem 4.1.

Corollary 4.3. *Let Hypothesis* 1.2 *hold and let B be $\mathcal{Q}$-invariant. Then*

$$\langle z, n_j\rangle\langle y, r_j\rangle \ge 0$$

for all $z \in B, y \in \mathcal{N}_B(z)$ and $j \in J$.

Proof. Let $j \in J, z \in B$ and $y \in \mathcal{N}_B(z)$ be given. We have $\langle y, z - w\rangle \ge 0$ for every $w \in B$. Using Theorem 4.1 we obtain the assertion by putting $w = (I - Q_j)z$ ∎

The following result is immediate and we leave the proof to the reader.

Corollary 4.4. *Let B be a $\mathcal{Q}$-invariant set. Then the sets $\varrho B = \{\varrho x : x \in B\}$ are $\mathcal{Q}$-invariant for every $\varrho \in \mathbb{R}$ with $|\varrho| \geq 1$. Moreover, if B_1 and B_2 are $\mathcal{Q}$-invariant, then $B^* = \operatorname{conv}(B_1 \cup B_2)$ and $B_* = B_1 \cap B_2$ are $\mathcal{Q}$-invariant. In particular, to every $\mathcal{Q}$-invariant set B there exists a symmetric $\mathcal{Q}$-invariant set $B_{\text{sym}} = B \cap (-B)$.*

We now give an explicit description of the minimal $\mathcal{Q}$-invariant set. The construction is illustrated on Figure 2 in Section 8.

Corollary 4.5. *Let Λ denote the set of all finite sequences $\lambda = (j_0, \dots, j_{m-1})$ $(m \in \mathbb{N})$ such that $j_k \in J$ for $k = 0, \dots, m-1$. Let $s_\lambda = (x_0, \dots, x_m)$ be the sequence*

$$\left.\begin{aligned} x_0 &= 0 \\ x_{k+1} &= (I - Q_{j_k})x_k \pm Q_{j_k} n_{j_k} \quad (k = 0, \dots, m-1) \end{aligned}\right\} \tag{4.4}$$

and put $x^\omega_\lambda = x_m$. Further, let B^ω be the set $B^\omega = \overline{\operatorname{conv}}\{x^\omega_\lambda : \lambda \in \Lambda\}$. Then:

(i) *B^ω is a symmetric $\mathcal{Q}$-invariant set.*

(ii) *Every $\mathcal{Q}$-invariant set B contains B^ω.*

Proof. To prove assertion (i) it suffices to check that B^ω satisfies $(\mathcal{A})$. By definition of B^ω, $(I - Q_j)x^\omega_\lambda \pm Q_j n_j \in B^\omega$ for every $j \in J$ and every $\lambda \in \Lambda$. In a similar way, for every convex combination

$$x = \sum_{i=1}^{n} \alpha_i x^\omega_{\lambda_i} \in B^\omega \qquad \text{with } \sum_{i=1}^{n} \alpha_i = 1 \text{ and } \alpha_i \geq 0$$

we have

$$(I - Q_j)x \pm Q_j n_j = \sum_{i=1}^{n} \alpha_i \big((I - Q_j)x^\omega_{\lambda_i} \pm Q_j n_j\big) \in B^\omega$$

and the closedness of B^ω yields assertion (i).

Assertion (ii) is an immediate consequence of Theorem 4.1: if B is a $\mathcal{Q}$-invariant set, then by induction $x^\omega_\lambda \in B$ for every $\lambda \in \Lambda$. Since B is convex and closed, the assertion follows ∎

Remark 4.6. A sequence s_λ of form (4.4) is called an 1-*trajectory associated* to $\lambda \in \Lambda$. We will see below in Theorem 5.8 that the Lipschitz constant of the Skorokhod map is related to the diameter of the set B from Theorem 4.1. According to Corollary 4.5, B^ω is the minimal set with the desired property. An upper bound for all possible 1-trajectories will therefore yield an upper bound for the Lipschitz constant.

In particular, we have to ask whether B^ω is bounded. We first state a necessary condition in terms of the vectors n_j and r_j. For each $J' \subset J$ we define the spaces

$$\left.\begin{aligned} R_{J'} &= \operatorname{span}\{r_j : j \in J'\} \\ N_{J'} &= \operatorname{span}\{n_j : j \in J'\} \end{aligned}\right\}. \tag{4.5}$$

Lemma 4.7. *For every $J' \subset J$ we have*

$$R_{J'} \cap N_{J'}^{\perp} \subset B^{\omega} \subset R_J$$

where $R_{J'}$ and $N_{J'}$ are defined by (4.5) and $N_{J'}^{\perp}$ denotes the orthogonal complement to $N_{J'}$.

Proof. The fact that $B^{\omega} \subset R_J$ is obvious. Let $J' \subset J$ and $x \in R_{J'} \cap N_{J'}^{\perp}$ be arbitrarily chosen and assume $x \neq 0$. We find real numbers $a_i \ \ (i \in J')$ such that $x = \sum_{i \in J'} a_i r_i$ and put $b_i = \langle n_i, r_i \rangle a_i$ and $c = \sum_{i \in J'} |b_i|$. Then the point

$$\frac{1}{c}\, x = \sum_{i \in J'} \frac{1}{c}\, b_i Q_i n_i$$

belongs to B^{ω} by definition. Moreover, if $kx \in B^{\omega}$ for some $k \in \mathbb{R}$, then by Theorem 4.1 and Corollary 4.5

$$(I - Q_j)kx \pm \operatorname{sign}(b_j) Q_j n_j \in B^{\omega} \qquad \forall\, j \in J'.$$

By hypothesis, $Q_j x = 0$ for every $j \in J'$, and the convexity of B^{ω} yields

$$\begin{aligned} \sum_{j \in J'} \frac{1}{c}\, |b_j| \big((I - Q_j)\, kx \pm \operatorname{sign}(b_j) Q_j n_j \big) &= \sum_{j \in J'} \frac{1}{c}\, |b_j| \big(kx \pm \operatorname{sign}(b_j)\, Q_j n_j \big) \\ &= \Big(k \pm \frac{1}{c} \Big) x \in B^{\omega} \end{aligned}$$

hence B^{ω} contains the whole line $\operatorname{span}\{x\}$ ∎

Corollary 4.8. *Let B^{ω} be bounded. Then*

$$R_{J'} \cap N_{J'}^{\perp} = \{0\} \tag{4.6}$$

for all $J' \subset J$.

In the sequel, condition (4.6) will be referred to as the *transversality condition.* It is obviously satisfied in the case of normal reflection and, obviously as well, it is not robust with respect to small changes of reflection vectors. This is indeed a drawback, but we show below in Corollary 5.3 that in combination with ℓ-paracontractivity the transversality condition is equivalent to the condition

$$\dim N_{J'} = \dim R_{J'} \qquad \forall\, J' \subset J \tag{4.7}$$

which is simply a linear constraint to the robustness of the ℓ-paracontractivity.

For the reader's convenience, we give here the proof of the following Lipschitz estimate which basically follows the lines of [3: Theorem 2.2]. We however do not assume explicitly here that the set B has non-empty interior.

Theorem 4.9. *Let Hypothesis* 1.2 *hold and let there exist a symmetric $\mathcal{Q}$-invariant set B. Let $m_B : X \to \mathbb{R}^+ \cup \{+\infty\}$ be the Minkowski functional of the set B, that is,*

$$m_B(x) = \inf\left\{s > 0 : \frac{1}{s}\, x \in B\right\} \qquad (x \in X).$$

Let $u_1, u_2 \in C([0,T]; X)$ be two input functions for which there exist respective solutions $(\xi_1, x_1), (\xi_2, x_2)$ to the Skorokhod problem. For $t \in [0,T]$ put

$$\begin{aligned} \bar\xi(t) &= \xi_1(t) - \xi_2(t) \\ \bar x(t) &= x_1(t) - x_2(t) \\ \bar u(t) &= u_1(t) - u_2(t). \end{aligned}$$

Then for every $t \in [0,T]$ we have

$$m_B(\bar\xi(t)) \le \max\left\{m_B(\bar\xi(0)), |\bar u|_{[0,t]}\right\}. \tag{4.8}$$

Proof. Put $X_B = \{x \in X : m_B(x) < \infty\}$. Then X_B is a subspace of X, and since $\pm Q_j n_j \in B^\omega$ for every $j \in J$, we obtain from Corollary 4.5 that $R_J \subset X_B$.

The statement is empty if $\bar\xi(0) \notin X_B$. Let us assume therefore that $\bar\xi(0) \in X_B$, and for $t \in [0,T]$ put $\gamma(t) = |\bar u|_{[0,t]}$. For every $t \in [0,T]$ we have by definition

$$\bar\xi(t) - \bar\xi(0) \in \mathcal{R}_{[0,t]}(x_1) - \mathcal{R}_{[0,t]}(x_2) \subset X_B,$$

hence we can restrict our considerations to the reduced Minkowski functional

$$\tilde m_B = m_B|_{X_B}.$$

For $t \in [0,T]$ put $\psi(t) = \tilde m_B(\bar\xi(t))$ and assume the assertion of Theorem 4.9 does not hold. We can find $t_0 \in (0,T)$ such that $\gamma_0 := \psi(t_0) > \gamma(t_0)$ and $\psi(t) < \psi(t_0)$ for $t \in [0, t_0)$. Put $z = \bar\xi(t_0)/\gamma_0$. Then $z \in B$ and, for every $y \in \partial \tilde m_B(z)$ where $\partial \tilde m_B$ is the subdifferential of $\tilde m_B$, by definition

$$\langle y, z - \tilde z\rangle \ge \tilde m_B(z) - \tilde m_B(\tilde z) \qquad \forall\, \tilde z \in X_B. \tag{4.9}$$

In particular, $y \in \mathcal{N}_B(z)$, and putting $\tilde z = \bar\xi(t_0 - h)/\gamma_0$ in (4.9) for small positive h we obtain

$$\big\langle y, \bar\xi(t_0) - \bar\xi(t_0 - h)\big\rangle \ge \gamma_0\big(\psi(t_0) - \psi(t_0 - h)\big) > 0. \tag{4.10}$$

By (1.3) choose h sufficiently small such that

$$\left.\begin{aligned} \tilde J(x_1(t)) \subset \tilde J(x_1(t_0)) \\ \tilde J(x_2(t)) \subset \tilde J(x_2(t_0)) \end{aligned}\right\} \qquad (t \in [t_0 - h, t_0]). \tag{4.11}$$

By (1.7), we have

$$\begin{aligned} \xi_1(t_0) - \xi_1(t_0 - h) &\in \mathcal{C}(\tilde J(x_1(t_0))) \\ \xi_2(t_0) - \xi_2(t_0 - h) &\in \mathcal{C}(\tilde J(x_2(t_0))). \end{aligned}$$

We thus infer from (4.10) that there exists either some $j \in \tilde J(x_1(t_0))$ such that $\langle y, r_j\rangle > 0$, or some $i \in \tilde J(x_2(t_0))$ such that $\langle y, r_i\rangle < 0$. Both cases are symmetric, let us assume therefore $\langle y, r_j\rangle > 0$ for some $j \in \tilde J(x_1(t_0))$. Then Corollary 4.3 yields $\langle z, n_j\rangle \ge 0$. On the other hand, by definition of $\tilde J(x_1(t_0))$ we have $\langle \bar x(t_0), n_j\rangle \ge 0$. We conclude with

$$0 \le \langle z, n_j\rangle = \frac{1}{\gamma_0}\langle \bar u(t_0), n_j\rangle - \frac{1}{\gamma_0}\langle \bar x(t_0), n_j\rangle \le \frac{1}{\gamma_0}\langle \bar u(t_0), n_j\rangle \le \frac{\gamma(t_0)}{\gamma_0} < 1.$$

This violates property $(\mathcal{B})$ from Theorem 4.1, which is indeed a contradiction. Theorem 4.9 is proved ∎

For practical purposes, formula (4.8) is more convenient to work with if the set B has non-empty interior. The following straightforward argument shows that this condition represents no restriction.

Proposition 4.10. *Let B be a $\mathcal{Q}$-invariant set and let $B_1(0)$ denote the unit ball in X. Then $B' = 2B + B_1(0)$ is also a $\mathcal{Q}$-invariant set.*

Proof. Let $x' \in B'$ and $y \in \mathcal{N}_{B'}(x')$ be given such that $|\langle x', n_j\rangle| < 1$ for some $j \in J$. There exist $x \in B$ and $h \in B_1(0)$ such that $x' = 2x + h$. By definition of the normal cone, $\langle y, x' - (2\,b + h)\rangle \geq 0$ for every $b \in B$, hence $y \in \mathcal{N}_B(x)$. On the other hand, $|\langle x, n_j\rangle| = \frac{1}{2}\,|\langle x' - h, n_j\rangle| < 1$. Since B is $\mathcal{Q}$-invariant we obtain $\langle y, r_j\rangle = 0$ and the proof is complete ∎

Corollary 4.11. *If there exists a bounded $\mathcal{Q}$-invariant set, then there exists a bounded $\mathcal{Q}$-invariant set with non-empty interior.*

Theorem 4.9 implies uniqueness of solutions and a Lipschitz continuous dependence with respect to the sup-norm provided the set B is bounded. Existence (in $W^{1,1}(0,T;X)$) and uniqueness (in $C([0,T];X)$) thus have been proved under different hypotheses. In the next Section 5 we show (Theorem 5.5) that the ℓ-paracontractivity together with transversality of the system $\mathcal{Q}$ ensures the existence of a bounded $\mathcal{Q}$-invariant set. This will enable us to characterize a class of Skorokhod problems for which existence, uniqueness and Lipschitz continuous dependence hold.

5. Paracontractivity and invariant sets

Keeping the notation from Corollary 4.5, we assume that $\mathcal{Q}$ is an ℓ-paracontracting system, and that $x \in X$ and $\lambda \in \Lambda$, $\lambda = (j_0, \ldots, j_{m-1})$ are given. Let us consider the sequence

$$\left.\begin{aligned} x_0 &= x \\ x_{k+1} &= (I - Q_{j_k})x_k \quad (k = 0, 1, \ldots, m-1) \end{aligned}\right\}. \tag{5.1}$$

We define the mapping $\omega_\lambda : X \to X$ by

$$\omega_\lambda(x) = x_m. \tag{5.2}$$

By definition of ℓ-paracontractivity,

$$|x_{k+1} - x_k| \leq \|x_k\| - \|x_{k+1}\| \qquad (k = 0, 1, \ldots, m-1), \tag{5.3}$$

hence

$$|x - \omega_\lambda(x)| \leq \|x\| - \|\omega_\lambda(x)\|. \tag{5.4}$$

We now introduce some further notation. For $J' \subset J$ put

$$\Lambda_{J'} = \left\{ \lambda \in \Lambda : \lambda = (j_0, \ldots, j_{m-1}) \text{ with } \bigcup_{k=0}^{m-1} \{j_k\} = J' \right\}. \tag{5.5}$$

We start with two auxiliary results.

Lemma 5.1. *Let $\mathcal{Q}$ be an ℓ-paracontracting system, and let $J' \subset J$ and $\lambda \in \Lambda_{J'}$ be given. Then $\omega_\lambda(x) = x$ if and only if $x \in N_{J'}^\perp$.*

Proof. We have indeed $\omega_\lambda(x) = x$ for $x \in N_{J'}^\perp$. Conversely, let $\omega_\lambda(x) = x$ for some $x \in X$ and $\lambda \in \Lambda_{J'}$, $\lambda = (j_0, \dots, j_{m-1})$. From (5.3) we infer $x = x_1 = \dots = x_{m-1}$ and $Q_j x = 0$ for all $j \in J'$, hence $x \in N_{J'}^\perp$ ∎

Lemma 5.2. *Let $\mathcal{Q}$ be an ℓ-paracontracting system. Then $R_{J'}^\perp \cap N_{J'} = \{0\}$ for every $J' \subset J$.*

Proof. For arbitrary $z \in R_{J'}^\perp \cap N_{J'}$ and $\lambda \in \Lambda_{J'}$ we define recursively the sequence

$$\left.\begin{aligned} z_0 &= z \\ z_n &= \omega_\lambda(z_{n-1}) \quad (n \in \mathbb{N}) \end{aligned}\right\}.$$

By (5.4), $|z_n - z_{n+1}| \le \|z_n\| - \|z_{n+1}\|$, hence $\{z_n\}$ is a convergent sequence, $z_n \to z^*$. On the other hand, for every $j \in J'$ and $x \in X$ we have $\langle Q_j x, z\rangle = 0$, hence $\langle z_n, z\rangle = |z|^2$ for every $n \in \mathbb{N}$. Passing to the limit as $n \to \infty$ we obtain $z^* = \omega_\lambda(z^*)$ and $\langle z^*, z\rangle = |z|^2$, hence by Lemma 5.1 $z^* \in N_{J'}^\perp$ and $z = 0$ ∎

As an immediate consequence of Lemma 5.2 we get

Corollary 5.3. *Let $\mathcal{Q}$ be an ℓ-paracontracting system. Then the following conditions are equivalent:*

- **(i)** *Transversality condition* (4.6) *holds.*
- **(ii)** *Condition* (4.7) *holds.*
- **(iii)** $R_{J'}^\perp \oplus N_{J'} = R_{J'} \oplus N_{J'}^\perp = X$ *for every $J' \subset J$.*

The next statement is the key point of this section and illustrates the meaning of paracontractivity. We see that for every $J' \subset J$ and $\lambda \in \Lambda_{J'}$ the mapping ω_λ leaves invariant both complementary subspaces $R_{J'}$ and $N_{J'}^\perp$, and that it reduces to the identity on $N_{J'}^\perp$ and to a contraction on $R_{J'}$ with respect to the norm $\|\cdot\|$.

Proposition 5.4. *Let $\mathcal{Q}$ be an ℓ-paracontracting system and let transversality condition* (4.6) *hold. Then for every $J' \subset J$ there exists $\delta_{J'} \in [0,1)$ such that*

$$\omega_\lambda(x) \in R_{J'}, \qquad \|\omega_\lambda(x)\| \le \delta_{J'}\|x\|.$$

for all $x \in R_{J'}$ and all $\lambda \in \Lambda_{J'}$.

Proof. Let $J' \subset J$ be given. The fact that $\omega_\lambda(x) \in R_{J'}$ for $x \in R_{J'}$ and $\lambda \in \Lambda_{J'}$ is obvious. Put

$$\delta_{J'} = \sup\Big\{\|\omega_\lambda(x)\| : \lambda \in \Lambda_{J'} \text{ and } x \in R_{J'} \text{ with } \|x\| = 1\Big\}.$$

By (5.4) we have $\delta_{J'} \le 1$. Assume $\delta_{J'} = 1$. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $R_{J'}$ with $\|x_n\| = 1$ and a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ in $\Lambda_{J'}$ such that

$$\|\omega_{\lambda_n}(x_n)\| \ge 1 - \frac{1}{n} \qquad (n \in \mathbb{N}). \tag{5.6}$$

We may assume $x_n \to x$ with $\|x\| = 1$. Let us fix an arbitrary $j \in J'$. For each $n \in \mathbb{N}$, the sequence $\lambda_n = (j_0^{(n)}, \dots, j_{m_n-1}^{(n)})$ contains j, say $j = j_{k_n}^{(n)}$ for some $k_n \le m_n - 1$. Put $\lambda_n' = (j_0^{(n)}, \dots, j_{k_n-1}^{(n)})$ and $z_n = \omega_{\lambda_n'}(x_n)$. Then, by (5.3),

$$\|\omega_{\lambda_n}(x_n)\| \le \|(I - Q_j) z_n\| \tag{5.7}$$

$$\|z_n\| \le \|x_n\| = 1 \tag{5.8}$$

for every $n \in \mathbb{N}$, hence

$$\|z_n\| - |Q_j\, z_n| \ge \|(I - Q_j) z_n\| \ge 1 - \frac{1}{n} \qquad (n \in \mathbb{N}). \tag{5.9}$$

Therefore $\lim_{n\to\infty} \|z_n\| = 1$ and $\lim_{n\to\infty} |Q_j\, z_n| = 0$, and (5.3) entails

$$|x_n - z_n| \le \|x_n\| - \|z_n\| \qquad (n \in \mathbb{N}). \tag{5.10}$$

We conclude that $\lim_{n\to\infty} z_n = x$ and $Q_j x = 0$ for all $j \in J'$, which contradicts transversality condition (4.6) ∎

The main result of this section can be stated as follows.

Theorem 5.5. *Let $\mathcal{Q}$ be an ℓ-paracontracting system and let transversality condition* (4.6) *hold. Then the minimal $\mathcal{Q}$-invariant set B^ω from Corollary* 4.5 *is contained in the ball centered at* 0 *of radius K with respect to the norm $\|\cdot\|$, where*

$$K \le \frac{C}{\delta}\Big(\Big(\frac{1}{1-\delta}\Big)^p - 1\Big) \tag{5.11}$$

with $C = \max\{\frac{\|r_j\|}{\langle n_j, r_j\rangle} : j \in J\}$ and any $\delta \in (0,1)$ with $\delta \ge \max\{\delta_{J'} : J' \subset J\}$.

We postpone the proof of Theorem 5.5 and prove first an auxiliary statement.

Proposition 5.6. *Let the assumptions of Theorem* 5.5 *hold, and let $J' \subset J$ and $\lambda \in \Lambda_{J'}$ be given,* $\operatorname{card} J' = q \in J$ *and $\lambda = (j_0, \dots, j_{m-1})$. Let $\hat{n}_{j_k} = \pm n_{j_k}$ be arbitrarily chosen for each $k = 0, \dots, m-1$ and let us define the sequence*

$$\left.\begin{aligned} z_k &= (I - Q_{j_{m-1}}) \cdots (I - Q_{j_k})\, Q_{j_{k-1}} \hat{n}_{j_{k-1}} \quad (k = 1, \dots, m-1) \\ z_m &= Q_{j_{m-1}} \hat{n}_{j_{m-1}} \end{aligned}\right\}. \tag{5.12}$$

Then

$$\Big\|\sum_{k=1}^{m} z_k\Big\| \le \frac{C}{\delta}\Big(\Big(\frac{1}{1-\delta}\Big)^q - 1\Big). \tag{5.13}$$

The proof of Proposition 5.6 is based on the following induction step.

Lemma 5.7. *Let the assertion of Proposition* 5.6 *hold for some* $q < p$, *and let* $J' \subset J$ *and* $\lambda \in \Lambda_{J'}$ *be given,* $\operatorname{card} J' = q+1$ *and* $\lambda = (j_0, \ldots, j_{m-1})$, *such that* $\lambda' = (j_1, \ldots, j_{m-1}) \notin \Lambda_{J'}$. *Let* z_k *be defined by* (5.12). *Then*

$$\Big\| \sum_{k=1}^{m} z_k \Big\| \le C\Big(1 + \frac{1}{\delta}\Big(\Big(\frac{1}{1-\delta}\Big)^q - 1\Big)\Big). \tag{5.14}$$

Proof. By induction hypothesis, we have

$$\Big\| \sum_{k=2}^{m} z_k \Big\| \le \frac{C}{\delta}\Big(\Big(\frac{1}{1-\delta}\Big)^q - 1\Big) \tag{5.15}$$

while

$$\|z_1\| \le \|Q_{j_0}\hat{n}_{j_0}\| \le C, \tag{5.16}$$

and formula (5.14) follows easily ∎

Proof of Proposition 5.6. For $q = 1$ we have $z_k = 0$ for $k < m$, hence

$$\Big\| \sum_{k=1}^{m} z_k \Big\| = \|z_m\| \le C \tag{5.17}$$

and (5.13) holds. Assume now that the assertion holds for some $q \ge 1$, $q < p$ and fix some $J' \subset J$, $\operatorname{card} J' = q+1$, and $\lambda \in \Lambda_{J'}$, $\lambda = (j_0, \ldots, j_{m-1})$. We define the numbers $d(0), d(1), \ldots, d(\ell)$ recurrently according to the following recipe:

$$\left.\begin{aligned} d(0) &= m \\ d(1) &= \max\big\{k < m : (j_k, \ldots, j_{m-1}) \in \Lambda_{J'}\big\} \\ &\ \vdots \\ d(n+1) &= \max\big\{k < d(n) : (j_k, \ldots, j_{d(n)-1}) \in \Lambda_{J'}\big\} \end{aligned}\right\}$$

until $(j_0, \ldots, j_{d(\ell)-1}) \notin \Lambda_{J'}$. For $n = 0, \ldots, \ell$ and $k = 1, \ldots, d(n)-1$ put

$$\left.\begin{aligned} \zeta_k^n &= (I - Q_{j_{d(n)-1}}) \cdots (I - Q_{j_k}) Q_{j_{k-1}} \hat{n}_{j_{k-1}} \\ \zeta_{d(n)}^n &= Q_{j_{d(n)-1}} \hat{n}_{j_{d(n)-1}} \end{aligned}\right\}. \tag{5.18}$$

Then for $d(n+1)+1 \le k \le d(n)$ we have

$$z_k = (I - Q_{j_{m-1}}) \cdots (I - Q_{j_{d(n)}}) \zeta_k^n. \tag{5.19}$$

The inequality

$$\Big\| \sum_{k=d(n+1)+1}^{d(n)} \zeta_k^n \Big\| \le C\Big(1 + \frac{1}{\delta}\Big(\Big(\frac{1}{1-\delta}\Big)^q - 1\Big)\Big), \tag{5.20}$$

where we put $d(\ell+1)=0$, is valid for $n=0,\dots,\ell-1$ according to Lemma 5.7 and for $n=\ell$ according to the induction hypothesis. Proposition 5.4 now yields for $n=0,\dots,\ell$

$$\begin{aligned}\Big\|\sum_{k=d(n+1)+1}^{d(n)} z_k\Big\| &= \Big\|(I-Q_{j_{m-1}})\cdots(I-Q_{j_{d(n)}})\sum_{k=d(n+1)+1}^{d(n)} \zeta_k^n\Big\| \\ &\le C\delta^n\Big(1+\frac{1}{\delta}\Big(\Big(\frac{1}{1-\delta}\Big)^q-1\Big)\Big).\end{aligned} \tag{5.21}$$

Summing up the above inequalities over $n=0,\dots,\ell$ we obtain

$$\Big\|\sum_{k=1}^{m} z_k\Big\| \le \frac{C}{1-\delta}\Big(1+\frac{1}{\delta}\Big(\Big(\frac{1}{1-\delta}\Big)^q-1\Big)\Big) = \frac{C}{\delta}\Big(\Big(\frac{1}{1-\delta}\Big)^{q+1}-1\Big) \tag{5.22}$$

and the induction step is complete. Proposition 5.6 is proved ∎

We are now ready to conclude this section by proving Theorem 5.5.

Proof of Theorem 5.5. Let $\lambda\in\Lambda$, $\lambda=(j_0,\dots,j_{m-1})$ be arbitrary, and let s_λ be the corresponding 1-trajectory defined by (4.4). Then

$$\left.\begin{aligned} x_1 &= Q_{j_0}\hat n_{j_0} \\ x_k &= \sum_{i=1}^{k-1}(I-Q_{j_{k-1}})\cdots(I-Q_{j_i})\,Q_{j_{i-1}}\hat n_{j_{i-1}} \quad (k=2,\dots,m-1)\end{aligned}\right\} \tag{5.23}$$

for some $\hat n_{j_i}=\pm n_{j_i}$ $(i=0,\dots,m-1)$. Using Proposition 5.6 we obtain

$$\sup_{\lambda\in\Lambda}\|x_\lambda^\omega\| \le \frac{C}{\delta}\Big(\Big(\frac{1}{1-\delta}\Big)^p-1\Big), \tag{5.24}$$

hence inequality (5.11) holds ∎

Theorem 5.8. *Let the associated projection system $\mathcal{Q}$ be ℓ-paracontracting and transversal. Then the Skorokhod map $\mathcal{S}$ is well defined and of Lipschitz type both as a map from $Z\times W^{1,1}(0,T;X)$ to $W^{1,1}(0,T;X)$ and from $Z\times C([0,T];X)$ to $C([0,T];X)$.*

Proof. Theorem 3.1 guarantees that the Skorokhod problem admits a solution for every $u\in W^{1,1}(0,T;X)$ and every initial condition. By Theorem 5.5, the set B^ω is bounded. There exists therefore $M>0$ such that B^ω is contained in a ball centered at 0 with radius M. Using the fact that the space $W^{1,1}(0,T;X)$ is dense in $C([0,T];X)$, we obtain the existence and Lipschitz continuity in $C([0,T];X)$ immediately from Theorem 4.9, from the upper semicontinuity property (1.6) and from the inequality $m_{B^\omega}(x)\ge\frac{|x|}{M}$ for every $x\in X$. The Lipschitz continuity in $Z\times W^{1,1}(0,T;X)\to W^{1,1}(0,T;X)$ follows immediately from Remark 3.2 ∎

6. A bounded variation result

Similarly as in the normal reflection case, one might expect that, if the set B in Theorem 4.9 is bounded and Z has non-empty interior, the extension of the Skorokhod map onto $C([0,T];X)$ has a regularizing effect, namely that for inputs $u \in C([0,T];X)$ the outputs ξ belong to $C([0,T];X) \cap BV(0,T;X)$.

Assume that there exists $z_0 \in Z$ and $\varrho > 0$ such that the whole ball $B_\varrho(z_0)$ is contained in Z. We prove the following result (which subsequently immediately implies the desired BV-estimate).

Proposition 6.1. *Let the associated projection system $\mathcal{Q}$ be ℓ-paracontracting and transversal. Let $u \in C([0,T];X)$ be given and let $\xi, x \in C([0,T];X)$ be the corresponding solution to the Skorokhod problem for a given initial condition $x_0 \in Z$. Let $\delta > 0$ be such that the implication*

$$|t_2 - t_1| < \delta \quad \Longrightarrow \quad |u(t_2) - u(t_1)| < \frac{\varrho}{2}$$

holds for every $t_1, t_2 \in [0,T]$. Then for every $0 \le s < t \le T$ such that $|t-s| < \delta$ we have

$$\mathrm{Var}_{[s,t]}\xi \le \|x(\cdot) - z_0\|_{[0,t]}$$

where $\|\cdot\|_{[0,t]}$ denotes the sup-norm with respect to the norm $\|\cdot\|$ over the interval $[0,t]$.

Proof. We approximate the function u uniformly by functions from $W^{1,1}(0,T;X)$, and for each of these approximating functions we apply the discretization procedure from Section 3. By diagonalization we obtain, according to Theorem 5.8 and to the construction in the proof of Theorem 3.1, discrete sequences $\{u_i\}, \{x_i\}, \{\xi_i\}$ satisfying (3.1) such such that the piecewise linear interpolates $\{u^{(n)}\}, \{x^{(n)}\}, \{\xi^{(n)}\}$ given by (3.6) converge uniformly to u, x, ξ, respectively.

Let $\varepsilon > 0$ be arbitrarily given. We find n_0 sufficiently large such that for $n > n_0$ we have $|u^{(n)} - u|_{[0,T]} < \frac{\varrho}{4}$ and $\|x^{(n)} - x\|_{[0,T]} < \varepsilon$, and there exist $t_{j-1}^{(n)} \le s < t \le t_k^{(n)}$ such that $t_k^{(n)} - t_{j-1}^{(n)} < \delta$. For $i = j, \ldots, k$ we have by hypothesis

$$|u_i - u_{j-1}| \le 2\,|u^{(n)} - u|_{[0,T]} + \frac{\varrho}{2} \le \varrho,$$

hence $z_i = z_0 + u_i - u_{j-1} \in Z$ for every $i = j, \ldots, k$. Inequality (3.3) for $z = z_i$ yields

$$|\xi_i - \xi_{i-1}| \le \|x_{i-1} - u_{i-1} + u_{j-1} - z_0\| - \|x_i - u_i + u_{j-1} - z_0\|$$

for all $i = j, \ldots, k$. Summing up the above inequalities we obtain

$$\mathrm{Var}_{[s,t]}\xi^{(n)} \le \sum_{i=j}^{k} |\xi_i - \xi_{i-1}| \le \|x_j - z_0\| \le \varepsilon + \|x(\cdot) - z_0\|_{[0,t]}.$$

Passing to the limit as $n \to \infty$ and using the fact that ε has been chosen arbitrarily, we complete the proof ∎

7. An upper bound for the invariant sets

According to Lemma 2.2, the ℓ-paracontracting property is robust with respect to small changes of vectors r_i if the vectors n_i do not change. This allows us to extend the Lipschitz continuity results from the normal reflection case to the case of Skorokhod problems with reflection vectors r_i that are close to the normals n_i under the transversality constraint. This argument, however, does not provide an efficient estimate of the corresponding Lipschitz constant. In this section, we show an algorithm which gives at least an upper bound.

Put $N = \dim N_J$ and for $k = 1, \dots, N$ denote

$$\mathcal{L}_k = \Big\{ J' \subset J : \operatorname{card} J' = k \text{ and } \{n_i\}_{i\in J'} \text{ linearly independent} \Big\} \tag{7.1}$$

$$\varepsilon_k = \min \Big\{ \Big| \sum_{i\in J'} \alpha_i n_i \Big| : \sum_{i\in J'} \alpha_i^2 = 1 \quad (J' \in \mathcal{L}_k) \Big\}. \tag{7.2}$$

Note that $0 < \varepsilon_N \le \varepsilon_{N-1} \le \dots \le \varepsilon_1 \le 1$.

We make the following

Hypothesis 7.1. For every $j \in J$, $|n_j - r_j| \le \frac{\varepsilon_N}{2\sqrt{N}}$ and (4.7) holds.

The above hypothesis implies in particular $|n_j - r_j|^2 \le \frac{1}{4}$ for every j, hence $\langle n_j, r_j\rangle \ge \frac{7}{8} > 0$.

Notation 7.2. For an arbitrary subspace $X' \subset X$ we denote by $P_{X'}$ the orthogonal projection onto X'. In particular, $P_X = I$ is the identity operator. We further denote by $\mathcal{D}_k$ $(0 \le k \le N)$ the system of all k-dimensional subspaces of R_J generated by the vectors $r_1, \dots, r_p$, that is

$$\begin{aligned}
\mathcal{D}_0 &= \big\{\{0\}\big\} \\
\mathcal{D}_k &= \Big\{ X' \subset R_J : X' = \operatorname{span}\{r_{i_1}, \dots, r_{i_m}\} \ \ (i_j \in J \text{ for } j = 1, \dots, m), \dim X' = k \Big\} \\
&\qquad (k = 1, \dots, N-1) \\
\mathcal{D}_N &= \{R_J\}.
\end{aligned}$$

We need in the sequel the following elementary properties of projections.

Lemma 7.3. *Let $X' \subset X'' \subset X$ be subspaces of X. Then:*

(i) $P_{X''} P_{X'} = P_{X'} P_{X''} = P_{X'}$.

(ii) $|\langle z, v\rangle| \le |P_{X'} z| \le |z|$ *for all $z \in X$ and all $v \in X'$ with $|v| \le 1$.*

According to Hypothesis 7.1, every system $\{r_i\}_{i\in J'}$ for $J' \in \mathcal{L}_k$ is linearly independent and we may put

$$\delta_k = \min \Big\{ \Big| \sum_{i\in J'} \alpha_i r_i \Big| : \sum_{i\in J'} \alpha_i^2 = 1 \quad (J' \in \mathcal{L}_k) \Big\} \tag{7.3}$$

and have again $0 < \delta_N \le \delta_{N-1} \le \ldots \le \delta_1 \le 1$. Moreover, from Hypothesis 7.1 it follows that

$$\frac{1}{2}\varepsilon_k \le \delta_k \le \frac{3}{2}\varepsilon_k \qquad (k = 1, \ldots, N). \tag{7.4}$$

Indeed, we have for $J' \in \mathcal{L}_k$

$$\sum_{i \in J'} |\alpha_i|\,|r_i - n_i| \le \frac{\varepsilon_N}{2\sqrt{N}} \sum_{i \in J'} |\alpha_i| \le \frac{\varepsilon_N}{2}\sqrt{\frac{k}{N}}$$

and inequalities (7.4) follow.

We first prove an auxiliary estimate.

Lemma 7.4. *Let $k \in \{0, 1, \ldots, N-1\}$, $X' \in \mathcal{D}_k$, $v \in X'$ and $r_j \notin X'$ be given such that $|v| = 1$. Put*

$$\left.\begin{aligned} &\eta_0 = 0 \\ &\eta_k = 1 - \frac{1}{2}\Big(1 + \frac{1}{k}\Big)\delta_{k+1}^2 \quad (k = 1, \ldots, N-1) \end{aligned}\right\}. \tag{7.5}$$

Then

$$|\langle r_j, v\rangle| \le \eta_k. \tag{7.6}$$

Proof. The case $k = 0$ is trivial. For $k \ge 1$ we find $J' \in \mathcal{L}_k$ and real numbers α_i $(i \in J')$ such that $\operatorname{span}\{r_i : i \in J'\} = X'$ and $v = \sum_{i \in J'} \alpha_i r_i$. We have indeed $J'' := J' \cup \{j\} \in \mathcal{L}_{k+1}$ (note that Hypothesis 7.1 has been used here), and

$$\left.\begin{aligned} 1 + |v|^2 - 2\langle r_j, v\rangle = |r_j - v|^2 \ge \delta_{k+1}^2\Big(1 + \sum_{i \in J'} \alpha_i^2\Big) \ge \Big(1 + \frac{1}{k}\Big)\delta_{k+1}^2 \\ 1 + |v|^2 + 2\langle r_j, v\rangle = |r_j + v|^2 \ge \delta_{k+1}^2\Big(1 + \sum_{i \in J'} \alpha_i^2\Big) \ge \Big(1 + \frac{1}{k}\Big)\delta_{k+1}^2 \end{aligned}\right\} \tag{7.7}$$

and the assertion follows ∎

Let $\eta_0, \ldots, \eta_{N-1}$ be defined as in Lemma 7.4. For arbitrary $s \ge 0$ and $k = 0, \ldots, N$ we define the sequence $M_k(s)$ by the recurrent formula

$$\left.\begin{aligned} &M_0(s) = 0 \\ &M_k^2(s) = M_{k-1}^2(s) + \frac{1}{1 - \eta_{k-1}^2}\big(1 + s + \eta_{k-1}M_{k-1}(s)\big)^2 \end{aligned}\right\}. \tag{7.8}$$

Note that for all $s > 0$ and $k = 1, \ldots, N$

$$\Big(\frac{M_k(s)}{s}\Big)^2 = \Big(\frac{M_{k-1}(s)}{s}\Big)^2 + \frac{1}{1 - \eta_{k-1}^2}\Big(\frac{1}{s} + 1 + \eta_{k-1}\frac{M_{k-1}(s)}{s}\Big)^2, \tag{7.9}$$

hence each of the functions $s \mapsto \frac{M_k(s)}{s}$ $(k = 1, \ldots, N)$ is decreasing in $(0, \infty)$ and

$$\lim_{s \to \infty} \frac{M_k(s)}{s} = M_k(0) \qquad (k = 1, \ldots, N). \tag{7.10}$$

For every $s \geq 0$ define a functional $V_s : X \to \mathbb{R}^+$ by the formula

$$V_s(z) = \max\Big\{M_k^2(s) + |(P_{R_J} - P_{X'})z|^2 : X' \in \mathcal{D}_k \quad (k = 0, \ldots, N-1)\Big\}. \tag{7.11}$$

Obviously, V_s is convex and the set

$$B_s = \left\{z \in R_J : V_s(z) \leq M_N^2(s)\right\} \tag{7.12}$$

is convex and closed for every $s \geq 0$.

Our main goal is to prove the following

Theorem 7.5. *Let Hypothesis* 7.1 *hold. Assume moreover that*

$$\sigma := \max_{j \in J} |n_j - r_j| < \frac{1}{M_N(0)} \tag{7.13}$$

and let $s \geq 0$ satisfy the equation

$$\frac{s}{M_N(s)} = \sigma. \tag{7.14}$$

Then the set $B = B_s$ defined by (7.12) *satisfies condition* ($\mathcal{B}$).

Indeed, from (7.10) it follows that condition (7.14) is meaningful and the value of s is uniquely determined. Moreover, for every $z \in X$ we have

$$V_s(z) \geq M_0^2(s) + |(P_{R_J} - P_{\{0\}})z|^2 = |P_{R_J} z|^2, \tag{7.15}$$

hence by (7.12)

$$|z| \leq M_N(s) \qquad (z \in B). \tag{7.16}$$

In particular, the set B in Theorem 7.5 is contained in the ball centered at the origin with radius $M_N(s)$.

The proof of Theorem 7.5 is based on the following lemma.

Lemma 7.6. *Let the hypotheses of Theorem* 7.5 *hold. Further, assume that for some $z \in B$ and $X' \in \mathcal{D}_k$ $(k \in \{0, \ldots, N-1\})$ we have $M_k^2(s) + |(I - P_{X'})z|^2 = M_N^2(s)$, and that there exists $i \in J$ such that $|\langle z, n_i\rangle| < 1$. Then $r_i \in X'$.*

Proof. Assume $r_i \notin X'$ and put $X'' = X' \oplus \operatorname{span}\{r_i\}$. We find $v \in X'$ with $|v| = 1$ and real numbers a, b such that

$$P_{X''} z = a r_i + b v. \tag{7.17}$$

Putting $\eta = \langle r_i, v\rangle \in [-\eta_k, \eta_k]$ we have

$$|P_{X''} z|^2 = a^2 + b^2 + 2ab\eta \tag{7.18}$$

$$|P_{X'} z| \geq |\langle P_{X''} z, v\rangle| = |a\eta + b| \tag{7.19}$$

and, by hypothesis,

$$|a + b\eta| = |\langle P_{X''} z, r_i \rangle| = |\langle z, r_i \rangle| \leq |\langle z, n_i \rangle| + |z|\,|n_i - r_i| < 1 + \sigma|z|. \tag{7.20}$$

According to (7.14), we conclude from (7.20) and (7.16) that

$$|a + b\eta| < 1 + s. \tag{7.21}$$

The assumption $z \in B$ moreover yields

$$M_{k+1}^2(s) + |(I - P_{X''})z|^2 \leq M_k^2(s) + |(I - P_{X'})z|^2 \tag{7.22}$$

(note that for $k = N - 1$ we have $(I - P_{X''})z = 0$), and we obtain

$$M_{k+1}^2(s) - M_k^2(s) \leq |P_{X''} z|^2 - |P_{X'} z|^2 \tag{7.23}$$

where

$$M_{k+1}^2(s) - M_k^2(s) = \frac{1}{1 - \eta_k^2} \big(1 + s + \eta_k M_k(s)\big)^2 \tag{7.24}$$

and

$$\begin{aligned} |P_{X''} z|^2 &= (a\eta + b)^2 + a^2(1 - \eta^2) \\ &= (a\eta + b)^2 + \frac{1}{1 - \eta^2} \big(a + b\eta - \eta(a\eta + b)\big)^2 \\ &< |P_{X'} z|^2 + \frac{1}{1 - \eta^2} \big(1 + s + |\eta|\,|P_{X'} z|\big)^2 \\ &\leq |P_{X'} z|^2 + \frac{1}{1 - \eta_k^2} \big(1 + s + \eta_k |P_{X'} z|\big)^2. \end{aligned} \tag{7.25}$$

Combining (7.23) - (7.25) we obtain

$$M_k(s) < |P_{X'} z|, \tag{7.26}$$

hence

$$M_k^2(s) + |(I - P_{X'})z|^2 < |z|^2 \leq M_N^2(s) \tag{7.27}$$

which is a contradiction. Lemma 7.6 is proved ∎

We now pass to the proof of Theorem 7.5.

Proof of Theorem 7.5. Assume $z \in B$ is given and $|\langle z, n_i \rangle| < 1$ for some $i \in J$. For $\mu_0 > 0$ and $\mu \in [-\mu_0, \mu_0]$ put $z_\mu = z + \mu\, r_i$. Then $z_\mu \in R_J$ and for every $X' \in \mathcal{D}_k$ $(k = 1, \ldots, N-1)$ we either have $M_k^2(s) + |(I - P_{X'})z|^2 = M_N^2(s)$, hence by Lemma 7.6 $M_k^2(s) + |(I - P_{X'})z_\mu|^2 = M_N^2(s)$, or $M_k^2(s) + |(I - P_{X'})z|^2 < M_N^2(s)$, hence $\mu_0 > 0$ can be chosen in such a way that $z_\mu \in B$ for every $\mu \in [-\mu_0, \mu_0]$. For every $y \in \mathcal{N}_B(z)$ and every $\mu \in [-\mu_0, \mu_0]$ then $\langle y, z - z_\mu \rangle \geq 0$, hence $\langle y, r_i \rangle = 0$ and Theorem 7.5 is proved ∎

8. Example

We illustrate our theory on an example motivated by a problem from queuing theory. Let us consider a service point, where two kinds of customers are served: the ordinary ones and the privileged ones. The waiting room has maximal capacity $K > 0$, and the counters O for ordinary and P for privileged customers have maximal capacities c_o and c_p, respectively.

The incoming customer flow can be described by a time-dependent vector $v(t) = \binom{v_o(t)}{v_p(t)}$ where $v_o(t)$ and $v_p(t)$ represent the number of ordinary and privileged customers, respectively, arrived during the time interval $[0, t]$. We denote by $\eta(t) = \binom{\eta_o(t)}{\eta_p(t)}$ the respective number of customers who have left the service point during the interval $[0, t]$, and $x(t) = v(t) - \eta(t) = \binom{x_o(t)}{x_p(t)}$ is the queue at time t. The waiting room is then described by the condition

$$x(t) \in Z := \left\{ \begin{pmatrix} x_o \\ x_p \end{pmatrix} : x_o, x_p \geq 0 \text{ and } x_o + x_p \leq K \right\} \tag{8.1}$$

for every $t \in [0, T]$ (see Figure 1).

Figure 1: The Skorokhod diagram

We consider the following service rules:

- **A.** All customers are served at their respective counters.
- **B.** Both counters work at their maximal capacity.
- **C.** If there is an unused capacity at the counter O, it can also be used by privileged customers.
- **D.** If the capacity of the waiting room is exceeded, for each refused privileged customer there must be ϱ refused ordinary customers for some $\varrho > 0$.

This can be formalized in the following way:

(i) $x_o > 0, x_p > 0, x_o + x_p < K \quad \Longrightarrow \quad \dot{\eta}_o = c_o, \dot{\eta}_p = c_p.$

(ii) $x_o > 0, x_p = 0, x_o + x_p < K \quad \Longrightarrow \quad \dot{\eta}_o = c_o, \dot{\eta}_p = \min\{c_p, \dot{v}_p\}.$

(iii) $x_o = 0, x_p > 0, x_o + x_p < K \quad \Longrightarrow \quad \dot{\eta}_o = \min\{c_o, \dot{v}_o\}, \dot{\eta}_p = c_p + c_o - \dot{\eta}_o.$

(iv) $x_o > 0, x_p > 0, x_o + x_p = K \quad \Longrightarrow \quad \dot{\eta}_o - c_o = \varrho(\dot{\eta}_p - c_p).$

We normalize the problem by putting

$$u(t) = v(t) - t\begin{pmatrix} c_o \\ c_p \end{pmatrix}, \quad \xi(t) = \eta(t) - t\begin{pmatrix} c_o \\ c_p \end{pmatrix}. \tag{8.2}$$

Hence we are in the situation of (1.7) - (1.8) with normal and reflection vectors

$$\begin{aligned} n_1 &= \begin{pmatrix} -1 \\ 0 \end{pmatrix},\ r_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix},\ n_2 = r_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ n_3 &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix},\ r_3 = \frac{1}{\sqrt{\varrho^2+1}}\begin{pmatrix} \varrho \\ 1 \end{pmatrix} \end{aligned} \tag{8.3}$$

(see Figure 1). The projections Q_j have the form

$$Q_1\begin{pmatrix} x_o \\ x_p \end{pmatrix} = \begin{pmatrix} x_o \\ -x_o \end{pmatrix}, \quad Q_2\begin{pmatrix} x_o \\ x_p \end{pmatrix} = \begin{pmatrix} 0 \\ x_p \end{pmatrix}, \quad Q_3\begin{pmatrix} x_o \\ x_p \end{pmatrix} = \frac{1}{\varrho+1}(x_o + x_p)\begin{pmatrix} \varrho \\ 1 \end{pmatrix}. \tag{8.4}$$

Using the identity

$$(I - Q_3)\begin{pmatrix} x_o \\ x_p \end{pmatrix} = \begin{pmatrix} (1-\varepsilon)x_o - \varepsilon x_p - (\frac{\varrho}{\varrho+1} - \varepsilon)(x_o + x_p) \\ (1 - \varepsilon(1+\varepsilon))x_p - (1+\varepsilon)\varepsilon x_o - (\frac{1}{\varrho+1} - \varepsilon(1+\varepsilon))(x_o + x_p) \end{pmatrix}$$

and the triangle inequality we easily check that the system $\mathcal{Q}$ is ℓ-paracontracting with respect to the norm

$$\left\| \begin{pmatrix} x_o \\ x_p \end{pmatrix} \right\| = C\Big((1+\varepsilon)|x_o| + |x_p| + (1-\varepsilon)|x_o + x_p|\Big)$$

whenever $\varepsilon < \frac{\varrho}{\varrho+1}$, $\varepsilon(1+\varepsilon) < \frac{1}{\varrho+1}$ and $2C\varepsilon^2 \geq 1$. The transversality is indeed obvious. The construction of the set B^ω from Corollary 4.5 is shown on Figure 2.

Figure 2: Construction of the set B^ω

In the limit case $\varrho = +\infty$ the system is not ℓ-paracontracting any more, as

$$(I - Q_1)(I - Q_2)(I - Q_3)\begin{pmatrix} 0 \\ x_p \end{pmatrix} = \begin{pmatrix} 0 \\ -x_p \end{pmatrix}$$

for every $x_p \in \mathbb{R}$ which would contradict inequality (5.3). Similarly, for $\varrho = 0$ we have $r_3 = -r_2$, hence $R^{\perp}_{\{2,3\}} \cap N_{\{2,3\}} \neq \{0\}$ in contradiction with Lemma 5.2. In other words, difficulties arise if only ordinary customers or only privileged customers are refused in the case of exceeded capacity.

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