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Choquet Theory in Metric Spaces

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Abstract. This paper deals with a generalization of the classical Choquet theorem. We consider metric spaces which are endowed with an abstract notion of convexity. Convex combinations are obtained by the solutions of variational inequalities. A generalized Krein-Milman theorem is derived from our Choquet theorem. We end with an example based on hyperbolic geometry.

Keywords: *Abstract convexity in metric spaces, Choquet theorem, Krein-Milman theorem, hyperbolic geometry*

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1. Motivation and introduction

Let M be a convex subset of a normed linear space $(E, \|\cdot\|)$, let $a, b \in E$ and $\lambda \in [0,1]$. The element $z = (1-\lambda)\,a + \lambda\,b$ of M fulfils the variational inequality

$$\|z-u\| \le (1-\lambda)\|a-u\| + \lambda\|b-u\| \qquad (u \in M). \tag{1.1}$$

This inequality can be reformulated: Define a measure $\mu := (1-\lambda)\,\delta_a + \lambda\,\delta_b$, where δ_a and δ_b denote the Dirac measures of a and b, respectively, and note that the set $\mathrm{ex}\,(K)$ of extreme points of the convex and compact line segment $K = [a,b]$ consists precisely of a and b:

$$\|z-u\| \le \int_{\mathrm{ex}(K)} \|t-u\|\,d\mu(t) \qquad (u \in M). \tag{1.2}$$

By the classical Choquet theorem (see R. R. Phelps [10]), inequality (1.2) has a generalization to arbitrary compact convex sets $K \subseteq M$. More precisely, if $K \subseteq M$ is convex and compact, then every $z \in K$ is assigned to a representing probability measure μ_z such that

$$\|z-u\| \le \int_{\mathrm{ex}(K)} \|t-u\| d\mu_z(t) \qquad (u \in M). \tag{1.3}$$

For an arbitrary metric space (M,d), S. Doss [4] has introduced the notion of a mean $B_\lambda(a,b)$ of $a, b \in M$ (and $\lambda \in [0,1]$) by

$$B_\lambda(a,b) = \Big\{ x \in M :\, d(x,u) \le (1-\lambda)\,d(a,u) + \lambda\,d(b,u) \;\; (u \in M) \Big\}. \tag{1.4}$$

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This mean neither must be non-empty nor must be a singleton: Consider the spherical space and Example 1 in Section 4, respectively. M. Fréchet [7] asked whether this mean (he called the elements of $B_\lambda(a,b)$ *generalized means*) is a singleton in every Banach space. S. Gähler and G. Murphy [5] showed that $B_\lambda(a,b)$ is a singleton in every normed linear space and they gave a complete characterization of the metric d being induced by a norm. W. Takahashi [15] assumed non-emptyness of $B_\lambda(a,b)$ and considered metric spaces with a convex structure induced by a selector of the means. He used this to generalize the fixed point theorems of F. Browder and W. A. Kirk for non-expansive mappings (see [15] and the references therein). The developement continued by several authors (see, e.g., M. D. Guay, K. L. Singh and J. H. M. Whitfield [6], B. E. Rhoades, K. L. Singh and J. H. M. Whitfield [11], L. A. Talman [16], and I. Beg and A. Azam [2]). More general concepts of convexity can be found in C. D. Horvath [9] and A. Wieczorek [17, 18]. For general reference to abstract convexity see V. P. Soltan [14] and I. Singer [13].

Our considerations are based on an a priori arbitrary set-valued selector of Doss' mean. Section 2 supplies basic definitions related to convexity in metric spaces and some basic results we need. Section 3 contains the main result which is a direct extension of inequality (1.3) and, therefore, in some sense a Choquet theorem. As a corollary we get a theorem of Krein-Milman type. Section 4 illustrates the results by two examples.

2. Notation and basic concepts

M denotes always the underlying set of a metric space (M,d). By 2^M we denote the power set of M. By maps we mean set-valued maps with non-empty values. If the value of a map is a singleton, we indentify the singleton and its element. Purely single-valued maps are called functions. By a functional we mean a real-valued function.

For a compact set $\emptyset \neq K \subseteq M$ the space of all continuous functionals on K endowed with the supremum norm is denoted by $\mathcal{C}(K)$. The topological dual of $\mathcal{C}(K)$, the space of all finite signed Radon measures on K, is denoted by $\mathcal{M}(K)$. An element μ of $\mathcal{M}(K)$ is considered both as measure and functional. Especially, $\mu(K) = \mu(1)$ is the total mass of μ. Here, 1 denotes the functional with constant value 1. $\mathcal{M}^+(K)$ and $\mathcal{M}_1^+(K)$ denote the positive and probability measures in $\mathcal{M}(K)$, respectively. Especially, δ_x $(x \in K)$ denote the Dirac measures on K. Recall that $\mathcal{C}(K)$ is a lattice and that every $\mu \in \mathcal{M}^+(K)$ is a positive operator with norm $\|\mu\|$ equal to $\mu(1)$.

The basic concept of the following expositions is to consider a selector of Doss' mean. This selector induces a convex structure on the underlying metric space (M,d). To simplify notation, throughout this paper the selector under consideration is called *convex structure*, too. To be more precise we give

Definition 1. A map $E: [0,1] \times M \times M \to 2^M$ is called *convex structure* if for all $a,b \in M$, $\lambda \in [0,1]$ and $x \in E(\lambda,a,b)$ the variational inequality

$$d(x,u) \leq (1-\lambda)\, d(a,u) + \lambda\, d(b,u) \qquad (u \in M) \tag{2.1}$$

holds.

Direct consequences of inequality (2.1) are

$$E(0,a,b)=a \qquad \text{and} \qquad E(1,a,b)=b. \tag{2.2}$$

Every $x \in E(\lambda,a,b)$ lies between a and b in the metric sense. Precisely, we have

$$d(a,x)=\lambda\, d(a,b) \qquad \text{and} \qquad d(b,x)=(1-\lambda)\, d(a,b). \tag{2.3}$$

Indeed, in (2.1) consider $u=a$ and $u=b$. Then (2.1) reads as $d(x,a)\le \lambda\, d(b,a)$ and $d(x,b)\le(1-\lambda)\, d(a,b)$, respectively. Application of the triangle inequality to the left-hand sides of these estimates supplies the inverse estimates. Notice also that if $a\ne b$, we have

$$\lambda=\frac{d(a,x)}{d(a,b)}=1-\frac{d(b,x)}{d(a,b)}$$

for any $x\in E(\lambda,a,b)$.

In the following we suppose the existence of a convex structure on the metric space (M,d). We fix one of this structures and denote it by E. In particular, we suppose that E has non-empty values, i.e., for any $a,b\in M$ and $\lambda\in[0,1]$ inequality (2.1) has at least one solution.

Definition 2. A functional $f:\,M\to\mathbb{R}$ is called *convex* if for all $a,b\in M$, $\lambda\in[0,1]$ and $x\in E(\lambda,a,b)$

$$f(x)\le(1-\lambda)f(a)+\lambda f(b), \tag{2.4}$$

and it is called *concave* if $-f$ is convex. The functional f is called *affine* if it is convex and concave. If in (2.4) equality holds for $\lambda\in\{0,1\}$ or $a=b$ only, we call f *strictly convex* or *strictly concave*, respectively.

If the functionals f,g are convex, so is $f+g$. If f is convex and α is a positive (negative) real scalar, then αf is convex (concave). If (f_γ) is a family of convex (concave) functionals which is bounded from above (below), then $\sup_\gamma f_\gamma$ is convex ($\inf_\gamma f_\gamma$ is concave).

By $\mathcal{C}(M)$ we denote the set of all continuous functionals $f:\,M\to\mathbb{R}$, and by $\mathcal{C}_c(M)$ we denote all continuous functionals $f:\,M\to\mathbb{R}$ which are convex. So $\mathcal{A}(M)=(-\mathcal{C}_c(M))\cap\mathcal{C}_c(M)$ is the set of all continuous affine functionals. By (2.1) we have $\{d(\cdot,u):\,u\in M\}\subseteq\mathcal{C}_c(M)$. Therefore $\mathcal{C}_c(M)$ separates M.

An analogue of a theorem of Hervé, see e.g. [1], shows that compactness of (M,d) is sufficient for the existence of a continuous strictly convex functional.

Theorem 1. *Let the metric space (M,d) be compact. Then there exists a continuous strictly convex functional on M.*

Proof. Choose a dense sequence (x_n) in M. The functional $f:\,M\to\mathbb{R}$ defined by

$$f(x)=\sum_{n=1}^{\infty}2^{-n}d(x,x_n) \qquad (x\in M)$$

is continuous and convex. Suppose that f is not strictly convex, i.e., there are different $a,b\in M$, $\lambda\in(0,1)$ and $x\in E(\lambda,a,b)$ such that $f(x)=(1-\lambda)f(a)+\lambda f(b)$. Then

$$d(x,x_n)=(1-\lambda)\, d(a,x_n)+\lambda\, d(b,x_n) \qquad (n\in\mathbb{N}).$$

Now we get a contradiction by considering a subsequence (x_{n_k}) of (x_n) that converges to x ∎

All constant functionals are affine. In general nothing can be stated about the existence of non-constant affine continuous functionals. However, if there are enough functionals to separate M, we can stop here for the following reason:

Theorem 2. *Let (M,d) be a compact metric space. If $\mathcal{A}(M)$ separates the points of M, then the convex structure E is necessarily a function, i.e. single-valued, and there exists an affine embedding of (M,d) into a locally convex linear space.*

Proof. $\mathcal{A}(M)$ is a linear space over the scalar field $\mathbb{R}$. Let P be the algebraic dual of $\mathcal{A}(M)$ endowed with the linear topology induced by pointwise convergence. Define $\Phi : M \to P$ by $\Phi(x)(f) = f(x) \quad (f \in \mathcal{A}(M), x \in M)$. The map Φ is affine, i.e., for all $a,b \in M$ and $\lambda \in [0,1]$ we have $\Phi(x) = (1-\lambda)\Phi(a) + \lambda\Phi(b)$ for all $x \in E(\lambda,a,b)$. Since $\mathcal{A}(M)$ seperates the points of M, Φ is injective and E must be single-valued. Φ is continuous and a bijection onto its image. So, by the compactness of (M,d), Φ is an embedding ∎

Definition 3. Let the functional $f : M \to \mathbb{R}$ be bounded from below or above. We call the functionals $\widetilde{f}$ and $\widehat{f}$ defined by

$$\widetilde{f}(x) = \Big\{ \sup \varphi(x) : \varphi \in \mathcal{C}_c(M), \quad \varphi \le f \Big\} \qquad (x \in M)$$

and

$$\widehat{f}(x) = \Big\{ \inf \varphi(x) : \varphi \in -\mathcal{C}_c(M), \quad f \le \varphi \Big\} \qquad (x \in M)$$

the *lower envelope* and *upper envelope* of f, respectively.

A lower envelope is always a convex functional and an upper envelope a concave functional. If f is bounded, then $\widetilde{f}$ is lower semicontinuous and $\widehat{f}$ is upper semicontinuous. Especially, for bounded f both $\widetilde{f}$ and $\widehat{f}$ are measurable and $\mu(\widetilde{f})$ as well as $\mu(\widehat{f})$ are defined for any $\mu \in \mathcal{M}(K)$.

The proof of the following lemma can be carried through as in the linear setting, for which we refer to E. M. Alfsen [1: Proposition I.1.6].

Lemma 1. *Let the functionals $f,g : M \to \mathbb{R}$ be bounded from above and $\alpha \ge 0$. Then:*

- **(i)** $\widehat{\alpha f} = \alpha \widehat{f}$.
- **(ii)** $\widehat{f+g} \le \widehat{f} + \widehat{g}$.
- **(iii)** $\widehat{f} = -\big(\widetilde{-f}\big)$.

Definition 4. A subset $A \subseteq M$ is called *convex* (with respect to the given convex structure E) if $E([0,1] \times A \times A) \subseteq A$. The *convex hull*, in symbols $\operatorname{conv}(A)$, of A is the smallest convex subset of M that contains A. I.e., $\operatorname{conv}(A) = \cap \big\{ B : A \subseteq B \text{ and } B \text{ convex} \big\}$.

So a necessary condition that a subset A of M is convex is that the convex structure E of M restricted to $[0,1] \times A \times A$ is a convex structure for A.

Definition 5. Let $\emptyset \neq K \subseteq M$ be convex. A point $x \in K$ is called an *extreme point* (with respect to E) of K if, for all $a, b \in K$ and $\lambda \in (0,1)$, $x \in E(\lambda, a, b)$ implies $a = x = b$. By $\mathrm{ex}(K)$ we denote the set of all extreme points of K.

We arrive at our last preliminary lemma.

Lemma 2. *Let $f \in \mathcal{C}_c(K)$ be strictly convex. Then the measurable set $\{\widehat{f} = f\}$ is contained in* $\mathrm{ex}(K)$.

Proof. The measurability of $\{\widehat{f} = f\}$ results from the upper semicontinuity of $\widehat{f}$ and the continuity of f. To prove the inclusion let $x \notin \mathrm{ex}(K)$. Then there are different $a, b \in K$ and $\lambda \in (0,1)$ such that $x \in E(\lambda, a, b)$. We calculate

$$f(x) < (1-\lambda)f(a) + \lambda f(b) \leq (1-\lambda)\widehat{f}(a) + \lambda \widehat{f}(b) \leq \widehat{f}(x).$$

So x cannot be an element of $\{\widehat{f} = f\}$ ∎

3. Main results

Let $\emptyset \neq K \subseteq M$ be a compact convex subset of M.

3.1 A Choquet type theorem. The main result of our expositions is a generalization of the variational inequality (1.3) and the classical Choquet theorem (see R. R. Phelps [10]).

Theorem 3. *For all $z \in K$ there exists $\mu_z \in \mathcal{M}_1^+(K)$ such that*

$$\varphi(z) \leq \int_{\mathrm{ex}(K)} \varphi(t)\, d\mu_z(t) \qquad (\varphi \in \mathcal{C}_c(M)). \tag{3.1}$$

Remark. We note at this point that there is no need to show that $\mathrm{ex}(K)$ is a measurable subset of K with respect to the Borel σ-algebra induced by the restriction of d to K. Precisely, the measure μ_z above is induced by an element of $\mathcal{M}_1^+(K)$ and defined on the trace σ-algebra on $\mathrm{ex}\,(K)$.

The proof of the above theorem is given at the end of this section. We need some calculations beforehand.

Lemma 3 to Corollary 2 below can be easily proved by adapting the expositions given in Alfsen's monograph [1] to our nonlinear setting. However, to keep the paper self-contained we outline the proofs.

Set $\mathcal{P}(K) := \mathcal{C}_c(K) - \mathcal{C}_c(K)$.

Lemma 3. *$\mathcal{P}(K)$ is a real lattice in $\mathcal{C}(K)$ and, by the theorem of Stone, dense in $\mathcal{C}(K)$.*

Proof. By the remarks following Definition 2, $\mathcal{P}(K)$ is a real linear space. Constant functionals belong to $\mathcal{C}_c(K)$, so they belong to $\mathcal{P}(K)$. The functionals $d(\cdot, u)$ $(u \in K)$

belong to $\mathcal{P}(K)$, too, and they separate K. Since $\mathcal{C}_c(K)$ is closed under forming suprema and, for $f, g, h, i \in \mathcal{C}_c(K)$,

$$\sup\{f-g, h-i\} = \sup\{f+i, g+h\} - (g+i)$$
$$\inf\{f-g, h-i\} = (f+h) - \sup\{g+h, f+i\},$$

the hypotheses of the Stone theorem are fullfilled and, therefore, $\mathcal{P}(K)$ is dense in $\mathcal{C}(K)$ ∎

We define a partial order $\preceq$ on $\mathcal{M}(K)$, the dual of $\mathcal{C}(K)$, by

$$\mu \preceq \nu \quad :\Longleftrightarrow \quad \mu(f) \leq \nu(f) \;\; (f \in \mathcal{C}_c(K))$$

for $\mu, \nu \in \mathcal{M}(K)$. Indeed, $\preceq$ is a partial order: reflexivity and transitivity are obvious and antisymmetry follows from the foregoing lemma.

To motivate the following definition notice that if (3.1) holds for a measure $\mu_1 \in \mathcal{M}_1^+(K)$ (with integration over the whole space K instead of $\mathrm{ex}(K)$) and there exist $a, b \in K$, $\lambda \in (0,1)$ and $x \in E(\lambda, a, b)$ such that $\mu_1(\{x\}) > 0$, then for the measure $\mu_2 := \mu_1 + \mu_1(\{x\})\big((1-\lambda)\,\delta_a + \lambda\,\delta_b - \delta_x\big)$ inequality (3.1) also holds and, furthermore, $\mu_1 \preceq \mu_2$ and $\mu_2(\{x\}) = 0$. Obviously, x does not belong to $\mathrm{ex}(K)$, so we can in some way say that μ_2 has its mass more near to $\mathrm{ex}(K)$ than μ_1. As a matter of fact we will see that measures for which (3.1) holds are to be found among the maximal elements of the partial order $\preceq$.

So it is natural to give

Definition 6. A measure $\mu \in \mathcal{M}^+(K)$ is called a *boundary measure*, if it is maximal with respect to $\preceq$.

Note that a necessary condition for two measures μ_1 and μ_2 to be comparable with respect to $\preceq$ is that they have the same total mass: $\mu_1(1) = \mu_2(1)$.

The existence of maximal elements follows from Zorn's Lemma:

Theorem 4. *Let $\mu \in \mathcal{M}^+(K)$. Then there exists a boundary measure $\nu \in \mathcal{M}^+(K)$ such that $\mu \preceq \nu$.*

Proof. We consider $\mathcal{M}_\mu := \{\nu \in \mathcal{M}^+(K) : \mu \preceq \nu\}$ as a partial ordered subset of the dual of $\mathcal{C}(K)$. To apply Zorn's Lemma let $\mathcal{N}$ be a chain in $\mathcal{M}_\mu$. This chain is relatively weak*-compact and it turns out that the weak∗-accumulation points of $\mathcal{N}$ are upper bounds of $\mathcal{N}$ in $\mathcal{M}_\mu$. Zorn's Lemma provides the existence of maximal elements in $\mathcal{M}_\mu$ which are maximal in $\mathcal{M}^+(K)$, too, and therefore, boundary measures ∎

In the special case of a Dirac measure the above theorem reads as

Corollary 1. *For every $z \in K$ there exists a boundary measure $\mu_z \in \mathcal{M}_1^+(K)$ such that $\delta_z \preceq \mu_z$.*

It remains to show that the mass of the measure μ_z of the last corollary is concentrated on the extreme points of K. We start with

Theorem 5. *Let $\mu \in \mathcal{M}^+(K)$ and $f \in \mathcal{C}(K)$. Then there exists $\nu \in \mathcal{M}^+(K)$ such that $\mu \preceq \nu$ and $\nu(f) = \mu(\widehat{f})$.*

Proof. Consider the functional $\Phi : \mathcal{C}(K) \to \mathbb{R}$ defined by

$$\Phi(g) = \mu(\widehat{g}) \qquad (g \in \mathcal{C}(K)).$$

By Lemma 1, Φ is subadditive and positive homogeneous. Define $v : \operatorname{span}(f) \to \mathbb{R}$ by the linear extension of $\Phi|_{\{f\}}$. It turns out that v is dominated by the sublinear functional $\Phi|_{\operatorname{span}\{f\}}$. By the Hahn-Banach theorem there exists a linear extension $\nu : \mathcal{C}(K) \to \mathbb{R}$ of v which is dominated by Φ. Precisely,

$$\nu(f) = \mu(\widehat{f})$$

and

$$\nu(g) \leq \mu(\widehat{g}) \qquad (g \in \mathcal{C}(K)). \tag{3.2}$$

Note that for $g \in -\mathcal{C}_c(K)$ we have $\widehat{g} = g$ and $0 \in -\mathcal{C}_c(K)$. Let $0 \leq g \in \mathcal{C}(K)$. Then $-g \leq 0$ and $\widehat{-g} \leq 0$. So $\nu(g) = -\nu(-g) \geq -\mu(\widehat{-g}) \geq 0$. I.e., ν is positive. By (3.2) $\nu(1) = \mu(1)$. So $\nu \in \mathcal{M}^+(K)$. It remains to show $\mu \preceq \nu$. This follows again from (3.2) for $-\mathcal{C}_c(K) \subseteq \mathcal{C}(K)$ ∎

Corollary 2. *For a measure $\mu \in \mathcal{M}^+(K)$ the following statements are equivalent:*

(i) *μ is maximal with respect to $\preceq$.*

(ii) *$\mu(\widehat{f}) = \mu(f)$ for all $f \in \mathcal{C}(K)$.*

(iii) *$\mu(\widehat{f}) = \mu(f)$ for all $f \in \mathcal{C}_c(K)$.*

Proof. *(ii)* $\Rightarrow$ *(iii)* is trivial and *(i)* $\Rightarrow$ *(ii)* follows directly from the last theorem. To prove *(iii)* $\Rightarrow$ *(i)* suppose *(iii)* and choose $\nu \in \mathcal{M}^+(K)$ such that $\mu \preceq \nu$. To show $\mu = \nu$ it is sufficient to show $\nu(f) \leq \mu(f)$ for $f \in \mathcal{C}_c(K)$. Note first that $\mu \preceq \nu$ means $\nu(g) \leq \mu(g)$ for $g \in -\mathcal{C}_c(K)$. So for any $f \in \mathcal{C}_c(K)$ we have $\nu(f) \leq \nu(\widehat{f}) \leq \mu(\widehat{f})$. By *(iii)* the latter one is equal to $\mu(f)$ which is the desired assertion ∎

We are now able to prove our main result.

Proof of Theorem 3. Let $z \in K$ be fixed and μ_z be the boundary measure given by Corollary 1. Formula (3.1) is equivalent to $\delta_z \preceq \mu_z$ if one can show that the mass of μ_z is concentrated on $\operatorname{ex}(K)$. Theorem 1 supplies the existence of a strictly convex functional $f \in \mathcal{C}_c(K)$ and Corollary 2 shows that the maximality of μ_z enforces the mass of μ_z to be concentrated on $\{\widehat{f} = f\}$ which is by Lemma 2 a measurable subset of $\operatorname{ex}(K)$ ∎

Recall that the functionals $d(\cdot, u)$ $(u \in M)$ are convex. So we obtain as a generalization of (1.3) the following

Corollary 3. *For all $z \in K$ there exists $\mu_z \in \mathcal{M}_1^+(K)$ such that*

$$d(z, u) \leq \int_{\operatorname{ex}(K)} d(t, u)\, d\mu_z(t) \qquad (u \in M). \tag{3.3}$$

Now we deal with the converse question: Given a measure $\mu \in \mathcal{M}_1^+(K)$, does there exist a point $z \in K$ such that (3.1) holds? And additionally, if the answer is affirmative, is this point uniquely determined as in the case of a compact convex subset K of a normed linear space?

The answer to the second question is negative: if we suppose that the set

$$\left\{ \sum_{i=1}^{n} \alpha_i \, d(x_i, \cdot) + \alpha : \, \alpha, \alpha_i \in [0, \infty), \; x_i \in K, \; n \in \mathbb{N} \right\}$$

is dense in $\mathcal{C}_c(K)$, what means that (3.1) and (3.3) are equivalent, then (see Example 1 in Section 4) z is in general not uniquely determined by the measure μ. The answer to the first question is positive:

Theorem 6. *Let $\emptyset \neq K \subseteq M$ be convex and compact and let $\mu \in \mathcal{M}_1^+(K)$. Then there exists $z \in K$ such that* (3.1) *holds with $\mu_z = \mu$.*

Proof. Define

$$M_\varphi := \left\{ x \in K : \, \varphi(x) \leq \int_K \varphi(t) \, d\mu(t) \right\} \qquad (\varphi \in \mathcal{C}_c(K)).$$

We have to show that $\cap_{\varphi \in \mathcal{C}_c(K)} M_\varphi \neq \emptyset$. For M_φ being closed subsets of the compact set K it is sufficient to verify that $(M_\varphi)_{\varphi \in \mathcal{C}_c(K)}$ has the finite intersection property.

Fix $\varphi_1, \ldots, \varphi_n \in \mathcal{C}_c(K)$ and define $\psi := (\varphi_1, \ldots, \varphi_n)$. Since $M_\varphi = M_{\varphi + c}$ for any constant $c \in \mathbb{R}$ we can suppose $\psi \geq 0$. To show that the point $p = \int_K \psi(t) \, d\mu(t)$ belongs to the image of $T : \, K \to 2^{\mathbb{R}^n}$ given by $T(x) = \{ y \in \mathbb{R}^n : \, y \geq \psi(x) \}$ $(x \in K)$ we suppose $p \notin T(K)$. It can be easily seen that $T(K)$ is convex and closed. Therefore there exists $a \in \mathbb{R}^n$, $a \geq 0$ and $\|a\| = 1$, such that $\inf_{y \in T(K)} \langle a, y \rangle > \langle a, p \rangle$. The functional $\eta := \langle a, \psi \rangle$ is an element of $\mathcal{C}_c(K)$ and we obtain

$$\mu(\eta) = \langle a, p \rangle < \inf_{y \in T(K)} \langle a, y \rangle \leq \left\langle a, \int_K \psi(t) \, d\mu(t) \right\rangle = \mu(\eta)$$

which is a contradiction ∎

3.2 A Krein-Milman type theorem. The existence of extremal points can be proved directly by the lemma of Zorn. However, Theorem 3 tells us that $\mathrm{ex}(K)$ is non-empty and allows us to omit the existence part in order to prove the Krein-Milman Theorem.

For the proof we need some additional property of the underlying convex structure which is an extension of the notion of negative curvature of W. Herer [8]. For the motivation of the following definition see also Example 2 in Section 4 and [12].

Definition 7. A metric space (M, d) with a convex structure E is said to be of *negative curvature* if for all $a, b, c, d \in M$, $\lambda \in [0, 1]$ and $x \in E(\lambda, a, b)$ there exists $x' \in E(\lambda, c, d)$ such that

$$d(x, x') \leq (1 - \lambda) \, d(a, c) + \lambda \, d(b, d). \tag{3.4}$$

Note that for $c = d = u$ this is precisely the variational inequality (2.1).

Recall that a map $A : \, M \times M \to 2^M$ is *lower semicontinuous* if for all $x \in M \times M$, $y \in A(x)$ and all sequences $x_n \to x$ in $M \times M$ there exists a sequence $y_n \to y$ in M such that $y_n \in A(x_n)$ $(n \in \mathbb{N})$.

Theorem 7. *Let (M,d) be a metric space endowed with a convex structure E which is of negative curvature, and suppose the maps $E(\lambda,\cdot,\cdot)$ $(\lambda \in [0,1])$ are lower semicontinuous. If $\emptyset \neq K \subseteq M$ is compact and convex, then $K = \overline{\operatorname{conv}\operatorname{ex}(K)}$.*

Proof. Suppose the contrary, i.e., there exists $x \in K \setminus H$ where $H = \overline{\operatorname{conv}\operatorname{ex}(K)}$. The functional $d(\cdot,H)$ defined by $d(y,H) = \inf\{d(y,h) : h \in H\}$ $(y \in K)$ is continuous. By the lower semicontinuity of $E(\lambda,\cdot,\cdot)$ $(\lambda \in [0,1])$ the set H is convex and, therefore, by the negative curvature of the space (K,d) the functional $d(\cdot,H)$ turns out to be convex. Indeed, for any $a,b \in K$, $\lambda \in [0,1]$ and $x \in E(\lambda,a,b)$ there exist $c,d \in H$ such that $d(a,c) = d(a,H)$ and $d(b,d) = d(b,H)$. If we choose x' as in the definition above we have

$$d(x,H) \le d(x,x') \le (1-\lambda)\,d(a,c) + \lambda\,d(b,d) = (1-\lambda)\,d(a,H) + \lambda\,d(b,H).$$

I.e., $d(\cdot,H) \in \mathcal{C}_c(K)$. Theorem 3 assures the existence of $\mu_x \in \mathcal{M}_1^+(K)$ such that (3.1) holds. Then with the contradiction

$$0 < d(x,H) \le \int_{\operatorname{ex}(K)} d(t,H)\,d\mu_x(t) = 0$$

the statement is proved ∎

4. Examples

As a first example we consider the case that there is more than one solution of (2.1).

Example 1 (a set-valued convex structure). If M is a linear space and its metric d is induced by a norm then, there exists precisely one convex structure which is given by $E(\lambda,a,b) = \{(1-\lambda)\,a + \lambda\,b\}$ $(\lambda \in [0,1];\ a,b \in M)$. This has been shown by S. Gähler and G. Murphy [5]. An essential asssumption to prove the uniqueness in this case is that u in (2.1) varies in elements which are not contained in the convex hull of $\{a,b\}$. If one considers only a convex subset K of M, the situation can change drastically as we will see now.

Let M be the Euclidean triangle

$$M = \Big\{(a_1,a_2) \in \mathbb{R}^2 : 0 \le a_2 \le a_1 \le 1\Big\}.$$

We endow M with the metric d which is induced by the maximum norm on $\mathbb{R}^2$. For $\lambda \in [0,1]$ and $a = (a_1,a_2), b = (b_1,b_2)$ in M define

$$E(\lambda,a,b) = \Big\{\big((1-\lambda)\,a_1 + \lambda\,b_1, c_2\big) : c_2 \in \Big\{(1-\lambda)\,a_2 + \lambda\,b_2, b_2'\Big\}\Big\},$$

where

$$b_2' = \max\Big\{(1-\lambda)\,a_2 + \lambda\,b_2, \min\{\lambda, 1-\lambda\}|a_1 - b_1|\Big\}.$$

Because of

$$\min\{\lambda,1-\lambda\}|a_1 - b_1| \le (1-\lambda)\,a_1 + \lambda\,b_1, \tag{4.1}$$

E maps on M. To show the variational inequality (2.1) let $u = (u_1, u_2) \in M$. We only have to consider the case where the definition of E differs from the usual affine combination in $\mathbb{R}^2$. So we can suppose that

$$(1-\lambda)\,a_2 + \lambda\,b_2 \le \min\{\lambda, 1-\lambda\}|a_1 - b_1|$$

and that the left-hand side of (2.1) is equal to

$$\big|u_2 - \min\{\lambda, 1-\lambda\}|a_1 - b_1|\,\big|.$$

Now (2.1) follows from

$$\begin{aligned}
\big|u_2 - \min\{\lambda, 1-\lambda\}|a_1 - b_1|\,\big| &= \begin{cases} u_2 - \min\{\lambda, 1-\lambda\}|a_1 - b_1| \\ \min\{\lambda, 1-\lambda\}|a_1 - b_1| - u_2 \end{cases} \\
&\le \begin{cases} u_2 - \big((1-\lambda)\,a_2 + \lambda\,b_2\big) \\ \min\{\lambda, 1-\lambda\}|a_1 - b_1| \end{cases} \\
&\le \begin{cases} (1-\lambda)|u_2 - a_2| + \lambda|u_2 - b_2| \\ (1-\lambda)|u_1 - a_1| + \lambda|u_1 - b_1| \end{cases} \\
&\le (1-\lambda)\,d(u,a) + \lambda\,d(u,b).
\end{aligned}$$

So E is a convex structure on (M, d).

Consider now the points $a = (0,0)$, $b = (1,0)$ and $c = \left(\frac{1}{2}, \frac{1}{2}\right)$ in M. We have $E\left(\frac{1}{2}, a, b\right) = \{\left(\frac{1}{2}, 0\right), c\}$. Define $K := \overline{\operatorname{conv}\{a,b\}}$. This is the Euclidean triangle with edges $\{a, b, c\}$. Indeed, for any $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in the triangle we have estimate (4.1) and

$$\min\{\lambda, 1-\lambda\}|a_1 - b_1| \le 1 - \big((1-\lambda)\,a_1 + \lambda\,b_1\big)$$

which means that $E([0,1], a, b)$ lies under the graphs of the identity and $x \mapsto 1 - x$, respectively. I.e., the triangle is a closed convex set and it contains K. That the triangle is contained in K is obvious.

By considering the first coordinate we see that a and b are extremal points of K. So we have $\operatorname{ex}(K) = \{a, b\}$ and z in Corollary 3 is not uniquely determined by the measure $\mu_z = \frac{1}{2}\delta_a + \frac{1}{2}\delta_b$.

Example 2 (spaces with curvature ≤ 0). This example deals with the inner metric structure of metric spaces. For definitions and a detailed background we refer to R. Rinow [12].

By (2.3) a necessary condition for a point x to be an element of $E(\lambda, a, b)$ is that x lies between a and b. I.e., the space (M, d) must be convex in the metric sense. By a theorem of K. Menger a complete metric space which is convex in the metric sense is a space with inner metric, and for every two different points $a, b \in M$ there exists a shortest curve $C = C(a, b)$ connecting them.

If we suppose the existence of a continuous function f from $M \times M$ to the space of all shortest curves in M (with reduced parametrization) and suppose (M, d) has curvature ≤ 0, then [12: Section 47/Subsection 1] tells us that for any $a, b, c, d \in M$ the function $\lambda \mapsto d\big(f(a,b)(\lambda), f(c,d)(\lambda)\big)$ is continuous and convex. So with $E(\lambda, a, b) := f(a,b)(\lambda)$ $(a, b \in M, \lambda \in [0,1])$ we obtain a convex structure and (M, d) is of negative curvature in the sense of Definition 7. Also, we have $E(\lambda, \cdot, \cdot)$ $(\lambda \in [0,1])$ is continuous since f is continuous.

To get a more concrete example consider a complete simple connected Riemannian manifold with geodesic metric which is of non-positive sectional curvature, e.g. the hyperbolic plane (see H. Busemann [3], S. Gähler and G. Murphy [5], respectively).

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