Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 19 (2000), No. 2, 489–510

Inverse Problems for Memory Kernels by Laplace Transform Methods

J. Janno and L. v. Wolfersdorf

Dedicated to Prof. E. Meister on the occasion of his 70th birthday

Abstract. Basic inverse problems for identification of memory kernels in linear heat conduction and viscoelasticity in the infinite time interval $(0, \infty)$ are treated by Laplace transform method in coupling with Fourier's method for the direct initial-boundary value problem of the corresponding integro-differential equation. Under suitable assumptions on the data existence and uniqueness of the memory kernel are shown.

Keywords: *Inverse problems, memory kernels, heat conduction, viscoelasticity, Laplace transform*

AMS subject classification: 35 R 30, 44 A 10, 45 K 05, 73 B 30, 73 F 15

0. Introduction

In recent time different methods are developed in dealing with inverse problems for identification of memory kernels in linear heat conduction and viscoelasticity. In the thermal case we especially refer to the papers Lorenzi and Sinestrari [19], Grasselli [3] and our papers [10 - 12], in the viscoelastic case to Grasselli, Kabanikhin and Lorenzi [6 - 7], Grasselli [4 - 5] and our papers [13 - 14]. See further the papers Lorenzi [17], Lorenzi and Paparoni [18], Bukhgeim [1] and the overview on our papers [10 - 11, 13] in [21]. In all these papers the inverse problems are formulated in a finite time interval $[0, T]$ and treated by a fixed point argument in a Banach space of functions defined on $[0, T]$.

If the memory kernels are defined (or extended) to the infinite time interval $[0, \infty)$, the method of Laplace transform can be used for solving corresponding inverse problems. This has been done in Janno [8 - 9] for a particular viscoelastic problem, see also the general remarks in Tobias and Engelbrecht [20].

In the present paper the method of Laplace transform is applied to two basic inverse problems for memory kernels in linear heat conduction and viscoelasticity where again

J. Janno: Tallinn Techn. Univ., Inst. Cybern., 12618 Tallinn, Estonia. The paper was written during a stay of the author at the Freiberg University of Mining and Technology with a grant of DFG from February 1999 to May 1999, also supported by the Estonian Science Foundation (Grant No. 3131).
L. v. Wolfersdorf: Techn. Univ. Bergakademie Freiberg, Fak. Math. & Inf, D-09596 Freiberg.

ISSN 0232-2064 / $ 2.50

the solution of the direct problem is constructed by the classical Fourier method as in our papers [10 - 11] and [13 - 14], respectively. Under suitable assumptions on the data we prove existence and uniqueness of the memory kernel and also estimate the kernel by an exponential function. The method should be applicable further to the cases of general linear heat flow [12], of weakly singular kernels in viscoelasticity [14], of thermal- and poro-viscoelasticity [15] and of viscoelasticity with dominating Newtonian viscosity [16].

1. Statement of problems in heat conduction

In the linear theory of heat flow in a rigid isotropic body consisting of material with thermal memory the generalized heat equation

$$\beta u_t - \operatorname{div}(\lambda \nabla u) + \int_0^t m(t-\tau)\operatorname{div}(\lambda \nabla u(x,\tau))\,d\tau = f \tag{1}$$

holds in the cylinder $\Omega = D \times \{t > 0\}$, where D is a bounded domain in $\mathbb{R}^N$ with piecewise smooth boundary S (cf. [10, 11] or [21]). Here u is the temperature of the body which we suppose as zero for $t < 0$ and ∇u denotes the gradient of u with respect to $x \in D$. Further, m is the memory kernel of heat flux and f the heat supply. The heat conduction coefficient λ and $\beta = c\rho$, where c is the specific caloric coefficient and ρ the mass density, are given positive continuous functions on D.

An analogous parabolic integro-differential equation as (1) occurs in the theory of flow in porous media where u now denotes the pressure, $\lambda = \kappa\rho$ with permeability κ and mass density ρ again, β is the reciprocal of Biot's modulus with the corresponding memory kernel m and f is the mass source density (cf. [15], for instance). Besides equation (1) the function u satisfies the initial and boundary conditions

$$u(x,0) = \varphi(x) \qquad \text{on } D \tag{2}$$

$$u(x,t) = 0 \;\text{ or }\; \lambda\frac{\partial u}{\partial n} + \mu u = 0 \qquad \text{on } \Sigma = S \times \{t > 0\} \tag{3}$$

with given continuous functions φ on D and $\mu \geq 0$ on S, where n is the outer normal to S.

In the *inverse problem* we have to find the kernel m such that the corresponding function u satisfies equations (1) - (3) and an additional condition of the form

$$\Psi[u](t) = h(t) \qquad (t > 0) \tag{4}$$

or

$$\Psi[u](t) - \int_0^t m(t-\tau)\Psi[u](\tau)\,d\tau = h(t) \qquad (t > 0) \tag{5}$$

where Ψ is a suitable linear observation functional on $u(\cdot,t)$, for instance $\Psi[u](t) = u(x_0,t)$ $(x_0 \in D)$ in the case of (4) and $\Psi[u](t) = \frac{\partial u}{\partial n}(x_0,t)$ $(x_0 \in S)$ in the case of

(5) corresponding to a measurement of the temperature and the flux in some point x_0, respectively. The given continuous and differentiable function h represents the output of the observation.

The solution u of the *direct problem* (1) - (3) is taken in form of the Fourier series

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t) v_k(x) \tag{6}$$

where v_k are the eigenfunctions of the problem

$$\operatorname{div}(\lambda \nabla v) + \mu \beta v = 0 \qquad \text{in } D \tag{7}$$

with (3) and the positive (non-negative) eigenvalues μ_k. In view of (1) and (2) the coefficient functions a_k in (6) satisfy the initial value problem

$$\left.\begin{aligned} \dot{a}_k(t) + \mu_k a_k(t) - \mu_k \int_0^t m(t-\tau) a_k(\tau)\, d\tau = r_k(t) \\ a_k(0) = \varphi_k \end{aligned}\right\} \tag{8}$$

where

$$\varphi(x) = \sum_{k=1}^{\infty} \varphi_k v_k(x) \qquad \text{and} \qquad r(x,t) \equiv \frac{f}{\beta} = \sum_{k=1}^{\infty} r_k(t) v_k(x). \tag{9}$$

By representation (6) for u the *additional conditions* (4) and (5) take the form

$$\sum_{k=1}^{\infty} \gamma_k a_k(t) = h(t) \qquad (t > 0) \tag{10}$$

and

$$\sum_{k=1}^{\infty} \gamma_k \left[a_k(t) - \int_0^t m(t-\tau) a_k(\tau)\, d\tau \right] = h(t) \qquad (t > 0), \tag{11}$$

respectively, where $\gamma_k = \Psi(v_k)$ are the "observation coefficients".

We assume the existence of the in $L_2(0)$ complete orthogonal system of eigenfunctions $\{v_k\}$ of (7) with (3). There are sufficient conditions on the data φ, r for the convergence of the Fourier series (9) and the existence of a generalized solution u to the initial-boundary value problem (1) - (3). For instance, for a continuous memory kernel m the conditions $\varphi \in L_2(D), r \in L_2(\Omega_T), \Omega_T = D \times (0,T)$ are sufficient for the convergence of (9) and the existence of a solution $u \in H^{1,0}(\Omega_T)$ with arbitrary $T > 0$. But we prefer here to make no concrete assumptions on the data φ, r beside the existence of the Fourier coefficients $\varphi_k, r_k(t)$ and working with the formal solution (6) of the direct problem (1) - (3). Concentrating on the inverse problems for their solvability indeed the convergence conditions (33), (34) in Theorem 1 below are natural assumptions which are coupled conditions on the data φ, r of the direct problem and on the observation functional Ψ in the additional conditions for the inverse problems. These conditions can be fulfilled by different pairs of conditions for φ, r on one side and for Ψ on the other side yielding a related generalized solution of the direct problem.

We now apply the Laplace transform

$$\mathcal{L}_{t\to p}(\omega) = \int_0^\infty e^{-pt}\omega(t)\,dt$$

to equations (8) and (10) - (11). Equation (8) is transformed to

$$pA_k(p) - \varphi_k + \mu_k A_k(p) - \mu_k M(p)A_k(p) = R_k(p) \tag{12}$$

where

$$\left.\begin{aligned} A_k &= \mathcal{L}_{t\to p}(a_k) \\ M(p) &= \mathcal{L}_{t\to p}(m) \\ R_k &= \mathcal{L}_{t\to p}(r_k). \end{aligned}\right\}$$

From (12) the relations for A_k

$$A_k(p) = \frac{\mu_k}{p+\mu_k} M(p)A_k(p) + \frac{\varphi_k + R_k(p)}{p+\mu_k} \tag{13}$$

and

$$A_k(p) = \frac{1}{1 - \frac{\mu_k}{p+\mu_k}M(p)} \cdot \frac{\varphi_k + R_k(p)}{p+\mu_k} \tag{14}$$

follow. Equations (10) and (11) go over into

$$\sum_{k=1}^{\infty} \gamma_k A_k(p) = H(p) \tag{15}$$

and

$$\sum_{k=1}^{\infty} \gamma_k A_k(p) - M(p)\sum_{k=1}^{\infty} \gamma_k A_k(p) = H(p) \tag{16}$$

where $H = \mathcal{L}_{t\to p}(h)$.

In the *first inverse problem* with condition (4), from (15) observing (13), we obtain the equation for $M(p)$

$$M(p)\sum_{k=1}^{\infty} \gamma_k \frac{\mu_k}{p+\mu_k} A_k(p) = H(p) - \sum_{k=1}^{\infty} \gamma_k \frac{\varphi_k + R_k(p)}{p+\mu_k}. \tag{17}$$

Multiplying this equation by p^2 and introducing

$$d_1 := \sum_{k=1}^{\infty} \gamma_k \mu_k \varphi_k \neq 0 \qquad \text{(by assumption)}, \tag{18}$$

we get for $M(p)$ the fixed point equation

$$M(p) = -\frac{1}{d_1} M(p) \sum_{k=1}^{\infty} \gamma_k \mu_k B_k(p) + G_1(p) \tag{19}$$

where

$$B_k(p) = B_k[M](p) = \frac{p^2}{p+\mu_k} A_k(p) - \varphi_k \tag{20}$$

with $A_k = A_k[M]$ given by (14) and

$$G_1(p) = \frac{p^2}{d_1} \left(H(p) - \sum_{k=1}^{\infty} \gamma_k \frac{\varphi_k + R_k(p)}{p+\mu_k} \right). \tag{21}$$

By (20) and (14) the explicit expression for B_k

$$B_k(p) = \frac{1}{1 - \frac{\mu_k}{p+\mu_k} M(p)} \left[\frac{\mu_k}{p+\mu_k} \varphi_k M(p) + \Phi_k(p) \right] \tag{22}$$

follows where

$$\Phi_k(p) = \frac{p^2}{(p+\mu_k)^2} R_k(p) - \frac{2\mu_k p + \mu_k^2}{(p+\mu_k)^2} \varphi_k. \tag{23}$$

In the *second inverse problem* with condition (5), from (16) and (13) the equation for $M(p)$ writes

$$M(p) \sum_{k=1}^{\infty} \gamma_k \left[\frac{\mu_k}{p+\mu_k} - 1 \right] A_k(p) = H(p) - \sum_{k=1}^{\infty} \gamma_k \frac{\varphi_k + R_k(p)}{p+\mu_k}. \tag{24}$$

Multiplying this equation by p and introducing

$$d_0 := -\sum_{k=1}^{\infty} \gamma_k \varphi_k \neq 0 \qquad \text{(by assumption)}, \tag{25}$$

the fixed point equation for $M(p)$

$$M(p) = -\frac{1}{d_0} M(p) \sum_{k=1}^{\infty} \gamma_k B_k(p) + G_0(p) \tag{26}$$

follows where $B_k = B_k[M]$ are given by (22) and

$$G_0(p) = \frac{p}{d_0} \left(H(p) - \sum_{k=1}^{\infty} \gamma_k \frac{\varphi_k + R_k(p)}{p+\mu_k} \right). \tag{27}$$

2. Existence theorem in heat conduction

The fixed point equations (19) and (26) for $M(p)$ in both inverse problems have the form

$$M(p) = -\frac{1}{d_\nu} M(p) \sum_{k=1}^{\infty} \gamma_k \mu_k^\nu B_k[M](p) + G_\nu(p) \tag{28}$$

where $\nu = 1$ in the first inverse problem and $\nu = 0$ in the second one. The functions B_k are given by (22) with (23), the constants d_ν and functions G_ν are defined by (18), (25) and (21), (27), respectively.

We are looking for analytic solutions $M(p)$ of equation (28) in a half-plane $\operatorname{Re} p > \sigma$ $(\sigma > 0)$. For this end we introduce the norm

$$\|f\|_{\gamma,\sigma} := \sup_{\operatorname{Re} p > \sigma} [|p|^\gamma |f(p)|] \qquad (\gamma \geq 0, \sigma > 0) \tag{29}$$

and the Banach space of complex-valued functions in $\operatorname{Re} p > \sigma$ $(\sigma > 0)$

$$\mathcal{A}_{\gamma,\sigma} = \Big\{ f : f(p) \text{ holomorphic on } \operatorname{Re} p > \sigma \text{ with } \|f\|_{\gamma,\sigma} < \infty \Big\} \quad (\gamma \geq 0).$$

We note that $\mathcal{A}_{\gamma,\sigma} \subset \mathcal{A}_{\gamma,\sigma'}$ with $\|\cdot\|_{\gamma,\sigma'} \leq \|\cdot\|_{\gamma,\sigma}$ for $\sigma' > \sigma$.

Let now α and β be two real numbers such that

$$0 < \beta \leq 1 \qquad \text{and} \qquad 1 < \alpha \leq 1 + \beta. \tag{30}$$

Further, let $w = w(t)$ $(t > 0)$ be a real-valued function with Laplace transform $W(p) = \mathcal{L}_{t \to p} w$ satisfying the condition

$$W \in \mathcal{A}_{\beta,\sigma_0} \qquad \text{with some } \sigma_0 > 0. \tag{31}$$

For instance, w could have power-type singularities at $t = 0$ as in the example

$$w(t) = \sum_{i=1}^{n} c_i t^{-\beta_i} e^{-s_i t}, \qquad W(p) = \sum_{i=1}^{n} c_i \frac{\Gamma(1-\beta_i)}{(p+s_i)^{1-\beta_i}}$$

with real constants $c_i, s_i \geq 0$ and $0 \leq \beta_i \leq 1 - \beta$. Finally, we introduce the space for the solutions $M(p)$

$$\mathcal{M}_{W,\sigma} = \Big\{ M : M(p) = W(p) + N(p) \text{ where } N \in \mathcal{A}_{\alpha,\sigma} \Big\} \qquad (\sigma \geq \sigma_0).$$

Then the following *existence theorem* holds.

Theorem 1. *Let beside* (30) *and* (31) *the following assumptions be fulfilled for* $\nu \in \{0,1\}$*:*

$$d_\nu = (-1)^{1+\nu} \sum_{k=1}^{\infty} \gamma_k \mu_k^\nu \varphi_k \neq 0 \tag{32}$$

$$\sum_{k=1}^{\infty} |\gamma_k| \mu_k^{1+\nu} |\varphi_k| < \infty \tag{33}$$

$$\Phi_k \in \mathcal{A}_{1,\sigma_0}, \quad \sum_{k=1}^{\infty} |\gamma_k| \mu_k^\nu \|\Phi_k\|_{1,\sigma_0} < \infty \tag{34}$$

for Φ_k *defind by* (23) *and* $\sigma_0 > 0$ *from* (31) *and*

$$G_\nu = W + K_\nu \in \mathcal{M}_{W,\sigma_0}. \tag{35}$$

Then there exists $\sigma_1 \geq \sigma_0$ *such that equation* (28) *has a unique solution* $M = W + N \in \mathcal{M}_{W,\sigma_1}$*.*

Proof. At first we state some auxiliary inequalities for p from $\operatorname{Re} p > \sigma > 0$. We have the obvious inequalities

$$|p + \mu_k| \geq |p| > \sigma \qquad \text{if } \operatorname{Re} p > \sigma > 0 \tag{36}$$

$$\left| \frac{\mu_k}{p + \mu_k} \right| < \frac{\mu_k}{\sigma + \mu_k} < 1 \qquad \text{if } \operatorname{Re} p > \sigma > 0 \tag{37}$$

and from (36) it follows that

$$\left| \frac{1}{p + \mu_k} \right| |p|^{1-\beta} \leq \frac{1}{|p|^\beta} < \frac{1}{\sigma^\beta} \qquad \text{if } \operatorname{Re} p > \sigma > 0. \tag{38}$$

Now, denoting $N = M - W$ and observing $K_\nu = G_\nu - W$ from (35), equation (28) reduces to the equation for N

$$N = A\,N \tag{39}$$

where

$$(AN)(p) = -\frac{1}{d_\nu} \big(N(p) + W(p)\big) \sum_{k=1}^{\infty} \gamma_k \mu_k^\nu B_k[N + W](p) + K_\nu(p). \tag{40}$$

The theorem holds if we prove that equation (39) has a unique solution $N \in \mathcal{A}_{W,\sigma_1}$ for some $\sigma_1 \geq \sigma_0$.

In the sequel we show that the operator A is a contraction in the balls

$$D_{\alpha,\sigma}(\rho) = \big\{ N \in \mathcal{A}_{\alpha,\sigma} : \|N\|_{\alpha,\sigma} \leq \rho \big\}$$

for suitably chosen $\sigma \geq \sigma_0$ and $\rho > 0$. For $N \in D_{\alpha,\sigma}(\rho)$ we estimate the B_k from (22). Observing inequalities (36) - (38) and the assumption $\Phi_k \in \mathcal{A}_{1,\sigma_0}$, we obtain

$$\begin{aligned} \|B_k[N+W]\|_{1,\sigma} &\leq \frac{\mu_k |\varphi_k| \left(\frac{\|W\|_{\beta,\sigma}}{\sigma^\beta} + \frac{\|N\|_{\alpha,\sigma}}{\sigma^\alpha} \right) + \|\Phi_k\|_{1,\sigma}}{1 - \frac{\|W\|_{\beta,\sigma}}{\sigma^\beta} - \frac{\|N\|_{\alpha,\sigma}}{\sigma^\alpha}} \\ &\leq \frac{\mu_k |\varphi_k| \left(\frac{\|W\|_{\beta,\sigma}}{\sigma^\beta} + \frac{\rho}{\sigma^\alpha} \right) + \|\Phi_k\|_{1,\sigma}}{1 - \frac{\|W\|_{\beta,\sigma}}{\sigma^\beta} - \frac{\rho}{\sigma^\alpha}} \end{aligned} \tag{41}$$

if $\sigma \geq \sigma_0$ and $\frac{\|W\|_{\beta,\sigma}}{\sigma^\beta} + \frac{\rho}{\sigma^\alpha} < 1$. From (41) and assumptions (33) - (35) of the theorem the estimation of the norm of AN

$$\|AN\|_{\alpha,\sigma} \leq \frac{1}{|d_\nu|} \frac{1}{1 - \frac{\|W\|_{\beta,\sigma}}{\sigma^\beta} - \frac{\rho}{\sigma^\alpha}} \left(\frac{\|W\|_{\beta,\sigma}}{\sigma^{1+\beta-\alpha}} + \frac{\rho}{\sigma} \right)$$
$$\times \left[\left(\frac{\|W\|_{\beta,\sigma}}{\sigma^\beta} + \frac{\rho}{\sigma^\alpha} \right) \sum_{k=1}^\infty |\gamma_k| \mu_k^{1+\nu} |\varphi_k| + \sum_{k=1}^\infty |\gamma_k| \mu_k^\nu \|\Phi_k\|_{1,\sigma} \right] + \|K_\nu\|_{\alpha,\sigma}$$

follows if $\sigma \geq \sigma_0$ and $\frac{\|W\|_{\beta,\sigma}}{\sigma^\beta} + \frac{\rho}{\sigma^\alpha} < 1$. For every $\rho > \rho_0 = \|K_\nu\|_{\alpha,\sigma_0}$ we can then choose $\sigma_2 = \sigma_2(\rho) \geq \sigma_0$ such that

$$\|AN\|_{\alpha,\sigma} \leq \rho \qquad \text{if } \sigma \geq \sigma_2(\rho) \text{ and } \rho > \rho_0. \tag{42}$$

Moreover, AN is a holomorphic function on $\operatorname{Re} p > \sigma_2(\rho)$. This follows from the holomorphy of $N \in D_{\alpha,\sigma}(\rho), W \in \mathcal{A}_{\beta,\sigma_0}, \Phi_k \in \mathcal{A}_{1,\sigma_0}, K_\nu \in \mathcal{A}_{\alpha,\sigma_0}, B_k \in \mathcal{A}_{1,\sigma}$ and from the uniform convergence of the series in (40) which can be seen from (41) and assumptions (33) and (34). Therefore by (42) we have

$$A : D_{\alpha,\sigma}(\rho) \to D_{\alpha,\sigma}(\rho) \qquad \text{if } \sigma \geq \sigma_2(\rho), \rho > \rho_0. \tag{43}$$

Next we show that the operator A is a contraction in $D_{\alpha,\sigma}(\rho)$ for $\sigma \geq \sigma_3(\rho)$ with some $\sigma_3(\rho) \geq \sigma_0$. From (22) the difference of B_k for N_1 and N_2 is given by

$$B_k[N_1 + W] - B_k[N_2 + W] = \frac{\frac{\mu_k}{p+\mu_k}(\varphi_k + \Phi_k)(N_1 - N_2)}{\left[1 - \frac{\mu_k}{p+\mu_k}(W + N_1)\right]\left[1 - \frac{\mu_k}{p+\mu_k}(W + N_2)\right]}.$$

In view of (36) - (38) for $N_1, N_2 \in D_{\alpha,\sigma}(\rho)$ $(\sigma \geq \sigma_0)$ we obtain

$$\big\|B_k[N_1 + W] - B_k[N_2 + W]\big\|_{1,\sigma} \leq \frac{\frac{|\varphi_k|}{\sigma^{\alpha-1}} + \frac{\|\Phi_k\|_{1,\sigma}}{\sigma^\alpha}}{(1 - \frac{\|W\|_{\beta,\sigma}}{\sigma^\beta} - \frac{\rho}{\sigma^\alpha})^2} \|N_1 - N_2\|_{\alpha,\sigma} \tag{44}$$

if again $\sigma \geq \sigma_0$ and $\frac{\|W\|_{\beta,\sigma}}{\sigma^\beta} - \frac{\rho}{\sigma^\alpha} < 1$.

From (41) and (44) for the difference of AN_1 and AN_2 by (40) we derive the estimation

$$\begin{aligned}
&\|AN_1 - AN_2\|_{\alpha,\sigma} \\
&\quad \leq \frac{1}{|d_\nu|} \frac{\|N_1 - N_2\|_{\alpha,\sigma}}{\sigma} \sum_{k=1}^\infty |\gamma_k| \mu_k^\nu \, \|B_k[N_1 + W]\|_{1,\sigma} \\
&\quad\quad + \frac{1}{|d_\nu|} \left(\frac{\|W\|_{\beta,\sigma}}{\sigma^{1+\beta-\alpha}} + \frac{\|N_2\|_{\alpha,\sigma}}{\sigma} \right) \sum_{k=1}^\infty |\gamma_k| \mu_k^\nu \big\|B_k[N_1 + W] - B_k[N_2 + W]\big\|_{1,\sigma} \\
&\quad \leq q(\sigma, \rho) \|N_1 - N_2\|_{\alpha,\sigma}
\end{aligned}$$

where

$$q(\sigma,\rho)=\frac{1}{|d_\nu|}\left\{\frac{\left(\frac{\|W\|_{\beta,\sigma}}{\sigma^\beta}+\frac{\rho}{\sigma^\alpha}\right)\sum_{k=1}^{\infty}|\gamma_k|\mu_k^{1+\nu}|\varphi_k|+\sum_{k=1}^{\infty}|\gamma_k|\mu_k^{\nu}\,\|\Phi_k\|_{1,\sigma}}{\sigma\left(1-\frac{\|W\|_{\beta,\sigma}}{\sigma^\beta}-\frac{\rho}{\sigma^\alpha}\right)}\right.$$
$$\left.+\frac{\left(\frac{\|W\|_{\beta,\sigma}}{\sigma^{1+\beta-\alpha}}+\frac{\rho}{\sigma}\right)\left(\frac{1}{\sigma^{\alpha-1}}\sum_{k=1}^{\infty}|\gamma_k|\mu_k^{\nu}|\varphi_k|+\frac{1}{\sigma^{\alpha}}\sum_{k=1}^{\infty}|\gamma_k|\mu_k^{\nu}\,\|\Phi_k\|_{1,\sigma}\right)}{\left(1-\frac{\|W\|_{\beta,\sigma}}{\sigma^\beta}-\frac{\rho}{\sigma^\alpha}\right)^2}\right\}.$$

For every $\rho>0$ there exists $\sigma_3=\sigma_3(\rho)\geq\sigma_0$ such that

$$\frac{\|W\|_{\beta,\sigma}}{\sigma^\beta}+\frac{\rho}{\sigma^\alpha}<1 \quad\text{and}\quad q(\sigma,\rho)<1 \qquad \text{if } \sigma\geq\sigma_3(\rho)$$

and A is a contraction in $D_{\alpha,\sigma}(\rho)$ for $\sigma\geq\sigma_3(\rho)$. This together with (43) proves that equation (39) possesses a unique solution in every ball $D_{\alpha,\sigma}(\rho)$ for $\rho>\rho_0$ and $\sigma\geq\sigma_4(\rho)=\max\{\sigma_2(\rho),\sigma_3(\rho)\}$. Therefore a solution M of equation (28) in the space $\mathcal{M}_{\mathcal{W},\sigma_\infty}$ with $\sigma_1=\min_{\rho>\rho_0}\sigma_4(\rho)>0$ exists.

Finally, we show the uniqueness of the solution M of equation (28) in the whole space $\mathcal{M}_{W,\sigma_1}$. Let M_1 and M_2 be two solutions of equation (28) in $\mathcal{M}_{W,\sigma_1}$ and let $N_1=M_1-W$ and $N_2=M_2-W$ the corresponding solutions of equation (39) in $\mathcal{A}_{\alpha,\sigma_1}$. Let us fix some $\rho_1>\max\{\|N_1\|_{\alpha,\sigma_1},\|N_2\|_{\alpha;\sigma_1},\rho_0\}$. Then $\|N_i\|_{\alpha,\sigma_1}\leq\rho_1\ \ (i=1,2)$. By the monotonicity of the norm $\|\cdot\|_{\alpha,\sigma}$ with respect to σ and $\sigma_4(\rho_1)\geq\sigma_1$ we have $\|N_i\|_{\alpha,\sigma_4(\rho_1)}\leq\|N_i\|_{\alpha,\sigma_1}\leq\rho_1$. Hence $N_i\in D_{\alpha,\sigma_4(\rho_1)}(\rho_1)\ \ (i=1,2)$. In this ball uniqueness of the solution N has already been shown. This proves that $N_1=N_2$ and consequently $M_1=M_2$. Theorem 1 is completely proved ∎

Remark 1. If the relation

$$K_\nu(p)=\frac{1}{d_\nu}W(p)\sum_{k=1}^{\infty}\gamma_k\mu_k^\nu B_k[W](p)$$

between the functions $W(p)$ and $K_\nu(p)$, respectively $G_\nu(p)$ holds, equation (39) with (40) has the unique solution $N=0$, i.e. $M=W(p)$ is the solution to equation (28).

Corollary 1. *Under the assumptions of Theorem* 1 *the inverse problems* (1) - (4) *and* (1) - (3), (5) *have the unique solutions* m *of the form*

$$m(t)=w(t)+\frac{1}{2\pi i}\int_{\zeta-i\gamma}^{\zeta+i\gamma}e^{tp}N(p)dp \qquad (\zeta>\sigma_1) \tag{45}$$

with $N\in\mathcal{A}_{\alpha,\sigma_1}$.

Since $\alpha > 1$ by (30), this follows from a known inversion formula for the Laplace transform (cf. [2: Theorem 21.3]). From representation (43) we have an estimation for m of the form

$$|m(t) - w(t)| \le C_1 e^{t\sigma_1} \qquad (t > 0)$$

with positive constants C_1 and σ_1 and the limit (cf. [2: Theorem 24.5])

$$m(t) - w(t) \to 0 \qquad \text{for } t \to +0.$$

By (45) also stability estimates for m could be derived via stability estimates for N obtainable in usual way.

From the physical point of view the existence of memory kernels m with the stronger property of exponential decay (instead of the proven exponential growth) is important. For this it has to be shown that the solutions M of equations (28) can be analytically continued into a half-plane of the form $\mathrm{Re}\, p > \sigma_1$ with some $\sigma_1 < 0$. But no simple sufficient conditions on the data are known which guarantee this and also other basic physically relevant properties as positiveness and monotonous decreasing of the memory kernel m.

Further we discuss *assumptions* (34) and (35) of Theorem 1. For this end we state two lemmata which can be easily proved using integration by parts and Hölder's inequality.

Lemma 1. *Let g be an absolutely continuous function on $t \ge 0$ satisfying*

$$g(0) = 0 \qquad \text{and} \qquad e^{-\sigma t} g'(t) \in L_\gamma(0,\infty) \;\; (\gamma = \tfrac{1}{2-\alpha}) \tag{46}$$

for some $\sigma \ge 0$ and $\alpha > 1$. Then $\mathcal{L}_{t\to p}(g) \in \mathcal{A}_{\alpha,\sigma}$.

Lemma 2. *Let g be an absolutely continuous function on $t \ge 0$ satisfying*

$$e^{-\sigma t} g'(t) \in L_1(0,\infty) \tag{47}$$

for some $\sigma \ge 0$. Then $\mathcal{L}_{t\to p}(g) \in \mathcal{A}_{1,\sigma}$ and we have the estimation

$$\|\mathcal{L}_{t\to p}(g)\|_{1,\sigma} \le |g(0)| + \int_0^\infty e^{-\sigma t} |g'(t)|\, dt. \tag{48}$$

From definition (23) of Φ_k and (36) we obtain

$$\|\Phi_k\|_{1,\sigma_0} \le \|R_k\|_{1,\sigma_0} + 2\mu_k |\varphi_k|. \tag{49}$$

Let now the functions r_k be absolutely continuous on $t \ge 0$ and satisfying

$$e^{-\sigma_0 t} \dot{r}_k(t) \in L_1(0,\infty). \tag{50}$$

Then by Lemma 2 we have $R_k \in \mathcal{A}_{1,\sigma_0}$ and hence also $\Phi_k \in \mathcal{A}_{1,\sigma_0}$. Further, by (48) and (49)

$$\begin{aligned}\sum_{k=1}^\infty |\gamma_k| \mu_k^\nu \|\Phi_k\|_{1,\sigma_0} &\le \sum_{k=1}^\infty |\gamma_k| \mu_k^\nu \left(|r_k(0)| + \int_0^\infty e^{-\sigma_0 t} |\dot{r}_k(t)|\, dt \right) + 2 \sum_{k=1}^\infty |\gamma_k| \mu_k^{1+\nu} |\varphi_k| \\ &< \infty\end{aligned}$$

follows if (33) and the condition

$$\sum_{k=1}^{\infty} |\gamma_k| \mu_k^{\nu} \left(|r_k(0)| + \int_0^{\infty} e^{-\sigma_0 t} |\dot{r}_k(t)|\, dt \right) < \infty \tag{51}$$

hold. So (34) is satisfied if beside (33) the assumptions (50) and (51) are fulfilled.

Condition (35) means that $K_\nu \in \mathcal{A}_{\alpha,\sigma_0}$. For $\nu = 0$ by (27) this is equivalent to the condition

$$G_0(p) - W(p) \in \mathcal{A}_{\alpha,\sigma_0} \tag{52}$$

where

$$G_0(p) = \frac{1}{d_0} \left[\mathcal{L}_{t\to p}(h_0) + h(0) - \sum_{k=1}^{\infty} \gamma_k \varphi_k \right] \tag{53}$$

with

$$h_0(t) = h'(t) + \sum_{k=1}^{\infty} \gamma_k \mu_k \varphi_k e^{-\mu_k t} - \sum_{k=1}^{\infty} \gamma_k r_k(t) + \sum_{k=1}^{\infty} \gamma_k \mu_k \int_0^t e^{-\mu_k (t-\tau)} r_k(\tau)\, d\tau. \tag{54}$$

Here we used the known relation $\mathcal{L}(e^{-\mu_k t}) = \frac{1}{p+\mu_k}$ and the formulas for the Laplace transform of derivative and convolution.

By (31) with $\beta \le 1$ and (52) with $\alpha > 1$ it follows that it must be $\lim_{\mathrm{Re}\, p\to\infty} G_0(p) = 0$ which by (53) implies the compatibility condition (cf. [21])

$$h(0) = \sum_{k=1}^{\infty} \gamma_k \varphi_k. \tag{55}$$

If this is fulfilled, we can apply Lemma 1 to the function $g(t) = \frac{1}{d_0} h_0(t) - w(t)$. Therefore the absolute continuity of g and the further compatibility condition

$$\frac{1}{d_0} h_0(t) - w(t) \Big|_{t=0} = 0 \tag{56}$$

and the condition

$$e^{-\sigma_0 t} \left(\frac{1}{d_0} h_0(t) - w(t) \right)' \in L_\gamma(0,\infty) \qquad (\gamma = \tfrac{1}{2-\alpha}) \tag{57}$$

are sufficient for (52) and hence for (35) in the case $\nu = 0$.

Analogously, for $\nu = 1$ in (35) we have to insure the condition

$$G_1(p) - W(p) \in \mathcal{A}_{\alpha,\sigma_0} \tag{58}$$

where by (21)

$$G_1(p) = \frac{1}{d_1}\left[\mathcal{L}_{t\to p}(h_1) + p\left(h(0) - \sum_{k=1}^{\infty}\gamma_k\varphi_k\right)\right.$$
$$\left. + \left(h'(0) + \sum_{k=1}^{\infty}\gamma_k\mu_k\varphi_k - \sum_{k=1}^{\infty}\gamma_k r_k(0)\right)\right] \tag{59}$$

with

$$h_1(t) = h''(t) - \sum_{k=1}^{\infty}\gamma_k\mu_k^2\varphi_k e^{-\mu_k t} + \sum_{k=1}^{\infty}\gamma_k\mu_k r_k(t)$$
$$- \sum_{k=1}^{\infty}\gamma_k\dot{r}_k(t) - \sum_{k=1}^{\infty}\gamma_k\mu_k^2\int_0^t e^{-\mu_k(t-\tau)}r_k(\tau)d\tau. \tag{60}$$

This leads to the compatibility conditions (cf. [21] again)

$$\left.\begin{aligned} h(0) &= \sum_{k=1}^{\infty}\gamma_k\varphi_k \\ h'(0) + \sum_{k=1}^{\infty}\gamma_k\mu_k\varphi_k &= \sum_{k=1}^{\infty}\gamma_k r_k(0). \end{aligned}\right\} \tag{61}$$

If in addition the function $\hat{g}(t) = \frac{1}{d_1}h_1(t) - w(t)$ is absolutely continuous and the further compatibility condition

$$\frac{1}{d_1}h_1(t) - w(t)\Big|_{t=0} = 0 \tag{62}$$

and the summability condition

$$e^{-\sigma_0 t}\left(\frac{1}{d_1}h_1(t) - w(t)\right)' \in L_\gamma(0,\infty) \qquad (\gamma = \tfrac{1}{2-\alpha}) \tag{63}$$

are satisfied, relation (58) follows. This means that assumptions (61) - (63) are sufficient for (35) in the case $\nu = 1$.

We summarize these results in

Theorem 2. *Let beside* (30) *and* (31) *assumptions* (32), (33) *and* (50), (51) *for* $\nu = 0, 1$ *as well as conditions* (55) - (57) *with* (54) *in the case* $\nu = 0$ *and* (61) - (63) *with* (60) *in the case* $\nu = 1$ *be satisfied. Then the inverse problems* (1) - (4) *for* $\nu = 1$ *and* (1) - (3), (5) *for* $\nu = 0$ *have the unique solutions* m *of the form* (45) *where* $N \in \mathcal{A}_{\alpha,\sigma_1}$ *with some* $\sigma_1 \geq \sigma_0$*.*

3. Statement of problems in viscoelasticity

We deal with the linear hyperbolic integro-differential equation

$$\rho u_{tt} - \operatorname{div}(\lambda \nabla u) + \int_0^t m(t-\tau) \operatorname{div}(\lambda \nabla u(x,\tau))\, d\tau = f \tag{64}$$

in the cylinder $\Omega = D \times \{t > 0\}$, where again D is a bounded domain in $\mathbb{R}^N$ with piecewise smooth boundary S and ρ, λ are given positive continuous functions on D, f is a given continuous function on Ω (cf. [13] or [21]).

In the case $N = 1$ equation (64) appears for inelastic wave propagation in a material governed by the Boltzmann stress-strain relation

$$\sigma(x,t) = \lambda(x) \left(\varepsilon(x,t) - \int_0^t m(t-\tau) \varepsilon(x,\tau)\, d\tau \right)$$

between the strain ε and the stress σ, where $\varepsilon = \frac{\partial u}{\partial x}$ with the displacement u which is supposed as zero if $t < 0$. Then ρ denotes the mass density and f the force density.

Besides equation (64) the function u satisfies the initial and boundary conditions

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x) \qquad \text{on } D \tag{65}$$

$$u(x,t) = 0 \ \text{ or } \ \lambda \frac{\partial u}{\partial n} + \mu u = 0 \qquad \text{on } \Sigma = S \times \{t > 0\} \tag{66}$$

with given continuous functions φ, ψ on D and $\mu \geq 0$ on S, where n again denotes the outer normal to S.

In the *inverse problem* we have to find the kernel m such that the corresponding function u satisfies equations (64) - (66) and an additional condition of the form (4) or (5). In particular, (4) now contains the case of displacement observation and (5) the case of the observation of the traction.

The solution u of the *direct problem* (64) - (66) is again taken in the form of the Fourier series (6) with β replaced by ρ in equation (7) for the eigenfunctions. In view of (64) and (65) the coefficient functions a_k in (6) now satisfy the initial value problem

$$\left.\begin{aligned} \ddot{a}_k(t) + \mu_k a_k(t) - \mu_k \int_0^t m(t-\tau) a_k(\tau)\, d\tau = r_k(t) \\ a_k(0) = \varphi_k, \ \dot{a}_k(0) = \psi_k \end{aligned}\right\} \tag{67}$$

where

$$\left.\begin{aligned} \varphi(x) &= \sum_{k=1}^{\infty} \varphi_k v_k(x) \\ \psi(x) &= \sum_{k=1}^{\infty} \psi_k v_k(x) \\ r(x,t) \equiv \frac{f}{\rho} &= \sum_{k=1}^{\infty} r_k(t) v_k(x). \end{aligned}\right\}$$

The additional conditions (4) and (5) again take the form (10) and (11), respectively.

As above in the heat conduction problem at this moment we suppose only the existence of the Fourier coefficients φ_k, ψ_k and $r_k(t)$ shifting specifying conditions on them to assumptions (81), (82) for the inverse problems in Theorem 3. For an absolutely continuous kernel m the conditions $\varphi \in \overset{\circ}{H}{}^1(D)$ or $\varphi \in H^1(D)$, for the first or third boundary condition respectively, $\psi \in L_2(D)$ and $r \in L_2(\Omega_T)$ are sufficient for the existence of a solution $u \in H^1(\Omega_T)$ of the direct problem (64) - (66) for any $T > 0$.

Applying the Laplace transform to (67), we obtain the equation

$$p^2 A_k(p) - p\varphi_k - \psi_k + \mu_k A_k(p) - \mu_k M(p) A_k(p) = R_k(p) \tag{68}$$

where A_k, M and R_k are the Laplace transforms of a_k, m and r_k, respectively. From (68) the relations for A_k

$$A_k(p) = \frac{\mu_k}{p^2 + \mu_k} M(p) A_k(p) + \frac{p\varphi_k + \psi_k + R_k(p)}{p^2 + \mu_k} \tag{69}$$

and

$$A_k(p) = \frac{1}{1 - \frac{\mu_k}{p^2+\mu_k} M(p)} \cdot \frac{p\varphi_k + \psi_k + R_k(p)}{p^2 + \mu_k} \tag{70}$$

follow. Equations (10) and (11) for the additional conditions (4) and (5) in the inverse problems are transformed to the equations (15) and (16) again.

In the *first inverse problem* with condition (4), from (15) observing (69) we obtain the equation for $M(p)$

$$M(p) \sum_{k=1}^{\infty} \gamma_k \frac{\mu_k}{p^2 + \mu_k} A_k(p) = H(p) - \sum_{k=1}^{\infty} \gamma_k \frac{p\varphi_k + \psi_k + R_k(p)}{p^2 + \mu_k}.$$

Multiplying this equation by p^3 and introducing d_1 by (18) again, we get the equation for $M(p)$

$$M(p) = -\frac{1}{d_1} M(p) \sum_{k=1}^{\infty} \gamma_k \mu_k B_k(p) + G_1(p) \tag{71}$$

where

$$B_k(p) = B_k[M](p) = \frac{p^3}{p^2 + \mu_k} A_k(p) - \varphi_k \tag{72}$$

with $A_k = A_k[M]$ given by (70) and

$$G_1(p) = \frac{p^3}{d_1} \left(H(p) - \sum_{k=1}^{\infty} \gamma_k \frac{p\varphi_k + \psi_k + R_k(p)}{p^2 + \mu_k} \right). \tag{73}$$

By (72) and (70) the explicit expression for B_k

$$B_k(p) = \frac{1}{1 - \frac{\mu_k}{p^2+\mu_k} M(p)} \left[\frac{\mu_k}{p^2 + \mu_k} \varphi_k M(p) + \Phi_k(p) \right] \tag{74}$$

follows where

$$\Phi_k(p) = \frac{p^3}{(p^2+\mu_k)^2}\big(\psi_k + R_k(p)\big) - \frac{2\mu_k p^2 + \mu_k^2}{(p^2+\mu_k)^2}\varphi_k. \tag{75}$$

In the *second inverse problem* with condition (5), from (16) and (69) the equation for $M(p)$ writes

$$M(p)\sum_{k=1}^{\infty}\gamma_k\left[\frac{\mu_k}{p^2+\mu_k} - 1\right]A_k(p) = H(p) - \sum_{k=1}^{\infty}\gamma_k\frac{p\varphi_k + \psi_k + R_k(p)}{p^2+\mu_k}. \tag{76}$$

Multiplying this equation by p and introducing d_0 by (25) again, the equation for $M(p)$

$$M(p) = -\frac{1}{d_0}M(p)\sum_{k=1}^{\infty}\gamma_k B_k(p) + G_0(p) \tag{77}$$

follows where $B_k = B_k[M]$ are given by (74) and

$$G_0(p) = \frac{p}{d_0}\left(H(p) - \sum_{k=1}^{\infty}\gamma_k\frac{p\varphi_k + \psi_k + R_k(p)}{p^2+\mu_k}\right). \tag{78}$$

4. Existence theorem in viscoelasticity

Equations (71) and (77) for $M(p)$ have the fixed point form

$$M(p) = -\frac{1}{d_\nu}M(p)\sum_{k=1}^{\infty}\gamma_k\mu_k^\nu B_k[M](p) + G_\nu(p) \tag{79}$$

where again $\nu = 1$ in the first inverse problem and $\nu = 0$ in the second one. The functions B_k are given by (74) with (75), the constants d_ν and functions G_ν are defined by (18), (25) and (73), (78), respectively.

Let α be a real number with $1 < \alpha \le 2$. We introduce the solution space

$$\mathcal{N}_{c,\sigma} = \left\{M :\, M(p) = \frac{c}{p} + N(p) \text{ where } N \in \mathcal{A}_{\alpha,\sigma}\right\} \qquad (\sigma > 0)$$

for given real constant c, where the space $\mathcal{A}_{\alpha,\sigma}$ is defined as above in the case of heat conduction. The choice of $\mathcal{N}_{c,\sigma}$ means that in the viscoelastic case we deal with smooth memory kernels only, for reasons of simplicity.

Then the following *existence theorem* holds.

Theorem 3. *Let be* $1 < \alpha \le 2$ *and let the following assumptions be fulfilled for* $\nu \in \{0,1\}$:

$$d_\nu = (-1)^{1+\nu} \sum_{k=1}^{\infty} \gamma_k \mu_k^\nu \varphi_k \neq 0 \tag{80}$$

$$\sum_{k=1}^{\infty} |\gamma_k| \mu_k^{1+\nu} |\varphi_k| < \infty \tag{81}$$

$$\Phi_k \in \mathcal{A}_{1,\sigma_0}, \ \sum_{k=1}^{\infty} |\gamma_k| \mu_k^\nu \|\Phi_k\|_{1,\sigma_0} < \infty \tag{82}$$

for Φ_k *defined by* (75) *and some* $\sigma_0 > 0$, *and*

$$G_\nu = \frac{c}{p} + K_\nu \in \mathcal{N}_{c,\sigma_0}. \tag{83}$$

Then there exists $\sigma_1 \ge \sigma_0$ *such that equation* (79) *has a unique solution* $M = \frac{c}{p} + N \in \mathcal{N}_{c,\sigma_1}$.

Proof. At first we prove two further auxiliary inequalities for p from $\mathrm{Re}\, p > \sigma > 0$. We have

$$\begin{aligned} |p^2 + \mu_k|^2 &= (\mathrm{Re}^2 p - \mathrm{Im}^2 p + \mu_k)^2 + 4(\mathrm{Re}\, p \,\mathrm{Im}\, p)^2 \\ &= \mathrm{Re}^2 p(\mathrm{Re}^2 p + 2\mathrm{Im}^2 p + 2\mu_k) + (\mathrm{Im}^2 p - \mu_k)^2 \\ &\ge \mathrm{Re}^2 p(|p|^2 + 2\mu_k) \end{aligned}$$

implying the inequality

$$|p^2 + \mu_k| > \sigma \sqrt{|p|^2 + 2\mu_k} > \max\{\sigma\sqrt{2\mu_k}, \sigma^2\} \tag{84}$$

for $\mathrm{Re}\, p > \sigma > 0$. Further, by the first inequality of (84), $|p^2 + \mu_k| > \sigma |p|$ holds. Hence we obtain

$$\left| \frac{1}{p} \frac{\mu_k}{p^2 + \mu_k} \right| = \left| \frac{1}{p} - \frac{p}{p^2 + \mu_k} \right| \le \frac{1}{|p|} + \frac{|p|}{|p^2 + \mu_k|} < \frac{1}{|p|} + \frac{|p|}{\sigma |p|} < \frac{2}{\sigma},$$

i.e.

$$\left| \frac{1}{p} \frac{\mu_k}{p^2 + \mu_k} \right| < \frac{2}{\sigma} \qquad \text{if } \mathrm{Re}\, p > \sigma > 0. \tag{85}$$

Now, denoting $N = M - \frac{c}{p}$ and observing (83), equation (79) reduces to the equation for N

$$N = AN \tag{86}$$

where

$$(AN)(p) = -\frac{1}{d_\nu} \Big(N(p) + \frac{c}{p} \Big) \sum_{k=1}^{\infty} \gamma_k \mu_k^\nu B_k \Big[N + \frac{c}{p} \Big](p) + K_\nu(p). \tag{87}$$

As in the proof of Theorem 1 we will show that the operator A is a contraction in the balls $D_{\alpha,\sigma}(\rho)$ for suitably chosen parameter $\sigma \ge \sigma_0$ and radius $\rho > 0$.

For $N \in D_{\alpha,\sigma}(\rho)$ we estimate the B_k from (74). Due to the assumption $\Phi_k \in \mathcal{A}_{1,\sigma_0}$ and inequality (84) we have

$$\begin{aligned} \left\| B_k\Big[N + \frac{c}{p}\Big]\right\|_{1,\sigma} &\leq \frac{\mu_k|\varphi_k|\left(\frac{|c|}{\sigma^2} + \frac{\|N\|_{\alpha,\sigma}}{\sigma^{1+\alpha}}\right) + \|\Phi_k\|_{1,\sigma}}{1 - 2\left(\frac{|c|}{\sigma} + \frac{\|N\|_{\alpha,\sigma}}{\sigma^\alpha}\right)} \\ &\leq \frac{\mu_k|\varphi_k|\left(\frac{|c|}{\sigma^2} + \frac{\rho}{\sigma^{1+\alpha}}\right) + \|\Phi_k\|_{1,\sigma}}{1 - 2\left(\frac{|c|}{\sigma} + \frac{\rho}{\sigma^\alpha}\right)} \end{aligned} \tag{88}$$

if $\sigma \geq \sigma_0$ and $\frac{|c|}{\sigma} + \frac{\rho}{\sigma^\alpha} < \frac{1}{2}$. From (88) and assumptions (81) - (83) of the theorem the estimation of the norm of AN

$$\begin{aligned} \|AN\|_{\alpha,\sigma} \leq & \frac{1}{|d_\nu|}\frac{1}{1-2\left(\frac{|c|}{\sigma} + \frac{\rho}{\sigma^\alpha}\right)}\left(\frac{|c|}{\sigma^{2-\alpha}} + \frac{\rho}{\sigma}\right) \\ & \times\left[\left(\frac{|c|}{\sigma^2} + \frac{\rho}{\sigma^{1+\alpha}}\right)\sum_{k=1}^{\infty}|\gamma_k|\mu_k^{1+\nu}|\varphi_k| + \sum_{k=1}^{\infty}|\gamma_k|\mu_k^{\nu}\|\Phi_k\|_{1,\sigma}\right] + \|K_\nu\|_{\alpha,\sigma} \end{aligned}$$

follows if $\sigma \geq \sigma_0$ and $\frac{|c|}{\sigma} + \frac{\rho}{\sigma^\alpha} < \frac{1}{2}$. For every $\rho > \rho_0 = \|K_\nu\|_{\alpha,\sigma_0}$, we can then choose $\sigma_2 = \sigma_2(\rho) \geq \sigma_0$ such that

$$\|AN\|_{\alpha,\sigma} \leq \rho \qquad \text{if } \sigma \geq \sigma_2(\rho) \text{ and } \rho > \rho_0. \tag{89}$$

Furthermore, as in the proof of Theorem 1 we can show that AN is a holomorphic function on $\operatorname{Re} p > \sigma_2(\rho)$. Therefore, by (89) we again have

$$A:\, D_{\alpha,\sigma}(\rho) \to D_{\alpha,\sigma}(\rho) \qquad \text{if } \sigma \geq \sigma_2(\rho) \quad \text{and} \quad \rho > \rho_0. \tag{90}$$

For proving that A is a contraction in $D_{\alpha,\sigma}(\rho)$ for $\sigma \geq \sigma_3(\rho)$ with some $\sigma_3(\rho) \geq \sigma_0$, we estimate the difference of B_k for N_1 and N_2 by (74) and using the inequalities (84) and (85). This gives

$$\left\| B_k\Big[N_1 + \frac{c}{p}\Big] - B_k\Big[N_2 + \frac{c}{p}\Big]\right\|_{1,\sigma} \leq \frac{\frac{\mu_k|\varphi_k|}{\sigma^{1+\alpha}} + \frac{2\|\Phi_k\|_{1,\sigma}}{\sigma^\alpha}}{\left[1 - 2\left(\frac{|c|}{\sigma} + \frac{\rho}{\sigma^\alpha}\right)\right]^2}\|N_1 - N_2\|_{\alpha,\sigma} \tag{91}$$

if again $\sigma \geq \sigma_0$ and $\frac{|c|}{\sigma} + \frac{\rho}{\sigma^\alpha} < \frac{1}{2}$. From (88) and (91) for the difference of AN_1 and AN_2 in $D_{\alpha,\sigma}(\rho)$ by (87) we have the estimation

$$\|AN_1 - AN_2\|_{\alpha,\sigma} \leq q(\sigma,\rho)\|N_1 - N_2\|_{\alpha,\sigma}$$

where

$$\begin{aligned} q(\sigma,\rho) = \frac{1}{|d_\nu|}\Bigg\{ & \frac{\left(\frac{|c|}{\sigma^2} + \frac{\rho}{\sigma^{1+\alpha}}\right)\sum_{k=1}^\infty|\gamma_k|\mu_k^{1+\nu}|\varphi_k| + \sum_{k=1}^\infty|\gamma_k|\mu_k^\nu\|\Phi_k\|_{1,\sigma}}{\sigma\left[1 - 2\left(\frac{|c|}{\sigma} + \frac{\rho}{\sigma^\alpha}\right)\right]} \\ & + \frac{\left(\frac{|c|}{\sigma^{2-\alpha}} + \frac{\rho}{\sigma}\right)\left(\frac{1}{\sigma^{1+\alpha}}\sum_{k=1}^{\infty}|\gamma_k|\mu_k^{1+\nu}|\varphi_k| + \frac{2}{\sigma^\alpha}\sum_{k=1}^{\infty}|\gamma_k|\mu_k^{\nu}\|\Phi_k\|_{1,\sigma}\right)}{\left[1 - 2\left(\frac{|c|}{\sigma} + \frac{\rho}{\sigma^\alpha}\right)\right]^2}\Bigg\}. \end{aligned}$$

For every $\rho > 0$ there exists $\sigma_3 = \sigma_3(\rho) \geq \sigma_0$ such that

$$\frac{|c|}{\sigma} + \frac{\rho}{\sigma^\alpha} < \frac{1}{2} \quad \text{and} \quad q(\sigma,\rho) < 1 \qquad \text{if } \sigma \geq \sigma_3(\rho)$$

and A is a contraction in $D_{\alpha,\sigma}(\rho)$ for $\sigma \geq \sigma_3(\rho)$. This together with (90) implies that equation (86) possesses a unique solution in every ball $D_{\alpha,\sigma}(\rho)$ for $\rho > \rho_0$ and $\sigma \geq \sigma_4(\rho) = \max\{\sigma_2(\rho), \sigma_3(\rho)\}$. Therefore a solution M of equation (79) in the space $\mathcal{N}_{c,\sigma_1}$ with $\sigma_1 = \min_{\rho>\rho_0} \sigma_4(\rho) > 0$ exists for which uniqueness can be shown as in the proof of Theorem 1. So Theorem 3 is completely proved ∎

Corollary 2. *Under the assumptions of Theorem* 2 *the inverse problems* (64) - (66), (4) *and* (64) - (66), (5) *have the unique solution m of the form*

$$m(t) = c + \frac{1}{2\pi i} \int_{\zeta-i\infty}^{\zeta+i\infty} e^{tp} N(p)\, dp \qquad (\zeta > \sigma_1) \tag{92}$$

with $N \in \mathcal{A}_{\alpha,\sigma_1}$.

From (92) the relation $m(0) = c$ and the estimation $|m(t)| \leq C_1 e^{t\sigma_1}$ $(t > 0)$ follow with positive constants C_1 and σ_1. By (92) also stability estimates for m could be derived via deriving stability estimates for N.

As in the heat conduction case above no simple sufficient conditions on the data are known which guarantee positiveness and monotonous exponentially decreasing of m.

Further, we discuss *assumptions* (82) and (83) of Theorem 3. For estimating the function Φ_k defined in (75) we at first show that

$$\left| \frac{p^3}{(p^2+\mu_k)^2} \right| \leq C_2(1+\sqrt{\mu_k}) \qquad \text{for } \operatorname{Re} p > \sigma_0 \tag{93}$$

with some positive constants σ_0 and $C_2 = C_2(\sigma_0)$. Using inequalies (84) and (85), and $2|p|\sqrt{\mu_k} \leq |p|^2 + \mu_k$, we have

$$\begin{aligned} \left| \frac{p^3}{(p^2+\mu_k)^2} \right| &= \left| \frac{1}{p} - \frac{2p\mu_k}{(p^2+\mu_k)^2} - \frac{\mu_k^2}{p(p^2+\mu_k)^2} \right| \\ &\leq \frac{1}{|p|} + \frac{(|p|^2+\mu_k)\sqrt{\mu_k}}{\sigma_0^2(|p|^2+2\mu_k)} + \frac{\mu_k}{\sigma_0\sqrt{|p|^2+2\mu_k}} \left| \frac{\mu_k}{p(p^2+\mu_k)} \right| \\ &\leq \frac{1}{|p|} + \frac{\sqrt{\mu_k}}{\sigma_0^2} + \frac{\mu_k}{\sigma_0\sqrt{2\mu_k}} \cdot \frac{2}{\sigma_0} \\ &\leq \frac{1}{\sigma_0} + \frac{\sqrt{\mu_k}}{\sigma_0^2}(1+\sqrt{2}) \end{aligned}$$

from which (93) follows. By (93) and again (84) and (85) now we obtain

$$
\begin{aligned}
|p|\,|\Phi_k(p)| &= \left| \frac{p^3}{(p^2+\mu_k)^2} p\, R_k(p) + \left(1 - \frac{2p^2\mu_k + \mu_k^2}{(p^2+\mu_k)^2}\right)\psi_k - \frac{2\mu_k p^3 + \mu_k^2 p}{(p^2+\mu_k)^2}\varphi_k \right| \\
&\le C_2(1+\sqrt{\mu_k})|p|\,|R_k(p)| + \left(1 + \frac{2}{\sigma_0^2}\mu_k\right)|\psi_k| \\
&\quad + \mu_k \left[2C_2(1+\sqrt{\mu_k}) + \frac{1}{\sigma_0^2}\frac{1}{2}\sqrt{\mu_k}\right]|\varphi_k| \\
&\le C_3 \left[(1+\sqrt{\mu_k})|p|\,|R_k(p)| + (1+\mu_k)|\psi_k| + \mu_k^{\frac{3}{2}}|\varphi_k|\right].
\end{aligned}
$$

Consequently, we have the estimation

$$
\|\Phi_k\|_{1,\sigma_0} \le C_3[1+\sqrt{\mu_k})\|R_k\|_{1,\sigma_0} + (1+\mu_k)|\psi_k| + \mu_k^{\frac{3}{2}}|\varphi_k|] \tag{94}
$$

with some positive constant C_3.

Let now the functions r_k be absolutely continuous on $t \ge 0$ and satisfying (50). Then by Lemma 2 again we have $R_k \in \mathcal{A}_{1,\sigma_0}$ and hence also $\Phi_k \in \mathcal{A}_{1,\sigma_0}$. Further, by (48) and (94), $\sum_{k=1}^{\infty} |\gamma_k|\mu_k^{\nu}\|\Phi_k\|_{1,\sigma_0} < \infty$ follows if (81) and the conditions

$$
\left.
\begin{aligned}
\sum_{k=1}^{\infty} |\gamma_k|\mu_k^{\frac{1}{2}+\nu}\left(|r_k(0)| + \int_0^{\infty} e^{-\sigma_0 t}|\dot r_k(t)|dt\right) &< \infty \\
\sum_{k=1}^{\infty} |\gamma_k|\mu_k^{\frac{3}{2}+\nu}|\varphi_k| &< \infty \\
\sum_{k=1}^{\infty} |\gamma_k|\mu_k^{1+\nu}|\psi_k| &< \infty
\end{aligned}
\right\} \tag{95}
$$

hold. So (82) (and (81)) are satisfied if assumptions (50) and (95) are fulfilled.

Finally, as in the case of heat conduction we can show that (83) is satisfied if some compatibility conditions and a summability condition are fulfilled. In the case $\nu = 0$ we introduce the function

$$
\begin{aligned}
h_2(t) &= h'(t) + \sum_{k=1}^{\infty} \gamma_k \mu_k^{\frac{1}{2}}\varphi_k \sin\sqrt{\mu_k}t \\
&\quad - \sum_{k=1}^{\infty}\gamma_k\psi_k\cos\sqrt{\mu_k}t - \sum_{k=1}^{\infty}\gamma_k\int_0^t r_k(\tau)\cos\sqrt{\mu_k}(t-\tau)d\tau.
\end{aligned} \tag{96}
$$

Then in the case $\nu = 0$ the formula for c

$$
c = \frac{1}{d_0}\left(h'(0) - \sum_{k=1}^{\infty}\gamma_k\psi_k\right), \tag{97}
$$

the compatibility condition (cf. [21])

$$h(0) = \sum_{k=1}^{\infty} \gamma_k \varphi_k \tag{98}$$

and the summability condition

$$e^{-\sigma_o t} h_2'(t) \in L_\gamma(0,\infty) \qquad (\gamma = \tfrac{1}{2-\alpha}) \tag{99}$$

are sufficient for (83).

In the case $\nu = 1$ we define the function

$$\begin{aligned} h_3(t) = h'''(t) &- \sum_{k=1}^{\infty} \gamma_k \mu_k^{\frac{3}{2}} \varphi_k \sin \sqrt{\mu_k} t + \sum_{k=1}^{\infty} \gamma_k \mu_k \psi_k \cos \sqrt{\mu_k} t \\ &+ \sum_{k=1}^{\infty} \gamma_k \mu_k \int_0^t r_k(\tau) \cos \sqrt{\mu_k}(t-\tau)\, d\tau - \sum_{k=1}^{\infty} \gamma_k \dot{r}_k(t). \end{aligned} \tag{100}$$

Then in the case $\nu = 1$ the formula for c

$$c = \frac{1}{d_1} \left(h'''(0) + \sum_{k=1}^{\infty} \gamma_k \mu_k \psi_k - \sum_{k=1}^{\infty} \gamma_k \dot{r}_k(0) \right), \tag{101}$$

the compatibility conditions (cf. [21] again)

$$\left.\begin{aligned} h(0) &= \sum_{k=1}^{\infty} \gamma_k \varphi_k \\ h'(0) &= \sum_{k=1}^{\infty} \gamma_k \psi_k \\ h''(0) &= \sum_{k=1}^{\infty} \gamma_k [r_k(0) - \mu_k \varphi_k] \end{aligned}\right\} \tag{102}$$

and the summability condition

$$e^{-\sigma_0 t} h_3'(t) \in L_\gamma(0,\infty) \qquad (\gamma = \tfrac{1}{2-\alpha}) \tag{103}$$

are sufficient for (83).

Summing up we obtain

Theorem 4. *Let be $1 < \alpha \le 2$ and beside (80) the assumptions (50) and (95) for $\nu = 0, 1$ as well as the conditions (98) and (99) with (96) in the case $\nu = 0$ and the conditions (102) and (103) with (100) in the case $\nu = 1$ be satisfied. Then the inverse problems (64) - (66), (4) for $\nu = 1$ and (64) - (66), (5) for $\nu = 0$ have the unique solution m of the form (92), where $N \in \mathcal{A}_{\alpha,\sigma_1}$ with some $\sigma_1 \ge \sigma_0$ and the constant $c = m(0)$ is given by (101) in the case $\nu = 1$ and (97) in the case $\nu = 0$.*

Acknowledgement. The authors thank the referee for several hints improving the readability of the paper.

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Received 18.06.1999