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Identity Surfaces

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Abstract. It is well-known that the zeros of holomorphic functions in more than one complex variable are not isolated. Nevertheless, there exist so-called identity surfaces such that a holomorphic function vanishes identically everywhere if only it equals zero on an identity surface. In the paper identity surfaces will be constructed using the technique of completely integrable overdetermined systems of partial differential equations. Moreover, identity surfaces will be constructed not only for holomorphic functions but also for solutions of more general first order systems of partial differential equations.
The present paper deals only with systems with real-analytic coefficients and, therefore, the classical Cauchy-Kovalevskaya and Holmgren theorems are applicable (while many recent papers deal with infinitely differentiable coefficients or they solve initial value problems with generalized analytic initial functions). Using the compatibility conditions of an overdetermined system, in the present paper the construction of identity surfaces (of minimal dimension) is carried out as some kind of inverse problem to an initial value problem.

Keywords: *Overdetermined systems of partial differential equations, compatibility conditions, initial value problems*

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1. Preliminaries on overdetermined first order systems

Let $x = (x_1, ..., x_m)$ be a point in $\mathbb{R}^m$, and let u_σ $(\sigma = 1, ..., s)$ be real- or complex-valued functions depending on x. Let k be a given natural number. Suppose the functions $u_\sigma = u_\sigma(x)$ satisfy $r = k \cdot s$ linear differential equations of the form

$$\sum_{\sigma,\mu} A_{\varrho\sigma\mu}(x) \frac{\partial u_\sigma}{\partial x_\mu} = 0 \tag{1}$$

where $\varrho = 1, ..., r$, $\sigma = 1, ..., s$, $\mu = 1, ..., m$ and the given coefficients $A_{\varrho\sigma\mu}$ possess local power series representations in their variables. The natural number $k - 1$ is called the *degree of overdetermination.*

To be short consider system (1) for $s = 1$, i.e., consider

$$\sum_{\mu} A_{\varrho\mu}(x) \frac{\partial u}{\partial x_\mu} = 0$$

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where $\varrho = 1, ..., k$ and $u = u(x_1, ..., x_m)$ is a desired real- or complex-valued function. Then the degree of overdetermination equals $k - 1$. Introducing new variables

$$X_\mu = \Phi_\mu(x_1, ..., x_m) \qquad (\mu = 1, ..., m)$$

(with real-analytic Φ_μ), this system passes into

$$\sum_\nu B_{\varrho\nu}(X) \frac{\partial u}{\partial X_\nu} = 0 \qquad \text{where} \quad B_{\varrho\nu} = \sum_\mu A_{\varrho\mu} \frac{\partial \Phi_\nu}{\partial x_\mu}.$$

Now suppose that this system can be solved for $\frac{\partial u}{\partial X_\varrho}$ with $\varrho = 1, ..., r$. Then the system under consideration can be rewritten in the form

$$\frac{\partial u}{\partial X_\lambda} = \sum_{\mu=k+1}^{m} \alpha_{\lambda\mu}(X) \frac{\partial u}{\partial X_\mu} \qquad (\lambda = 1, ..., k, \quad k \leq m - 1). \tag{2}$$

Note that the $\alpha_{\lambda\mu}$ have local power-series representations. System (2) is overdetermined if $k > 1$. In the sequel we shall make use of the fact that the initial value problem

$$u(0, ..., 0, X_{k+1}, ..., X_m) = \varphi(X_{k+1}, ..., X_m) \tag{3}$$

is uniquely solvable provided the system is compatible (see, for instance, the book [2] of E. Goursat or the book [5]. [1)] In accordance with the general theory of overdetermined systems, the variables $X_1, ..., X_k$ in (2) are called *essential*, while $X_{k+1}, ..., X_m$ are called *parametric* ones. Let $u = u(X_1, ..., X_m) = u(X)$ be any twice continuously differentiable function, and consider the expressions

$$U_\lambda = \frac{\partial u}{\partial X_\lambda} - \sum_{\mu=k+1}^{m} \alpha_{\lambda\mu}(X) \frac{\partial u}{\partial X_\mu} \qquad (\lambda = 1, ..., k).$$

Obviously, $u = u(X)$ is a solution of system (2) if $U_\lambda \equiv 0$ for all $\lambda = 1, ..., k$.

If X_λ and X_κ are essential variables, then the expression

$$\frac{\partial U_\lambda}{\partial X_\kappa} - \frac{\partial U_\kappa}{\partial X_\lambda} \tag{4}$$

contains second order derivatives of $u(X)$ with respect to one parametric and one essential variable. Such second order derivatives, however, can be expressed by first order derivatives of the U_λ with respect to parametric variables. That way we see that expression (4) is a linear combination of parametric derivatives of the three functions U_κ, U_λ and u. An easy calculation shows that the coefficient of $\frac{\partial u}{\partial X_\mu}$ is given by

$$[*]_{\lambda,\kappa,\mu} = \left(\frac{\partial \alpha_{\kappa\mu}}{\partial X_\lambda} - \frac{\partial \alpha_{\lambda\mu}}{\partial X_\kappa} \right) + \sum_{\tilde{\mu}} \left(\alpha_{\kappa\tilde{\mu}} \frac{\partial \alpha_{\lambda\mu}}{\partial X_{\tilde{\mu}}} - \alpha_{\lambda\tilde{\mu}} \frac{\partial \alpha_{\kappa\mu}}{\partial X_{\tilde{\mu}}} \right). \tag{5}$$

System (2) is said to be *compatible* in the case $[*]_{\lambda,\kappa,\mu} \equiv 0$ for any $1 \leq \lambda, \kappa \leq k$ and any $k + 1 \leq \mu \leq m$.

The following statement is true.

[1)] Concerning compatible systems in several complex variables see, for instance, R. P. Gilbert and J. L. Buchanan [1] and [5].

Lemma. *The initial value problem* $(2)-(3)$ *is uniquely solvable provided the system is compatible and the initial function* $\varphi = \varphi(X_{k+1}, ..., X_m)$ *has local power-series representations.*

Sketch of the proof. The solution can be constructed by freezing the essential variables step by step, and by solving k Cauchy-Kovalevskaya problems in one essential variable each (see Goursat [2]). The uniqueness of the solution follows from Holmgren's theorem. The compatibility conditions $[*]_{\lambda,\kappa,\mu} \equiv 0$ imply that system (2) is satisfied (locally) not only in frozen variables but also in arbitrary ones. Concerning the classical Cauchy-Kovalevskaya and Holmgren theorems in one essential variable see, for instance, F. Treves [4], while the case of several essential variables can be reduced to the case of one variable by the above mentioned method of freezing variables ∎

Since $u \equiv 0$ is a solution of system (2) one has also the following statement which will be used for the construction of identity surfaces.

Corollary 1. *If the initial values* $\varphi(X_{k+1}, ..., X_m)$ *are identically equal to zero, then the solution* u *of the initial value problem* $(2)-(3)$ *vanishes everywhere.*

2. Identity surfaces for linear and homogeneous systems

Now consider system (1) for several desired (real- or complex-valued) functions $u_\sigma = u_\sigma(x)$. Introducing new coordinates

$$X_\mu = \Phi_\mu(x_1, ..., x_m),$$

one gets the new system

$$\sum_{\sigma,\mu} B_{\varrho\sigma\mu}(X)\frac{\partial u_\sigma}{\partial X_\mu} = 0$$

Suppose this system can be solved for the derivatives of all functions with respect to X_1, ..., X_k, $k \leq m-1$. Then one has

$$\frac{\partial u_\varrho}{\partial X_\lambda} = \sum_{\sigma,\mu} \alpha_{\varrho\lambda\sigma\mu}\frac{\partial u_\sigma}{\partial X_\mu}$$

where $\varrho, \sigma = 1, ..., s$, while $\lambda = 1, ..., k$ and $\mu = k+1, ..., m$. Denote the difference of the left- and right-hand sides of the last equation by $U_{\varrho\lambda}$. Then one has to calculate the expression

$$\frac{\partial U_{\varrho\lambda}}{\partial X_\kappa} - \frac{\partial U_{\varrho\kappa}}{\partial X_\lambda} \tag{6}$$

instead of (4). An easy calculation shows that $[*]_{\lambda,\kappa,\mu}$ is to be replaced by an expression $[*]_{\lambda,\kappa,\mu,\varrho,\sigma}$ which is given by

$$\left(\frac{\partial \alpha_{\varrho\kappa\sigma\mu}}{\partial X_\lambda} - \frac{\partial \alpha_{\varrho\lambda\sigma\mu}}{\partial X_\kappa}\right) + \sum_{\tilde\sigma,\tilde\mu}\left(\alpha_{\varrho\kappa\tilde\sigma\tilde\mu}\frac{\partial \alpha_{\tilde\sigma\lambda\sigma\mu}}{\partial X_{\tilde\mu}} - \alpha_{\varrho\lambda\tilde\sigma\tilde\mu}\frac{\partial \alpha_{\tilde\sigma\kappa\sigma\mu}}{\partial X_{\tilde\mu}}\right).$$

While $[*]_{\lambda,\kappa,\mu,\varrho,\sigma}$ comes from the coefficient $\frac{\partial u_\sigma}{\partial X_\mu}$ in expression (6), in the case of several desired functions an additional term $[**]_{\lambda,\kappa,\mu,\tilde{\mu},\varrho,\sigma}$ (coming from $\frac{\partial^2 u_\sigma}{\partial X_\mu \partial X_{\tilde{\mu}}}$) has to vanish identically. This expression is given by

$$\sum_{\tilde{\sigma}} \Big(\alpha_{\varrho\kappa\sigma\mu}\alpha_{\sigma\lambda\tilde{\sigma}\tilde{\mu}} - \alpha_{\varrho\lambda\sigma\mu}\alpha_{\sigma\kappa\tilde{\sigma}\tilde{\mu}} \Big).$$

Theorem. *Suppose the degree of overdetermination of the system*

$$\sum_{\sigma,\mu} A_{\varrho\sigma\mu}(x)\frac{\partial u_\sigma}{\partial x_\mu} = 0$$

is $k-1,\ k \geq 2$. *Suppose, further, that the compatibility conditions*

$$\left.\begin{aligned} [*]_{\lambda,\kappa,\mu,\varrho,\sigma} &= 0 \\ [**]_{\lambda,\kappa,\mu,\tilde{\mu},\varrho,\sigma} &= 0 \end{aligned}\right\}$$

are satisfied. Then the $(m-k)$*-dimensional surface defined by*

$$\Phi_\mu(x_1, ..., x_m) = 0 \qquad (\mu = 1, ..., k)$$

is an identity surface.

Notice that in the case of one desired (real- or complex-valued) function ($s = 1$) the conditions $[**]_{\lambda,\kappa,\mu,\tilde{\mu},\varrho,\sigma} = 0$ can be omitted, one has also the following

Corollary 2. *In the case* $s = 1$ *the above theorem is true if only the compatibility conditions* $[*]_{\lambda,\kappa,\mu,\varrho,\sigma} = 0$ *are satisfied.*

Moreover, since the compatibility conditions $[*]_{\lambda,\kappa,\mu,\varrho,\sigma} = 0$ are always satisfied if the $\alpha_{\varrho\lambda\sigma\mu}$ are constant, the following corollary is true, too.

Corollary 3. *Consider a system*

$$\sum_{\mu} A_{\varrho\mu}\frac{\partial u}{\partial x_\mu} = 0$$

with constant coefficients whose degree of overdetermination is equal to $k-1$, *i.e.,* $\varrho = 1, ..., k,\ k \geq 1$. *Choose constants* $c_{\mu\nu}$ $(\mu, \nu = 1, ..., k)$ *such that its determinant is different from zero. Then for any choice of further constants* $c_{0\mu}$ *the* $(m-k)$*-dimensional plane*

$$\sum_{\nu} c_{\mu\nu}x_\nu = c_{0\mu} \qquad (\mu = 1, ..., k)$$

is an identity plane.

The last corollary shows the following connexion between the degree of overdetermination and the dimension of an identity surface:

Corollary 4. *The higher the degree of overdetermination, the smaller the dimension of an identity surface.*

Note that this statement is true also for more general systems with non-constant coefficients.

3. Identity surfaces for holomorphic functions in several variables

Denote the variables in $\mathbb{C}^{2n}$ by $z_j = x_j + iy_j$ and $\zeta_j = \xi_j + i\eta_j \quad (j = 1, ..., n)$. Suppose the complex-valued function $w = w(z_1, ..., z_n, \zeta_1, ..., \zeta_n)$ is holomorphic, i.e., the Cauchy-Riemann system

$$\left.\begin{aligned} \frac{\partial w}{\partial x_j} + i\frac{\partial w}{\partial y_j} = 0 \\ \frac{\partial w}{\partial \xi_j} + i\frac{\partial w}{\partial \eta_j} = 0 \end{aligned}\right\}$$

is satisfied.

Corollary 5. *The complex-n-dimensional conjugate complex diagonal surface defined by* $z_j = \overline{\zeta}_j \quad (j = 1, ..., n)$ *is an identity plane.*

Proof. Introducing new (real) variables

$$\begin{aligned} X_j &= x_j - \xi_j \qquad & \Xi_j &= \xi_j \\ Y_j &= y_j & H_j &= y_j + \eta_j \end{aligned}$$

the above diagonal surface is given by the equations

$$\left.\begin{aligned} X_j = 0 \\ H_j = 0 \end{aligned}\right\}$$

while the Cauchy-Riemann system passes into

$$\frac{\partial w}{\partial X_j} = \frac{1}{2}\frac{\partial w}{\partial \Xi_j} - \frac{i}{2}\frac{\partial w}{\partial Y_j} \tag{7}$$

$$\frac{\partial w}{\partial H_j} = \frac{i}{2}\frac{\partial w}{\partial \Xi_j} - \frac{1}{2}\frac{\partial w}{\partial Y_j}. \tag{8}$$

The variables X_j and H_j are thus essential, whereas Ξ_j and Y_j turn out to be parametric. Denote the differences of the left- and right-hand sides of equations (7) and (8) by W_{1j} and W_{2j}, and consider the expressions

$$\frac{\partial W_{1j}}{\partial X_k} - \frac{\partial W_{1k}}{\partial X_j}, \qquad \frac{\partial W_{1j}}{\partial H_k} - \frac{\partial W_{2k}}{\partial X_j}, \qquad \frac{\partial W_{2j}}{\partial H_k} - \frac{\partial W_{2k}}{\partial H_j}$$

which are analogous to (6). Using system (7) - (8) it follows that these expression vanish identically for any twice continuously differentiable functions depending on X_j, Y_j, Ξ_j and H_j. Hence system (7) - (8) is compatible. This proves the corollary ∎

Obviously, an analytic set is not an identity surface for holomorphic functions in several complex variables. In order to explain why the method of compatible differential equations does not work in this case, consider the following

Example 1. Since $w(z,\zeta) = z - \zeta$ vanishes only on the complex-one-dimensional diagonal surface defined by $z = \zeta$, this plane cannot be an identity surface.

Indeed, introduce new variables

$$\begin{aligned} X &= x - \xi \qquad & \Xi &= \xi \\ Y &= y \qquad & H &= y - \eta \end{aligned}$$

where again $z = x+iy$ and $\zeta = \xi + i\eta$. Then the diagonal surface $z = \zeta$ can be described by $X = 0$ and $H = 0$. Rewriting the Cauchy-Riemann system in the new coordinates X, Y, Ξ, H, one gets

$$\left.\begin{aligned} \frac{\partial w}{\partial X} + i\frac{\partial w}{\partial Y} + i\frac{\partial w}{\partial H} &= 0 \\ -\frac{\partial w}{\partial X} + \frac{\partial w}{\partial \Xi} - i\frac{\partial w}{\partial H} &= 0. \end{aligned}\right\}$$

The method of compatible systems is not applicable because the last system cannot be solved for $\frac{\partial w}{\partial X}$ and $\frac{\partial w}{\partial H}$. On the contrary, the corresponding system (7) - (8) can be solved for these derivatives in the case one considers the conjugate complex diagonal surface $z = \overline{\zeta}$. Therefore, the question whether a surface is an identity surface or not is reduced to the question if the rewritten Cauchy-Riemann system can be solved for the corresponding derivatives or not.

The next example shows how this connexion between identity surfaces and solvability can be used for the construction of identity surfaces.

Example 2. For holomorphic functions in two complex variables z and ζ the plane defined by

$$2z = (1-c)\zeta + (1+c)\overline{\zeta} \tag{9}$$

is an identity surface if c is any real constant different from -1.

Indeed, introducing new variables

$$\begin{aligned} X &= x - \xi \qquad & \Xi &= \xi \\ Y &= y \qquad & H &= y + c\eta \end{aligned}$$

one sees that the Cauchy-Riemann system can be solved for $\frac{\partial w}{\partial X}$ and $\frac{\partial w}{\partial H}$ if and only if $c \neq -1$ and, further, that the new system is compatible. The statement of Example 2 is true because equation (9) is equivalent to

$$\left.\begin{aligned} x - \xi &= 0 \\ y + c\eta &= 0. \end{aligned}\right\}$$

Note, finally, that for $c = -1$ the plane (9) is the diagonal surface $z = \zeta$ which is not an identity surface in accordance with Example 1.

4. Non-overdetermined systems

The above considerations are also applicable to systems whose degree of overdetermination is equal to zero. This will be shown for linear systems of the form

$$A_{j1}(x,y)\frac{\partial u}{\partial x}+A_{j2}(x,y)\frac{\partial u}{\partial y}+A_{j3}(x,y)\frac{\partial v}{\partial x}+A_{j4}(x,y)\frac{\partial v}{\partial y}=0 \qquad (j=1,2)$$

for two desired (real- or complex-valued) functions $u=u(x,y)$ and $v=v(x,y)$ in the (x,y)-plane. Introducing new coordinates

$$\left.\begin{aligned} X&=\Phi(x,y)\\ Y&=\Psi(x,y)\end{aligned}\right\}$$

the new system can be solved for $\frac{\partial u}{\partial Y}$ and $\frac{\partial v}{\partial Y}$ in the case

$$d_{13}\left(\frac{\partial\Psi}{\partial x}\right)^2+(d_{14}+d_{23})\frac{\partial\Psi}{\partial x}\frac{\partial\Psi}{\partial y}+d_{24}\left(\frac{\partial\Psi}{\partial y}\right)^2\neq 0 \tag{10}$$

where

$$d_{ij}=\begin{vmatrix} A_{1i} & A_{1j}\\ A_{2i} & A_{2j}\end{vmatrix}.$$

Provided the A_{ij} have power-series representations in x and y, the above considerations lead to the following

Corollary 6. *If condition* (10) *is satisfied, the curve* $\Psi(x,y)=0$ *is an identity line.*

Condition (10) is satisfied if the quadratic expression on the left-hand side is positive definite, and thus the following result is true:

Corollary 7. *Suppose*

$$(d_{14}+d_{23})^2<4d_{13}d_{24}. \tag{11}$$

Then each curve defined by $\Psi(x,y)=0$ *is an identity line.*

Consider the system

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=0 \tag{12}$$

$$\frac{\partial u}{\partial y}-\varepsilon\frac{\partial v}{\partial x}=0 \tag{13}$$

where ε is any real constant. In this case one has

$$\left.\begin{aligned} d_{12}=d_{24}&=1\\ d_{13}&=\varepsilon\\ d_{23}=d_{14}&=0\end{aligned}\right\} \tag{14}$$

and condition (11) reads $0<\varepsilon$, i.e., (11) is satisfied for positive ε only. For $\varepsilon=1$ the system is the Cauchy-Riemann system, and then Corollary 7 is obvious because the zeros of a non-constant holomorphic function are isolated. [2] In the case $\varepsilon\leq 0$, however, the existence of non-identity lines can be expected. Easy calculations show the existence of non-identity lines which are defined by linear functions:

[2] The same is true for generalized analytic functions, cf. I.N. Vekua's book [7]. Hence similar arguments can be applied to elliptic systems which are more general than (12) - (13).

Example 3. In the case $\varepsilon = 0$ system (12) - (13) has the solution

$$\left.\begin{aligned} u(x,y) &= 0 \\ v(x,y) &= cx \end{aligned}\right\}$$

where c is any constant. This solution vanishes on the y-axis, while it does not vanish identically.

Example 4. In the case $\varepsilon = -1$ special solutions of (12) - (13) are given by

$$\left.\begin{aligned} u(x,y) &= c(x+y) \\ v(x,y) &= c(x+y) \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} u(x,y) &= c(x-y) \\ v(x,y) &= -c(x-y) \end{aligned}\right\}$$

where c is any constant. These solutions are not identically equal to zero, although they vanish on the straight lines which are defined by $y = -x$ and $y = x$, respectively.

In view of (14) condition (10) passes into

$$\varepsilon \left(\frac{\partial \Psi}{\partial x} \right)^2 + \left(\frac{\partial \Psi}{\partial y} \right)^2 \neq 0. \tag{15}$$

Choosing $\Psi(x,y) = k_1 x + k_2 y$, the left-hand side of (10) equals zero for $k_2^2 = -\varepsilon k_1^2$ in accordance with Examples 3 and 4.

5. Concluding remarks

The present paper is not aimed at theorems which are as general as possible. On the contrary, the main goal is to explain how the technique of compatibility conditions can be used for getting uniqueness theorems for partial differential equations. Of course, there are various possible generalizations and applications such as the following:

Remark 1. A linear or non-linear system for s desired real- or complex-valued functions u_σ $(\sigma = 1, ..., s)$ is said to be *homogeneous* if $u_\sigma \equiv 0$ for $\sigma = 1, ..., s$ is a solution. Since Corollary 1 is true for arbitrary homogeneous systems, the above Theorem is also true for arbitrary non-linear systems if only they are homogeneous and compatibel. Note, however, that the formulation of the compatibility conditions is more complicated as in the linear case.

Remark 2. Initial value problems for (hyperbolic) first order systems can also be solved by methods of Clifford Analysis (see E. Obolashvili's book [3]). Consequently, identity surfaces can also be constructed using the corresponding uniqueness theorems. Since the paper [6] generalizes the results of [3] in the hyperbolic case, the methods of the paper [6] lead also to identity surfaces.

Remark 3. Notice, finally, that the methods of the present paper can also be used for the construction of solutions with prescribed initial values on identity surfaces. Therefore, identity surfaces can be compared with surfaces which are non-characteristic. This is also expressed by relations such as (10).

Remark 4. Similar constructions can also be carried out for overdetermined systems whose coefficients are not real-analytic.

Remark 5. Finally consider the following application of the above results to second order differential equations:

Let $w = w(z_1, \zeta_1, z_2, \zeta_2)$ be a complex-valued and continuously differentiable solution of the differential equation $\frac{\partial^2 w}{\partial z_1 \partial \zeta_1} + \frac{\partial^2 w}{\partial z_2 \partial \zeta_2} = 0$ in the unit ball $|z_1|^2 + |\zeta_1|^2 + |z_2|^2 + |\zeta_2|^2 < 1$ of $\mathbb{C}^4$ which is still continuous on its boundary. Such a solution is uniquely determined by its (complex) values on the boundary of the intersection of the unit ball with the conjugate complex diagonal surface $\zeta_1 = \overline{z}_1, \zeta_2 = \overline{z}_2$. Indeed, on the diagonal surface the given differential equation passes into the Laplace equation and thus a solution can be characterized by its boundary values. The above statement then follows from Corollary 5. Note, in addition, that to arbitrary (continuous) boundary values in the intersection there exists a uniquely determined solution in the complex-four-dimensional (open) unit ball having the prescribed boundary values in the real-four-dimensional intersection. The extension can be constructed by replacing the four real variables x_j and y_j $(j = 1, 2)$ by the four complex variables z_j and ζ_j where $2x_j = z_j + \zeta_j$ and $2y_j = i(\zeta_j - z_j)$.

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