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Asymptotics of Zeros of the Wright Function

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Abstract. The paper deals with the asymptotics of zeros of the Wright function

$$\phi(\rho,\beta;z)=\sum_{k=0}^{\infty}\frac{z^k}{k!\,\Gamma(\rho k+\beta)}\qquad(\rho>-1)$$

in the case the parameter β is a real number. The exact formulae for the order, the type and the indicator function of the entire function $\phi(\rho,\beta;z)$ are given for $\rho>-1$. On the basis of these results and using the obtained distribution of the zeros of the Wright function it is shown to be a function of completely regular growth.

Keywords: *Wright function, indicator function, asymptotics of zeros, entire functions of completely regular growth*

AMS subject classification: 33 E 20, 30 D 15, 30 E 15

1. Introduction

The entire function (of z)

$$\phi(\rho,\beta;z)=\sum_{k=0}^{\infty}\frac{z^k}{k!\,\Gamma(\rho k+\beta)}\qquad(\rho>-1,\beta\in\mathbb{C}),\tag{1}$$

named after the British mathematician Wright, was introduced by him for the first time in the case $\rho>0$ in the paper [17] in connection with his investigations on the asymptotic theory of partitions. In this paper and in the paper [18] he gave some elementary properties and the asymptotics of the function (1) in the case $\rho>0$. Later on, in the paper [19] Wright considered the entire function $\phi(\rho,\beta;z)$ in the case $-1<\rho<0$. In particular, he gave there its asymptotic behaviour in the complex plane $\mathbb{C}$ and showed that for $z\to\infty$ it is exponentially small in a suitable sector containing the negative real semi-axis, exponentially large in two neighbouring sectors and, if $-1<\rho<-\frac{1}{3}$, it has an algebraic expansion in a sector containing the positive real semi-axis.

The Wright function has found many other applications, first of all in the Mikusiński operational calculus and in the theory of integral transforms of Hankel type (see [6, 7,

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13 - 15]). Recently this function has appeared in papers related to partial differential equations of fractional order. Considering boundary-value problems for the fractional diffusion-wave equation, i.e., the linear partial integro-differential equation obtained from the classical diffusion or wave equation by replacing the first- or second-order time derivative by a fractional derivative of order α with $0 < \alpha \leq 2$, it was found that the corresponding Green functions can be represented in terms of the Wright function (1). A very informative survey of these results can be found in the paper by Mainardi [10]. The paper [11] contains a detailed discussion of the properties of the function $\phi(-\frac{\alpha}{2}, 1-\frac{\alpha}{2}; -z)$ $(0 < \alpha \leq 2)$ which was shown to be the Green function for the Cauchy problem for the time-fractional of order α diffusion-wave equation. Finally, in the recent papers [1, 5, 9] the scale-invariant solutions of some partial differential equations of fractional order have been given in terms of the Wright and the generalized Wright functions.

The above mentioned applications show the importance of the Wright function in different areas of mathematics. In this paper we obtain some new properties of the function $\phi(\rho, \beta; z)$ including the distribution of its zeros. Making use of the asymptotic formulae obtained by Wright in [18, 19] we give the explicit formulae for the indicator function of the Wright function showing that the Wright function is a function of completely regular growth. Taking as a pattern the analysis of the asymptotics of zeros of the generalized Mittag-Leffler function given by Djrbashian in [2: Chapter 1.2] we consider the problem of distribution of zeros of the Wright function. It turns out that in dependence of the value of the parameter $\rho > -1$ and the real parameter β there are the following five different situations:

1) For $\rho > 0$ all zeros with large enough absolute values are simple and are lying on the negative real semi-axis.

2) In the case $\rho = 0$ the Wright function is reduced to an exponential function with a constant factor (equal to zero if $\beta = -n, n \in \mathbb{N}_0$) and it has no zeros.

3) For $-\frac{1}{3} \leq \rho < 0$ all zeros with large enough absolute values are simple and are lying on the positive real semi-axis.

4) In the cases $\rho = -\frac{1}{2}, \beta = -n$ $(n \in \mathbb{N}_0)$ and $\rho = -\frac{1}{2}, \beta = \frac{1}{2} - n$ $(n \in \mathbb{N}_0)$ the Wright function has exactly $2n+1$ and $2n$ zeros, respectively.

5) For $-1 < \rho < -\frac{1}{3}$ (excluding the case 4) all zeros with large enough absolute values are simple and are lying in the neighbourhoods of the rays $\arg z = \pm\frac{1}{2}\pi(-1-3\rho)$.

In the cases 1, 3 and 5 the asymptotics of zeros of the Wright function is given. In the case 4 the function $\phi(\rho, \beta; z)$ is expressed as a product of an exponential function with a polynomial.

2. The indicator function of $\phi(\rho, \beta; z)$

We give at first the formulae for the order and type of the Wright function (1). In the case $-1 < \rho < 0$ they have been presented in [3: Chapter 1.3]; the case $\rho > 0$ is even more simpler. To get the result in this case we use the standard formulae for the order p and the type σ of an entire function f defined by the power series $f(z) = \sum_{k=0}^{\infty} c_n z^n$:

$$p = \limsup_{n\to\infty} \frac{n \log n}{\log \frac{1}{|c_n|}}, \qquad (\sigma e p)^{\frac{1}{p}} = \limsup_{n\to\infty} n^{\frac{1}{p}} \sqrt[n]{c_n}$$

and the Stirling asymptotic formula

$$\Gamma(z) = \sqrt{2\pi} z^{z-\frac{1}{2}} e^{-z} \big[1 + O\big(\tfrac{1}{z}\big)\big] \qquad \big(|\arg z| \le \pi - \varepsilon, \varepsilon > 0, |z| \to \infty\big).$$

After straightforward evaluations we arrive at the following result.

Theorem 1. *The Wright function $\phi(\rho, \beta; z)$ $(\rho > -1;\ \beta \ne -n\ (n \in \mathbb{N}_0)$ if $\rho = 0)$ is an entire function of finite order with the order p and the type σ given by*

$$p = \frac{1}{1+\rho}, \qquad \sigma = (1+\rho)|\rho|^{-\frac{\rho}{1+\rho}}. \tag{2}$$

Remark 1. In the case $\rho = 0$ the Wright function is reduced to the exponential function with the constant factor $\frac{1}{\Gamma(\beta)}$:

$$\phi(0, \beta; z) = \frac{\exp(z)}{\Gamma(\beta)} \tag{3}$$

which turns out to vanish identically for $\beta = -n, n \in \mathbb{N}_0$. For all other values of the parameter β and $\rho = 0$ formulas (2) (with $\sigma = \lim_{\rho\to 0}(1+\rho)|\rho|^{-\frac{\rho}{1+\rho}} = 1$) are still valid.

The basic characteristic of the growth of an entire function $f = f(z)$ of finite order p in different directions is its indicator function $h = h(\theta)$ $(|\theta| \le \pi)$ defined by

$$h(\theta) = \limsup_{r\to+\infty} \frac{\log |f(re^{i\theta})|}{r^p}. \tag{4}$$

The indicator function $h_\rho(\theta)$ of the entire function $\phi(\rho, \beta; z)$ is given by the following results.

Theorem 2. *Let $\rho > -1$ $(\beta \ne -n, n \in \mathbb{N}_0$ if $\rho = 0)$. Then the indicator function $h_\rho(\theta)$ of the Wright function $\phi(\rho, \beta; z)$ is given as follows:*

(a) *In the case $\rho \ge 0$ by*

$$h_\rho(\theta) = \sigma \cos p\theta \qquad (|\theta| \le \pi). \tag{5}$$

(b) *In the cases*

(i) $-\frac{1}{3} \le \rho < 0$

(ii) $\rho = -\frac{1}{2}, \beta = -n$ $(n \in \mathbb{N}_0)$

(iii) $\rho = -\frac{1}{2}, \beta = \frac{1}{2} - n \ (n \in \mathbb{N}_0)$

by

$$h_\rho(\theta) = \begin{cases} -\sigma \cos p(\pi + \theta) & \textit{for } -\pi \le \theta \le 0 \\ -\sigma \cos p(\theta - \pi) & \textit{for } 0 \le \theta \le \pi \end{cases} . \tag{6}$$

(c) *In the case* $-1 < \rho < -\frac{1}{3}$ $(\beta \neq -n, n \in \mathbb{N}_0$ *and* $\beta \neq \frac{1}{2} - n, n \in \mathbb{N}_0$ *if* $\rho = -\frac{1}{2})$ *by*

$$h_\rho(\theta) = \begin{cases} -\sigma \cos p(\pi + \theta) & \textit{for } -\pi \le \theta \le \frac{3}{2}\frac{\pi}{p} - \pi \\ 0 & \textit{for } |\theta| \le \pi - \frac{3}{2}\frac{\pi}{p} \\ -\sigma \cos p(\theta - \pi) & \textit{for } \pi - \frac{3}{2}\frac{\pi}{p} \le \theta \le \pi \end{cases} . \tag{7}$$

Here p *and* σ *are the order and type of the Wright function, respectively, defined by* (2).

Proof. To get formula (5) for $|\theta| < \pi$ we use the asymptotic expansion of the Wright function given in [18]:

$$\phi(\rho, \beta; z) = H(Z) \qquad (\rho > 0) \tag{8}$$

where

$$H(Z) = Z^{\frac{1}{2}-\beta} e^{\frac{1+\rho}{\rho} Z} \left\{ \sum_{m=0}^{M} (-1)^m \frac{a_m}{Z^m} + O\Big(\frac{1}{|Z|^{M+1}}\Big) \right\} \qquad (Z \to \infty) \tag{9}$$

with $Z = (\rho|z|)^{\frac{1}{\rho+1}} e^{i\frac{\theta}{\rho+1}}$ and $a_0 = (2\pi(1+\rho))^{-\frac{1}{2}} > 0$. This formula is valid if $\arg z = \theta \ \ (|\theta| \le \pi - \varepsilon, \varepsilon > 0)$. We then have

$$\begin{aligned} h_\rho(\theta) &= \limsup_{r \to +\infty} \frac{\log \left|\phi(\rho, \beta; re^{i\theta})\right|}{r^p} \\ &= \limsup_{r \to +\infty} \frac{\log \left|H((\rho r)^{\frac{1}{\rho+1}} e^{i\frac{\theta}{\rho+1}})\right|}{r^p} \\ &= \lim_{r \to +\infty} \frac{\log \Big((\rho r)^{\frac{\frac{1}{2}-\beta}{\rho+1}} e^{\frac{1+\rho}{\rho}(\rho r)^{\frac{1}{\rho+1}} \cos\frac{\theta}{\rho+1}} \big\{a_0 + O\big(r^{-\frac{1}{\rho+1}}\big)\big\}\Big)}{r^p} \\ &= \frac{1+\rho}{\rho} \rho^{\frac{1}{\rho+1}} \cos \frac{\theta}{\rho+1} \\ &= \sigma \cos(p\theta). \end{aligned} \tag{10}$$

Due to the fact that the indicator function of an entire function of finite order is continuous [4: Chapter 2.5] the obtained formula (5) is also valid for $|\theta| = \pi$.

To get formula (6) in the case (i) we use exactly the same reasoning as in the case $\rho > 0$ and the asymptotic expansion of the Wright function given in [19]:

$$\phi(\rho, \beta; z) = I(Y) \qquad (-1 < \rho < 0) \tag{11}$$

where

$$I(Y) = Y^{\frac{1}{2}-\beta} e^{-Y} \left\{ \sum_{m=0}^{M-1} A_m Y^{-m} + O(Y^{-M}) \right\} \qquad (Y \to \infty) \tag{12}$$

with

$$Y = (1+\rho)((-\rho)^{-\rho} y)^{\frac{1}{1+\rho}}, \qquad y = -z \;\; (-\pi < \arg z \le \pi, -\pi < \arg y \le \pi)$$

and $A_0 = (2\pi)^{-\frac{1}{2}}(-\rho)^{\frac{1}{2}-\beta}(1+\rho)^{\beta} > 0$. This formula is valid if $|\arg y| \le \min\{\frac{3}{2}\pi(1+\rho), \pi\} - \varepsilon, \varepsilon > 0$.

Let us consider the cases (ii) and (iii). We have in the case (ii)

$$\phi(-\tfrac{1}{2}, -n; z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\,\Gamma(-\frac{1}{2}k - n)} = \sum_{l=0}^{\infty} \frac{z^{2l+1}}{(2l+1)!\,\Gamma(-\frac{1}{2} - l - n)}. \tag{13}$$

We now use the Gauss-Legendre and the supplement formulae for the gamma function to represent the last series in formula (13) in terms of the hypergeometric function ${}_1F_1(z)$:

$$\begin{aligned} \phi(-\tfrac{1}{2}, -n; z) &= \sum_{l=0}^{\infty} \frac{\sqrt{\pi}}{2^{2l+1} l!\,\Gamma(l+\frac{3}{2})} \frac{\sin(\pi(n+l+\frac{3}{2}))\,\Gamma(\frac{3}{2}+n+l)}{\pi} z^{2l+1} \\ &= \frac{(-1)^{n+1} z}{2\sqrt{\pi}} \frac{\Gamma(\frac{3}{2}+n)}{\Gamma(\frac{3}{2})} \sum_{l=0}^{\infty} \frac{(\frac{3}{2}+n)_l}{(\frac{3}{2})_l} \frac{\left(-\frac{z^2}{4}\right)^l}{l!} \\ &= \frac{(-1)^{n+1} z}{\pi} \Gamma(\tfrac{3}{2}+n)\ {}_1F_1(\tfrac{3}{2}+n; \tfrac{3}{2}; -\tfrac{z^2}{4}). \end{aligned} \tag{14}$$

In the case (iii) using the same transformations we get

$$\phi(-\tfrac{1}{2}, \tfrac{1}{2} - n; z) = \frac{(-1)^n}{\pi} \Gamma(\tfrac{1}{2}+n)\ {}_1F_1(\tfrac{1}{2}+n; \tfrac{1}{2}; -\tfrac{z^2}{4}) \qquad (n \in \mathbb{N}_0). \tag{15}$$

We can rewrite formulas (14) - (15) by using the Kummer formula [12: Chapter 6]

$${}_1F_1(a;c;z) = e^z\ {}_1F_1(c-a;c;-z)$$

in the form

$$\phi(-\tfrac{1}{2}, -n; z) = e^{-\frac{1}{4}z^2} z P_n(z^2) \qquad (n \in \mathbb{N}_0) \tag{16}$$

$$\phi(-\tfrac{1}{2}, \tfrac{1}{2} - n; z) = e^{-\frac{1}{4}z^2} Q_n(z^2) \qquad (n \in \mathbb{N}_0) \tag{17}$$

where P_n, Q_n are polynomials of degree n defined as

$$\left.\begin{aligned} P_n(z) &= \frac{(-1)^{n+1}}{\pi} \Gamma(\tfrac{3}{2}+n)\ {}_1F_1(-n; \tfrac{3}{2}; \tfrac{z}{4}) \\ Q_n(z) &= \frac{(-1)^n}{\pi} \Gamma(\tfrac{1}{2}+n)\ {}_1F_1(-n; \tfrac{1}{2}; \tfrac{z}{4}) \end{aligned}\right\}.$$

Formulas (16) - (17) gives us the indicator function of the Wright function in the cases (ii) and (iii) in the form

$$h_\rho(\theta) = -\tfrac{1}{4}\cos(2\theta) \qquad (|\theta| \le \pi)$$

which is in accordance with formula (6).

Finally, to get formula (7) we use the standard formula (4), the asymptotic expansion (11) in the sector $|\arg(-z)| \le \frac{3}{2}\pi(1+\rho) - \varepsilon, \varepsilon > 0$ and the asymptotic expansion (see Wright [19])

$$\phi(\rho, \beta; z) = J(z) \qquad (-1 < \rho < -\tfrac{1}{3}) \tag{18}$$

where

$$J(z) = \sum_{m=0}^{M-1} \frac{z^{\frac{\beta-1-m}{-\rho}}}{(-\rho)\,\Gamma(m+1)\Gamma(1+\frac{\beta-m-1}{-\rho})} + O\big(z^{\frac{\beta-1-M}{-\rho}}\big) \qquad (z \to \infty) \tag{19}$$

in the sector $|\arg z| \le \frac{1}{2}\pi(-1-3\rho)-\varepsilon, \varepsilon > 0$. We note here that in the cases (ii) and (iii) (and only in these cases) all coefficients of the algebraic asymptotic expansion (19) are zeros. As we have seen (see formulas (16) - (17)) the Wright function is exponentially small in a suitable sector for these values of parameters ∎

Remark 2. It can be seen from formulas (5) - (6) that the indicator function $h_\rho(\theta)$ of the Wright function $\phi(\rho, \beta; z)$ is reduced to the function $\cos\theta$ (the indicator function of the exponential function e^z) if $\rho \to 0$. This property is not valid for another generalization of the exponential function – the Mittag-Leffler function defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \qquad (\alpha > 0, z \in \mathbb{C}). \tag{20}$$

Even though

$$E_1(z) = e^z,$$

the indicator function of the Mittag-Leffler function given for $0 < \alpha < 2, \alpha \ne 1$ by (see [4: Chapter 2.7])

$$h(\theta) = \begin{cases} \cos\frac{\theta}{\alpha} & \text{for } |\theta| \le \frac{\pi\alpha}{2} \\ 0 & \text{for } \frac{\pi\alpha}{2} \le |\theta| \le \pi \end{cases}$$

does not coincide with the indicator function of e^z if $\alpha \to 1$.

3. Asymptotics of zeros of $\phi(\rho, \beta; z)$

In the case $\rho = 0$ the Wright function is an exponential function with a constant factor (equal to zero if $\beta = -n, n \in \mathbb{N}_0$) and it has no zeros. For $\rho = -\frac{1}{2}, \beta = -n$ $(n \in \mathbb{N}_0)$ and $\rho = -\frac{1}{2}, \beta = \frac{1}{2} - n$ $(n \in \mathbb{N}_0)$ the Wright function is reduced to a product of an exponential function and a polynomial of the degree $2n + 1$ and $2n$ (see formulas (16) - (17)) and it has exactly $2n + 1$ and $2n$ zeros in the complex plane, respectively. For all other values of parameters the Wright function has an infinite number of zeros. We give the asymptotics of zeros of the Wright function in Theorem 3 (the case $\rho \geq -\frac{1}{3}$) and Theorem 4 (the case $-1 < \rho < -\frac{1}{3}$ ($\beta \neq -n$ $(n \in \mathbb{N}_0)$ and $\beta \neq \frac{1}{2} - n$ $(n \in \mathbb{N}_0)$ if $\rho = -\frac{1}{2}$)).

Theorem 3. *Let $\{\gamma_k\}_{k=1}^{\infty}$ be the sequence of zeros of the function $\phi(\rho, \beta; z)$ $(\rho \geq -\frac{1}{3}$ but $\rho \neq 0; \beta \in \mathbb{R})$, where $|\gamma_k| \leq |\gamma_{k+1}|$ and each zero is counted according to its multiplicity. Then:*

(A) *In the case $\rho > 0$ all zeros with large enough k are simple and are lying on the negative real semi-axis. The asymptotic formula*

$$\gamma_k = -\left(\frac{\pi k + \pi(p\beta - \frac{p-1}{2})}{\sigma \sin \pi p}\right)^{\frac{1}{p}} \left(1 + O(k^{-2})\right) \qquad (k \to +\infty) \tag{21}$$

is true. Here and in the next formulae p and σ are the order and type of the Wright function given by (2), *respectively.*

(B) *In the case $-\frac{1}{3} \leq \rho < 0$ all zeros with large enough k are simple, lying on the positive real semi-axis and the asymptotic formula*

$$\gamma_k = \left(\frac{\pi k + \pi(p\beta - \frac{p-1}{2})}{-\sigma \sin \pi p}\right)^{\frac{1}{p}} \left(1 + O(k^{-2})\right) \qquad (k \to +\infty)$$

is true.

Proof. We consider at first the case (A). It follows from the asymptotic formula (8) that all zeros of the function $\phi(\rho, \beta; z)$ with large enough index are lying in the sector $|\arg(-z)| \leq \varepsilon, \varepsilon > 0$ containing the semi-axis $(-\infty, 0]$.

We now prove the fact that the function $\phi(\rho, \beta; z)$ has exactly a countable set of zeros on the negative real half-axis. To do this we use the asymptotic expansion of the Wright function given by Wright [19]:

$$\phi(\rho, \beta; z) = H(Z_1) + H(Z_2) \tag{23}$$

where

$$\left.\begin{aligned} Z_1 &= (\rho|z|)^{\frac{1}{\rho+1}} e^{i\frac{\xi+\pi}{\rho+1}} \\ Z_2 &= (\rho|z|)^{\frac{1}{\rho+1}} e^{i\frac{\xi-\pi}{\rho+1}} \end{aligned}\right\}$$

$\arg(-z) = \xi$ ($|\xi| \le \pi$) and $H(Z)$ is given by (9). In particular, if $\beta \in \mathbb{R}$, we deduce from formula (23) the asymptotic expansion of the Wright function $\phi(\rho, \beta; -r)$ for $r \to +\infty$ in the form

$$\begin{aligned}&(\rho r)^{-p(\frac{1}{2}-\beta)} e^{-\sigma r^p \cos \pi p} \phi(\rho, \beta; -r) \\ &\qquad = 2a_0 \cos\big(\pi p(\tfrac{1}{2} - \beta) + \sigma r^p \sin \pi p\big) + O(r^{-p})\end{aligned} \tag{24}$$

where p and σ are the order and type of the Wright function, respectively, given by (2). Since the function

$$(\rho r)^{-p(\frac{1}{2}-\beta)} e^{-\sigma r^p \cos \pi p} \phi(\rho, \beta; -r) \qquad (0 < r < +\infty)$$

takes only real values and the function $\cos\big(\pi p(\frac{1}{2} - \beta) + \sigma r^p \sin \pi p\big)$ vanishes if $\pi p(\frac{1}{2} - \beta) + \sigma r^p \sin \pi p = \frac{\pi}{2} + \pi k$ ($k \in \mathbb{N}_0$), the function $\phi(\rho, \beta; -r)$ has a countable set of zeros r_k ($k \ge k_0$) and in a small enough neighbourhood of the point $\frac{\pi}{2} + \pi k$ ($k \ge k_0$) there is only one point of the form $\pi p(\frac{1}{2} - \beta) + \sigma r_k^p \sin \pi p$. Summarizing the above-mentioned reasoning we arrive at the fact that the function $\phi(\rho, \beta; z)$ has a countable set of zeros $\{-r_k\}_{k_0}^{\infty}$ and

$$\pi p(\tfrac{1}{2} - \beta) + \sigma r_k^p \sin \pi p = \tfrac{\pi}{2} + \pi k + \alpha_k \qquad (k \ge k_0) \tag{25}$$

where $\alpha_k = O(1)$ ($k \to +\infty$).

Substituting the expression for r_k from (25) into (24) we get $\alpha_k = O(r_k^{-p})$. It follows from (25) that $r_k^p \asymp k$ and, consequently, the asymptotic formula (25) can be rewritten in the form

$$\pi p(\tfrac{1}{2} - \beta) + \sigma r_k^p \sin \pi p = \tfrac{\pi}{2} + \pi k + O(\tfrac{1}{k}) \qquad (k \to +\infty). \tag{26}$$

From this last formula we get the representation for the zeros of the function $\phi(\rho, \beta; z)$ as

$$\begin{aligned}-r_k &= -\left(\frac{\pi k + \pi(p\beta - \frac{p-1}{2}) + O(\frac{1}{k})}{\sigma \sin \pi p}\right)^{\frac{1}{p}} \\ &= -\left(\frac{\pi k + \pi(p\beta - \frac{p-1}{2})}{\sigma \sin \pi p}\right)^{\frac{1}{p}} \big(1 + O(k^{-2})\big)\end{aligned} \qquad (k \to +\infty). \tag{27}$$

Let us establish the fact that these zeros $-r_k$ are simple if $k \ge k_1 \ge k_0$. Indeed, differentiation term by term of the series (1) gives us the formula

$$\frac{d}{dr}\phi(\rho, \beta; -r) = -\phi(\rho, \beta + \rho; -r).$$

We multiply now both parts of this identity by $(\rho r)^{-p(\frac{1}{2}-\beta-\rho)} e^{-\sigma r^p \cos \pi p}$ and use the asymptotic formula (24) thus obtaining the relation

$$\begin{aligned}&(\rho r)^{-p(\frac{1}{2}-\beta-\rho)} e^{-\sigma r^p \cos \pi p} \frac{d}{dr}\phi(\rho, \beta; -r) \\ &\qquad = -2a_0 \cos\big(\pi p(\tfrac{1}{2} - \beta - \rho) + \sigma r^p \sin \pi p\big) + O(r^{-p}).\end{aligned}$$

Setting $r = r_k$ $(k \geq k_0)$ in the last formula and using (25) with $\alpha_k = O(\frac{1}{k})$ we get

$$(\rho r_k)^{-p(\frac{1}{2}-\beta-\rho)} e^{-\sigma r_k^p \cos \pi p} \frac{d}{dr}\phi(\rho,\beta;-r)\Big|_{r=r_k} = (-1)^k 2a_0 \sin(\pi \rho p) + O(\tfrac{1}{k})$$

as $k \to +\infty$. Since $0 < \rho p < 1$ for $\rho > 0$, it follows from the last relation that all zeros $-r_k$ beginning with some index $k_1 \geq k_0$ are simple.

To finish the proof of statement (A) of Theorem 3, we have to establish the fact that the set of all zeros of the function $\phi(\rho,\beta;z)$ with large enough index coincides with the set of numbers $\{-r_k\}_{k_2}^{\infty}$ $(k_2 \geq k_1)$. Indeed, let us define a subdomain $D_\varepsilon^{(k)}$ of the sector $D_\varepsilon = \{z \in \mathbb{C} : |\arg(-z)| \leq \varepsilon\}, \varepsilon > 0$ by

$$\left(\frac{\pi k + \pi p(\beta - \frac{1}{2})}{\sigma \sin \pi p}\right)^{\frac{1}{p}} \leq |z| \leq \left(\frac{\pi (k+1) + \pi p(\beta - \frac{1}{2})}{\sigma \sin \pi p}\right)^{\frac{1}{p}}$$

and let $l_k(\varepsilon)$ be a boundary curve of this subdomain. Then we have

$$\min_{z \in l_k(\varepsilon)} \left|\cos\left(\pi p(\frac{1}{2} - \beta) + \sigma(-z)^p \sin \pi p\right)\right| = m(\varepsilon) > 0$$

where $m(\varepsilon)$ is some constant not depending on k. Further, using (23) we arrive at $(z \in D_\varepsilon)$

$$(-\rho z)^{-p(\frac{1}{2}-\beta)} e^{-\sigma(-z)^p \cos \pi p} \phi(\rho,\beta;z) - 2a_0 \cos\left(\pi p(\tfrac{1}{2} - \beta) + \sigma(-z)^p \sin \pi p\right)$$
$$= O(|z|^{-p}|).$$

Since the left part of the last relation tends to zero if $z \to \infty$, it follows from the Rouché theorem that the function $\phi(\rho,\beta;z)$ has inside the domain $D_\varepsilon^{(k)}$ $(k \geq k_2 \geq k_1)$ as many zeros as the function $\cos\left(\pi p(\frac{1}{2} - \beta) + \sigma(-z)^p \sin \pi p\right)$, that means, a single zero only. In this case this zero of the function $\phi(\rho,\beta;z)$ is a point $-r_k \in D_\varepsilon^{(k)}$. It follows from the above-mentioned arguments that $\gamma_k = -r_{k-N}$ $(k \geq N + k_2)$ for some fixed $N \geq 1$. Using this fact and (27) we arrive at the asymptotic formula (21).

Statement (B) of Theorem 3 is proved by using exactly the same technique as statement (A) and we omit the details. We only note that in this case we use instead of (23) - (24) the asymptotic formula given by Wright [19]

$$\phi(\rho,\beta;z) = I(Y_1) + I(Y_2) \qquad (-\tfrac{1}{3} < \rho < 0, |\arg z| \leq \pi(1+\rho) - \varepsilon\ (\varepsilon > 0)) \tag{28}$$

where $I(Y)$ is defined by (12),

$$\left.\begin{aligned} Y_1 &= (1+\rho)\left((-\rho)^{-\rho} z\, e^{\pi i}\right)^{\frac{1}{1+\rho}} \\ Y_2 &= (1+\rho)\left((-\rho)^{-\rho} z\, e^{-\pi i}\right)^{\frac{1}{1+\rho}} \end{aligned}\right\} \tag{29}$$

and its modified form $(r \to +\infty, \beta \in \mathbb{R})$

$$(\sigma r^p)^{\beta - \frac{1}{2}} e^{\sigma r^p \cos \pi p} \phi(\rho,\beta;r) = 2A_0 \cos\left(\pi p(\tfrac{1}{2} - \beta) - \sigma r^p \sin \pi p\right) + O(r^{-p}). \tag{30}$$

In the case $\rho = -\frac{1}{3}$ the asymptotic formula from Wright [19] is used:

$$\phi(\rho,\beta;z) = I(Y_1) + I(Y_2) + J(z) \qquad (\rho = -\tfrac{1}{3}, |\arg z| \leq \pi(1+\rho) - \varepsilon\ (\varepsilon > 0)) \tag{31}$$

where $I(Y)$ is defined by (12), Y_1, Y_2 by (29) and $J(z)$ by (19) ∎

Remark 3. Due to the relation

$$J_\nu(z) = (\tfrac{z}{2})^\nu \phi(1, \nu+1; -\tfrac{1}{4}z^2) \tag{32}$$

Wright considered the function $\phi(\rho, \beta; z)$ as a generalization of the Bessel function $J_\nu(z)$. Combining the representation (32) with the asymptotic formula (21) we get the known formula (see, for example, [16: p. 506]) for the asymptotic expansion of the large zeros r_k of the Bessel function $J_\nu(z)$:

$$r_k = \pi(k + \tfrac{1}{2}\nu - \tfrac{1}{4}) + O(k^{-1}) \qquad (k \to \infty).$$

We consider now the case $-1 < \rho < -\frac{1}{3}$. It follows from the asymptotic formulas (11) and (18) that in this case all zeros of the function $\phi(\rho, \beta; z)$ with large enough absolute value are lying inside of the angular domains

$$\Omega_\varepsilon^{(\pm)} = \left\{ z \in \mathbb{C} : \left| \arg z \mp \left(\pi - \tfrac{3\pi}{2p}\right) \right| < \varepsilon \right\} \tag{33}$$

where ε is any number of the interval $(0, \min\{\pi - \frac{3\pi}{2p}, \frac{3\pi}{2p}\})$. Consequently, the function $\phi(\rho, \beta; z)$ has on the real axis only finitely many zeros. Let

$$\left. \begin{array}{l} \{\gamma_k^{(+)}\}_{k=1}^\infty \in G^{(+)} = \{z \in \mathbb{C} : \Im(z) > 0\} \\ \{\gamma_k^{(-)}\}_{k=1}^\infty \in G^{(-)} = \{z \in \mathbb{C} : \Im(z) < 0\} \end{array} \right\}$$

be sequences of zeros of the function $\phi(\rho, \beta; z)$ in the upper and lower half-plane, respectively, such that $|\gamma_k^{(+)}| \le |\gamma_{k+1}^{(+)}|$ and $|\gamma_k^{(-)}| \le |\gamma_{k+1}^{(-)}|$ and each zero is counted according to its multiplicity. We have the following result.

Theorem 4. *In the case $-1 < \rho < -\frac{1}{3}$ ($\beta \neq -n$ ($n \in \mathbb{N}_0$) and $\beta \neq \frac{1}{2} - n$ ($n \in \mathbb{N}_0$) if $\rho = -\frac{1}{2}$) all zeros of the function $\phi(\rho, \beta; z)$ ($\beta \in \mathbb{R}$) with large enough k are simple and the asymptotic formula*

$$\gamma_k^{(\pm)} = e^{\pm i(\pi - \frac{3\pi}{2p})} \left(\frac{2\pi k}{\sigma}\right)^{\frac{1}{p}} \left(1 + O\left(\tfrac{\log k}{k}\right)\right) \qquad (k \to +\infty) \tag{34}$$

is true.

Proof. If $-1 < \rho < -\frac{1}{3}$ and $z \in \Omega_\varepsilon^{(\pm)}$ the asymptotics of the Wright function is given in [19] as

$$\phi(\rho, \beta; z) = I(Y) + J(z) \tag{35}$$

where $I(Y)$ and $J(z)$ are defined by (12) and (19), respectively. Let us introduce the notations ($\beta \neq -n$ ($n \in \mathbb{N}_0$) and $\beta \neq \frac{1}{2} - n$ ($n \in \mathbb{N}_0$) if $\rho = -\frac{1}{2}$)

$$\tau_\rho = \begin{cases} \frac{\beta-1}{-\rho} & \text{if } 1 + \frac{\beta-1}{-\rho} \notin -\mathbb{N}_0 \\ -1 - k + \frac{1}{\rho} & \text{if } 1 + \frac{\beta-1}{-\rho} = -k \ (k \in \mathbb{N}_0) \end{cases}$$

$$c_\rho = \begin{cases} \frac{1}{(-\rho)\,\Gamma(1 + \frac{\beta-1}{-\rho})} & \text{if } 1 + \frac{\beta-1}{-\rho} \notin N_0 \\ \frac{1}{(-\rho)\,\Gamma(-k + \frac{1}{\rho})} & \text{if } 1 + \frac{\beta-1}{-\rho} = -k \ (k \in \mathbb{N}_0). \end{cases}$$

With these notations formula (35) gives us

$$z^{-\tau_\rho}\phi(\rho,\beta;z) = z^{-\tau_\rho}(\sigma(-z)^p)^{\frac{1}{2}-\beta}e^{-\sigma(-z)^p}\left(A_0 + O(|z|^{-p})\right) + c_\rho + O(z^{\frac{1}{\rho}}) \qquad (36)$$

where $-\pi < \arg z \leq \pi$ and $-\pi < \arg(-z) \leq \pi$. We consider now the equation

$$A_0 z^{-\tau_\rho}(\sigma(-z)^p)^{\frac{1}{2}-\beta}e^{-\sigma(-z)^p} + c_\rho = 0. \qquad (37)$$

Let us define the curve

$$L_0 = \left\{z \in \mathbb{C} : |A_0 z^{-\tau_\rho}(\sigma(-z)^p)^{\frac{1}{2}-\beta}e^{-\sigma(-z)^p}| = |c_\rho|\right\}. \qquad (38)$$

Assuming $z = re^{i\phi} \in L_0 \;\; (\pi \geq \phi > 0)$ we get $-z = re^{i(\phi-\pi)}$ and $(-z)^p = r^p e^{ip(\phi-\pi)}$. The equation of the branch of the curve L_0 in the domain $G^{(+)}$ can be rewritten in the form

$$|A_0|\sigma^{\frac{1}{2}-\beta}r^{-\tau_\rho+p(\frac{1}{2}-\beta)}e^{-\sigma r^p \cos p(\phi-\pi)} = |c_\rho| \qquad (39)$$

or, after some transformations, in the form

$$-\sigma\cos p(\phi-\pi) = (\tau_\rho - p(\tfrac{1}{2}-\beta))\frac{\log r}{r^p} + O(r^{-p}). \qquad (40)$$

Equation (40) gives us

$$\phi = \pi - \frac{3\pi}{2p} + \frac{\tau_\rho - p(\frac{1}{2}-\beta)}{p\sigma}\frac{\log r}{r^p} + O(r^{-p}), \qquad (41)$$

that is, the branch of the curve L_0 in the domain $G^{(+)}$ is in the sector $\Omega_\varepsilon^{(+)}$ for large enough r. If $z = re^{i\phi}$ $(\pi \geq \phi > 0)$, then

$$\arg(z^{-\tau_\rho}(-z)^{p(\frac{1}{2}-\beta)}e^{-\sigma(-z)^p}) = -\tau_\rho\phi + (\tfrac{1}{2}-\beta)p(\phi-\pi) - \sigma r^p \sin p(\phi-\pi). \qquad (42)$$

This means that there is a countable set of points $\lambda_k = r_k e^{i\phi_k}$ $(k \in \mathbb{N}_0, 0 \leq \phi_k \leq \pi)$ lying on the curve L_0 for which

$$-\tau_\rho\phi_k + (\tfrac{1}{2}-\beta)p(\phi_k-\pi) - \sigma r_k^p \sin p(\phi_k-\pi) = 2\pi k + C \qquad (k \in \mathbb{Z}) \qquad (43)$$

where $C = \arg(-c_\rho\sigma^{\beta-\frac{1}{2}}/A_0)$ is equal to π or to 0. Evidently, these points coincide with those solutions of equation (37) that lie in the domain $G^{(+)}$. Using (41) and (43) we arrive for $k \to \infty$ at

$$\sigma r_k^p = 2\pi k + O(1) \qquad (44)$$

which gives us the formula

$$r_k = \left(\frac{2\pi}{\sigma}k\right)^{\frac{1}{p}}(1 + O(\tfrac{1}{k})). \qquad (45)$$

This formula and (41) gives us

$$\phi_k = \pi - \frac{3\pi}{2p} + O(\tfrac{\log k}{k}) \tag{46}$$

which, together with (45), leads to

$$\lambda_k = r_k e^{i\phi_k} = \left(\frac{2\pi}{\sigma}k\right)^{\frac{1}{p}} e^{i(\pi - \frac{3\pi}{2p})}(1 + O(\tfrac{\log k}{k})).$$

Using the asymptotic formula (36) and exactly the same technique as in the proof of Theorem 3 we arrive at the representation (34) of the zeros $\gamma_k^{(+)}$ of the Wright function lying in the domain $G^{(+)}$.

Finally, if $\beta \in \mathbb{R}$, then $\phi(\rho, \beta; \overline{z}) = \overline{\phi(\rho, \beta; z)}$ and, consequently, $\gamma_k^{(-)} = \overline{\gamma_k^{(+)}}$ $(k \in \mathbb{N})$ which gives us the representation (34) in the domain $G^{(-)}$ ∎

As a consequence of Theorems 2 - 4 we get the following

Theorem 5. *The Wright function* $\phi(\rho, \beta; z)$ $(\rho > -1)$ *is an entire function of completely regular growth.*

We recall [8: Chapter 3] that an entire function $f(z)$ of finite order p is called a function of completely regular growth (CRG-function) if for all $\theta, |\theta| \le \pi$, there exist a set $E_\theta \subset \mathbb{R}_+$ and the limit

$$\lim_{\substack{r\to+\infty \\ r\in E_\theta^*}} \frac{\log|f(re^{i\theta})|}{r^p} \tag{47}$$

where

$$E_\theta^* = \mathbb{R}_+ \setminus E_\theta, \qquad \lim_{r\to+\infty} \frac{\mathrm{mes} E_\theta \cap (0,r)}{r} = 0.$$

It is known [4: Chapter 2.6] that the zeros of a CRG-function $f(z)$ are regularly distributed, namely, they possess the finite angular density

$$\lim_{r\to+\infty} \frac{n(r,\theta)}{r^p} = \nu(\theta) \tag{48}$$

where $n(r, \theta)$ is the number of zeros of $f(z)$ in the sector $0 < \arg z < \theta,\ |z| < r$ and p is the order of $f(z)$. From the other side, the angular density $\nu(\theta)$ is connected with the indicator function $h(\theta)$ of a CRG-function. In particular (see [4: Chapter 2.6]), the jump of $h'(\theta)$ at $\theta = \theta_0$ is equal to $2\pi p\Delta$, where Δ is the density of zeros of $f(z)$ in an arbitrary small angle containing the ray $\arg z = \theta_0$.

In our case we get from Theorem 2 that the derivative of the indicator function of the Wright function has the jump $2\sigma p \sin \pi p$ at $\theta = \pi$ for $\rho > 0$, the same jump at $\theta = 0$ for $-\frac{1}{3} \le \rho < 0$, and the jump σp at $\theta = \pm(\pi - \frac{3\pi}{2p})$ for $-1 < \rho < -\frac{1}{3}$ $(\beta \ne -n\ (n \in \mathbb{N}_0)$ and $\beta \ne \frac{1}{2} - n\ (n \in \mathbb{N}_0)$ if $\rho = -\frac{1}{2})$, where p and σ are the order and type of the Wright function, respectively, given by (2); if $\rho = 0$ or $\rho = -\frac{1}{2}$ and either $\beta = -n\ (n \in \mathbb{N}_0)$ or $\beta = \frac{1}{2} - n\ (n \in \mathbb{N}_0)$, the derivative of the indicator function has no jumps. As we see, the behaviour of the derivative of the indicator function of the Wright function is in accordance with the distribution of its zeros given by Theorems 3 and 4 as predicted by the general theory of the CRG-functions.

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