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On a Local Lipschitz Constant of the Maps Related to LU-Decomposition

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Abstract. Let $M(n,\mathbb{R})$ be the set of real positive definite symmetric $(n \times n)$-matrices equipped with the Euclidean norm, and let $A \in M(n,\mathbb{R})$. Let $L(n,\mathbb{R})$ be the set of all real non-degenerate lower-triangular $(n \times n)$-matrices equipped with the Euclidean norm, and let $L : M(n,\mathbb{R}) \to L(n,\mathbb{R})$ be a (differentiable) map assigning to a positive definite symmetric matrix its lower-triangular factor in the LU-decomposition. We give an effective upper estimate for $\|L'(A)\|$.

Keywords: *Lipschitz constant, LU-decomposition, Schur and Hadamard inequalities*

AMS subject classification: Primary 58E05, 58E09, secondary 35J20

1. Introduction

Frequently encountered problems in the field of numerical analysis are related to finding solutions of partial differential equations by solving large systems of linear algebraic equations with symmetric (or Hermitian) matrices. An important computational method for solving such large systems is based on the LU-decomposition of positive definite matrices. More precisely, in order to solve a system $Ax = b$ we can represent the matrix A as product of a lower-triangular matrix L and an upper-triangular matrix U which can be easily inverted, so the original system can be solved.

Accuracy of any numerical method is related to its stability with respect to small perturbations of the data (cf. [2, 3, 6 - 8]). In other words, as unavoidable errors associated for example with rounding or approximation of the data propagate, get amplified, and consequently, contaminate the results, it is important to effectively and accurately estimate the impact of small changes of the data (i.e. the right-hand side vector b of the system of linear equation $Ax = b$) on the computed solution.

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Another crucial problem related to finding numerical solutions of linear equations, from the more delicate standpoint of numerical performance, is associated with the so-called *coefficient stability*. In other words, we need to know how perturbations of the entries of the matrix A affect the solution of the corresponding linear system. It is our impression that this fundamental stability problem for the LU-decomposition method has not been given sufficient consideration in the existing literature.

The above stability problem can be studied using various information about the matrix A, for example its spectral properties, that is a standard approach to evaluate stability of a numerical method. However, the information about the spectrum of A is not always available, so it is more natural from the practical point of view to use an evaluation method depending directly on the coefficients of the matrix A.

In this paper we present an estimate for the stability of the LU-decomposition method expressed in terms of the norm of the matrix A and its principal minors. Namely, we present an estimation of the magnitude of the error in the calculated triangular factors of the matrix A as a function of the approximation error of the entrees of A. To be more specific, let us formulate this problem in a purely analytic way.

Let $M(n,\mathbb{R})$ be the set of real positive definite symmetric $(n\times n)$-matrices equipped with the Euclidean norm, and let $A=(a_{ij})\in M(n,\mathbb{R})$. Consider its LU-decomposition

$$A = L(A)U(A)$$

where $L(A)$ and $U(A)$ are a lower- and upper-triangular matrix, respectively, transposed to $L(A)$). Recall that the coefficients of $L(A)$ can be computed from the well-known Gauss formulas (cf. [6, 9, 10])

$$l_{sk} = \begin{cases} 0 & \text{for } 1 \le s \le k-1 \\ \frac{A_{sk}}{\sqrt{A_{k-1}A_k}} & \text{for } k \le s \le n \end{cases} \qquad (k=1,\dots,n) \tag{1.1}$$

where A_i is the principal minor of A of order i $(i=1,\dots,n)$ and A_{sk} is the minor of A obtained by intersecting the rows with the indices $1,2,\dots,k-1,s$ and the columns with the indices $1,2,\dots,k$.

Denote by $L(n,\mathbb{R})$ the set of all real lower-triangular $(n\times n)$-matrices equipped with the Euclidean norm. Let $L: M(n,\mathbb{R})\to L(n,\mathbb{R})$ be the map defined by $L: A\to L(A)$ where $L(A)$ is the above lower-triangular matrix corresponding to A. In spite of the fact that the map L is given by an explicit formula, its analytic properties were not studied in a rigorous way. It is clear that L is differentiable and positively homogeneous of degree $\frac{1}{2}$, i.e.

$$L(\lambda A) = \lambda^{\frac{1}{2}} L(A) \qquad \text{for } \lambda > 0. \tag{1.2}$$

Since L is differentiable, the problem of finding an estimate for the stability of the LU-decomposition method for a fixed matrix A can be reduced to estimating the local Lipschitz constant represented by the norm $\|L'(A)\|$. Consequently, the stability problem for the LU-decomposition method can be reformulated as follows: *Given a matrix A what is $L'(A)$?*

The main result of this paper is an effective upper estimate of $\|L'(A)\|$ presented in Theorem 2.5. We refer to [2, 3, 6 - 10] where several results related to our discussion can be found.

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2. Lipschitz constant estimates

The objective of this section is to analyze the Lipschitz continuity of the matrix function

$$L : M(n,\mathbb{R}) \to \mathbb{R}^{n\times n}$$

defined by (1.1). Our estimates are based on the following inequality expressing for $T \in M(n,\mathbb{R})$ the relation between $\det T$ and the Hilbert-Schmidt norm of the matrix T, denoted by $\|T\|$.

Lemma 2.1. *Let $T \in M(n,\mathbb{R})$. Then*

$$|\det T| \le n^{-\frac{n}{2}}\|T\|^n. \tag{2.1}$$

Proof. Inequality (2.1) is probably well-known, however we were not able to find any reference to a standard textbook in linear algebra. Therefore, it is appropriate to present its proof. For example, inequality (2.1) is a consequence of the well-known Schur inequality (cf. [1])

$$\sum_{k=1}^{n} |\lambda_k|^2 \le \|T\|^2$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of the matrix T. Indeed, using the fact that geometric mean is always less or equal than arithmetic mean we obtain

$$\begin{aligned}|\det T| &= \prod_{k=1}^{n} |\lambda_k| = \left(\left(\prod_{k=1}^{n} |\lambda_k|^2\right)^{\frac{1}{n}}\right)^{\frac{n}{2}} \\ &\le \left(\frac{1}{n}\sum_{k=1}^{n} |\lambda_k|^2\right)^{\frac{n}{2}} = n^{-\frac{1}{2}n}\left(\sum_{k=1}^{n} |\lambda_k|^2\right)^{\frac{n}{2}} \\ &\le n^{-\frac{1}{2}n}\|T\|^n\end{aligned}$$

and the statement is proved ∎

Let us notice that inequality (2.1) also follows from the well-known Hadamard inequality (cf. [1])

$$|\det T| \le \prod_{j=1}^{n}\left(\sum_{k=1}^{n} |a_{jk}|^2\right)^{\frac{1}{2}}.$$

Indeed, by similar arguments we have

$$|\det T| \le \left(\left(\prod_{j=1}^{n}\left(\sum_{k=1}^{n} |a_{jk}|^2\right)\right)^{\frac{1}{n}}\right)^{\frac{n}{2}} \le \left(\frac{1}{n}\sum_{j,k=1}^{n} |a_{jk}|^2\right)^{\frac{n}{2}} = n^{-\frac{1}{2}n}\|T\|^n.$$

Now, we can use Lemma 2.1 to estimate the partial derivatives of the entries of $L'(A)$.

Lemma 2.2. *The inequalities*

$$\left|\frac{\partial l_{sk}}{\partial a_{ij}}\right| \le \gamma_k \sqrt{n} \left(\prod_{k=1}^{n} k^k\right)^{-\frac{3}{4}} \frac{\|A\|^{\frac{3}{4}n^2-\frac{1}{4}n-\frac{1}{2}}}{A_1^{\frac{3}{2}} \cdots A_{n-1}^{\frac{3}{2}} A_n^{\frac{1}{2}}} \tag{2.2}$$

hold for $s = k, k+1, \ldots, n$ *and* $k = 1, \ldots, n$ *where*

$$\left.\begin{aligned} \gamma_1 &= \frac{3}{2} \\ \gamma_2 &= \frac{7}{2\sqrt{2}} \\ \gamma_k &= \frac{1}{2}\left(\frac{3k^{\frac{k}{4}}}{(k-1)^{\frac{k-1}{4}}} + \frac{(k-1)^{\frac{3(k-1)}{4}}}{(k-2)^{\frac{k-2}{2}} k^{\frac{k}{4}}}\right) \quad (k \ge 2) \end{aligned}\right\}.$$

Proof. First consider the case $2 \le k < n$. From (1.1) we have ($s \ge k$)

$$\frac{\partial l_{sk}}{\partial a_{ij}} = \frac{1}{A_{k-1}^{\frac{1}{2}} A_k^{\frac{1}{2}}} \frac{\partial A_{sk}}{\partial a_{ij}} - \frac{1}{2A_{k-1}^{\frac{3}{2}} A_k^{\frac{3}{2}}} A_{sk} A_{k-1} \frac{\partial A_k}{\partial a_{ij}} - \frac{1}{2A_{k-1}^{\frac{3}{2}} A_k^{\frac{3}{2}}} A_{sk} \frac{\partial A_{k-1}}{\partial a_{ij}} A_k. \tag{2.3}$$

Notice that, by Lemma 2.1,

$$|A_{sk}| \le \frac{1}{k^{\frac{k}{2}}} \|A\|^k \quad \text{and} \quad \left|\frac{\partial A_{sk}}{\partial a_{ij}}\right| \le \frac{1}{(k-1)^{\frac{k-1}{2}}} \|A\|^{k-1},$$

and since $A_k = |A_{kk}|$, inequality (2.3) implies

$$\left|\frac{\partial l_{sk}}{\partial a_{ij}}\right| \le \frac{1}{A_{k-1}^{\frac{1}{2}} A_k^{\frac{1}{2}}} \frac{\|A\|^{k-1}}{(k-1)^{\frac{k-1}{2}}} + \frac{\|A\|^{3k-2}}{2A_{k-1}^{\frac{3}{2}} A_k^{\frac{3}{2}}} \left(\frac{1}{(k-1)^{k-1} k^{\frac{k}{2}}} + \frac{1}{(k-2)^{\frac{k-2}{2}} k^k}\right). \tag{2.4}$$

By applying again inequality (2.1), we have

$$\left|\frac{\partial l_{sk}}{\partial a_{ij}}\right| \le \frac{\|A\|^{3k-2}}{2A_{k-1}^{\frac{3}{2}} A_k^{\frac{3}{2}}} \left(\frac{3}{(k-1)^{k-1} k^{\frac{k}{2}}} + \frac{1}{(k-2)^{\frac{k-2}{2}} k^k}\right). \tag{2.5}$$

In a similar way we can consider the case $k = n$. Since $l_{nn} = \sqrt{A_n / A_{n-1}}$ we obtain

$$\frac{\partial l_{nn}}{\partial a_{ij}} = \frac{1}{2A_{n-1}^{\frac{3}{2}} A_n^{\frac{1}{2}}} \frac{\partial A_n}{\partial a_{ij}} A_{n-1} - \frac{1}{2A_{n-1}^{\frac{3}{2}} A_n^{\frac{1}{2}}} A_n \frac{\partial A_{n-1}}{\partial a_{ij}} \tag{2.6}$$

and by applying (2.1) we get

$$\left|\frac{\partial l_{nn}}{\partial a_{ij}}\right| \le \frac{\|A\|^{2n-2}}{2A_{n-1}^{\frac{3}{2}} A_n^{\frac{1}{2}}} \left(\frac{1}{(n-1)^{n-1}} + \frac{1}{(n-2)^{\frac{n-2}{2}} n^{\frac{n}{2}}}\right). \tag{2.7}$$

Using (2.1) in the case $k < n$ we obtain the estimate

$$\begin{aligned} A_1^{\frac{3}{2}} \cdots A_{k-2}^{\frac{3}{2}} A_{k+1}^{\frac{3}{2}} \cdots A_{n-1}^{\frac{3}{2}} A_n^{\frac{1}{2}} &= \frac{A_1^{\frac{3}{2}} \cdots A_{k-2}^{\frac{3}{2}} A_{k-1}^{\frac{3}{2}} A_k^{\frac{3}{2}} A_{k+1}^{\frac{3}{2}} \cdots A_{n-1}^{\frac{3}{2}} A_n^{\frac{1}{2}}}{A_{k-1}^{\frac{3}{2}} A_k^{\frac{3}{2}}} \\ &\le \frac{\|A\|^{\frac{3}{2}} \cdots \|A\|^{\frac{3}{2}(n-1)} \|A\|^{\frac{1}{2}n}}{\left(\prod_{k=1}^{n-1} k^k\right)^{\frac{3}{4}} n^{\frac{1}{4}}} \frac{(k-1)^{\frac{3}{4}(k-1)} k^{\frac{3}{4}k}}{\|A\|^{\frac{3}{2}(k-1)} \|A\|^{\frac{3}{2}k}} \\ &= \frac{\|A\|^{\frac{3}{2}(1+2+\ldots n-1)+\frac{1}{2}n} n^{\frac{1}{2}}}{\left(\prod_{k=1}^{n} k^k\right)^{\frac{3}{4}}} \frac{(k-1)^{\frac{3}{4}(k-1)} k^{\frac{3}{4}k}}{\|A\|^{3k-\frac{3}{2}}} \\ &= \|A\|^{\frac{3}{4}n^2 - \frac{1}{4}n} \sqrt{n} \left(\prod_{k=1}^{n} k^k\right)^{-\frac{3}{4}} \frac{(k-1)^{\frac{3}{4}(k-1)} k^{\frac{3}{4}k}}{\|A\|^{3k-\frac{3}{2}}}. \end{aligned} \tag{2.8}$$

Combining (2.8) with (2.5) yields (2.2). Similarly, in the case where $k = n$ we have

$$A_1^{\frac{3}{2}} \cdots A_{n-2}^{\frac{3}{2}} \le \frac{\|A\|^{\frac{3}{4}(n-1)(n-2)}}{(1^1 \cdot 2^2 \cdots (n-2)^{n-2})^{\frac{3}{4}}}, \tag{2.9}$$

thus (2.2) follows from (2.7) and (2.9). The proof in the case where $k \le 2$ is similar with some evident simplifications ∎

Lemma 2.3. *Let*

$$\gamma_k = \frac{1}{2}\left(\frac{3k^{\frac{k}{4}}}{(k-1)^{\frac{k-1}{4}}} + \frac{(k-1)^{\frac{3(k-1)}{4}}}{(k-2)^{\frac{k-2}{2}} k^{\frac{k}{4}}}\right).$$

Then the inequality

$$\gamma_k \le 2e^{\frac{1}{4}} k^{\frac{1}{4}} \tag{2.10}$$

holds.

Proof. It is clear that

$$\frac{3k^{\frac{k}{4}}}{(k-1)^{\frac{k-1}{4}}} \le 3\left(1 + \frac{1}{k-1}\right)^{\frac{k}{4}} k^{\frac{1}{4}} \le 3e^{\frac{1}{4}} k^{\frac{k}{4}}$$

and

$$\begin{aligned} \frac{(k-1)^{\frac{3(k-1)}{4}}}{(k-2)^{\frac{k-2}{2}} k^{\frac{k}{4}}} &= (k-1)^{\frac{1}{4}} \frac{(k-1)^{\frac{k-2}{2}}}{(k-2)^{\frac{k-2}{2}}} \frac{(k-1)^{\frac{k}{4}}}{k^{\frac{1}{4}}} \\ &= (k-1)^{\frac{1}{4}} \left(1 + \frac{1}{k-2}\right)^{\frac{k-2}{2}} \left(1 - \frac{1}{k}\right)^{\frac{k}{4}} \\ &= (k-1)^{\frac{1}{4}} \left(1 + \frac{1}{k-2}\right)^{\frac{k-2}{2}} \left(1 - \frac{1}{k^2}\right)^{\frac{k}{4}} \frac{1}{(1+\frac{1}{k})^{\frac{k+1}{4}}} \left(\frac{k+1}{k}\right)^{\frac{1}{4}} \\ &\le e^{\frac{1}{4}} \left(\frac{k^2-1}{k}\right)^{\frac{1}{4}} \\ &\le e^{\frac{1}{4}} k^{\frac{1}{4}}, \end{aligned}$$

so inequality (2.10) follows ∎

Notice that for a fixed k an entry l_{sk} of the matrix $L(A)$ depends only on k^2 variables. Furthermore, for a fixed k there are only $n+1-k$ non-zero entries l_{sk} of the matrix $L(A)$, thus for a fixed k there can only be $(n+1-k)k^2$ non-zero partial derivatives $\frac{\partial l_{sk}}{\partial a_{ij}}$.

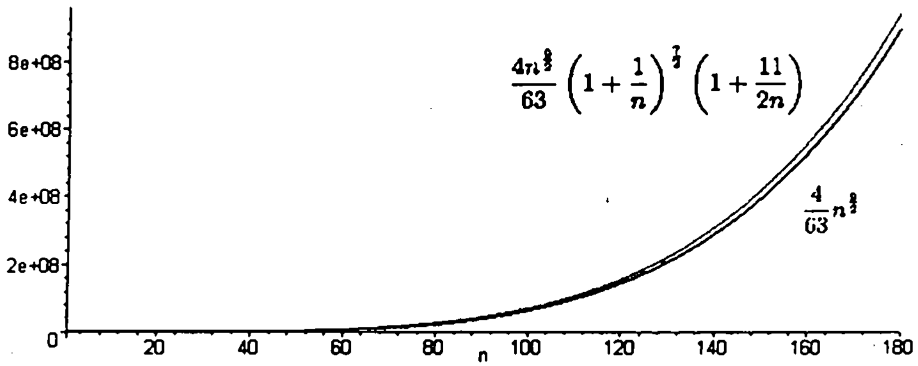
Lemma 2.4. *The inequality*

$$\frac{4}{63}n^{\frac{9}{2}} \leq \sum_{k=1}^{n}(n+1-k)k^{\frac{5}{2}} \leq \frac{4n^{\frac{9}{2}}}{63}\left(1+\frac{1}{n}\right)^{\frac{7}{2}}\left(1+\frac{11}{2n}\right)$$

holds.

Proof. The proof is straightforward and we omit it ∎

The estimates presented in Lemma 2.4 can be illustrated by the following graph.



We can summarize the estimates derived in Lemmas 2.2 - 2.4 in the following result.

Theorem 2.5. *Let A be a positive definite symmetric $(n \times n)$-matrix. Then the estimate*

$$\|L'(A)\| \leq \frac{4e^{\frac{1}{4}}n^{\frac{11}{4}}}{\sqrt{63}}\left(1+\frac{1}{n}\right)^{\frac{7}{4}}\left(1+\frac{11}{2n}\right)^{\frac{1}{2}}\left(\prod_{k=1}^{n}k^k\right)^{-\frac{3}{4}}\frac{\|A\|^{\frac{3}{4}n^2-\frac{1}{4}n-\frac{1}{2}}}{A_1^{\frac{3}{2}}\cdots A_{n-1}^{\frac{3}{2}}A_n^{\frac{1}{2}}} \tag{2.11}$$

holds.

Remark 2.6. Notice that the estimate standing on the right-hand side of inequality (2.11) is a homogeneous expression of order $-\frac{1}{2}$ with respect to A, as it was expected since $L'(A)$ is also homogeneous of order $-\frac{1}{2}$.

In general, however, estimate (2.11) is rather rough what can be seen using spectral properties of the matrix A. We have the following

Corollary 2.7. *Let m and M be the smallest and the largest eigenvalues of A, respectively. Then*

$$\|L'(A)\| \leq \frac{4e^{\frac{1}{4}}}{\sqrt{63}\,\Gamma^{\frac{3}{4}}}\left(1+\frac{1}{n}\right)^{\frac{7}{4}}\left(1+\frac{11}{2n}\right)^{\frac{1}{2}} e^{\frac{3}{16}n^2} n^{\frac{39}{16}-\frac{1}{2}n} \frac{1}{\sqrt{M}}\left(\frac{M}{m}\right)^{\frac{3}{4}n^2-\frac{1}{4}n} \tag{2.12}$$

where $\Gamma \approx 1.2824271\ldots$ is the so-called Gleisher constant.

Proof. We have the well known inequality $\|A\| \leq \sqrt{n}M$ and on the other hand by the classical Sturm separation theorem (cf. [1]) $A_1 \geq m, A_2 \geq m^2, \ldots, A_n \geq m^n$. Consequently, we have the estimate

$$\frac{\|A\|^{\frac{3}{4}n^2-\frac{1}{4}n-\frac{1}{2}}}{A_1^{\frac{3}{2}}\cdots A_{n-1}^{\frac{3}{2}}A_n^{\frac{1}{2}}} \leq n^{\frac{3}{8}n^2-\frac{3}{8}n-\frac{1}{4}} \frac{1}{M^{\frac{1}{2}}}\left(\frac{M}{m}\right)^{\frac{3}{4}n^2-\frac{1}{4}n}. \tag{2.13}$$

Furthermore, we have for the product $\prod_{k=1}^{n} k^k$ the estimate

$$\prod_{k=1}^{n} k^{-k} \leq \frac{1}{\Gamma}\,\frac{e^{\frac{n^2}{4}}}{n^{\frac{1}{2}n^2+\frac{1}{2}n+\frac{1}{12}}} \tag{2.14}$$

(cf. [4]). By Theorem 2.5 and inequalities (2.13) - (2.14) we obtain (2.12) ∎

3. Lipschitz constant estimates based on spectral properties of the matrix A

Accordingly to Corollary 2.7 both inequalities (2.11) and (2.12) are quite rough. However, if the information about the spectrum of the matrix A is available, the estimate of $\|L'(A)\|$ can be significantly improved.

Theorem 3.1. *Let A be a positive definite symmetric $(n \times n)$-matrix and suppose that m and M are the smallest and the largest eigenvalues of A, respectively. Then*

$$\|L'(A)\| \leq \frac{(n+1)^2}{\sqrt{3}}\,\frac{M^{2n-3}}{m^{2n-\frac{5}{2}}}. \tag{3.1}$$

Proof. In the case $3 \leq k < n$, by applying inequality (2.1) to (2.3) we obtain

$$\begin{aligned}\left|\frac{\partial l_{sk}}{\partial a_{ij}}\right| \leq\; & \frac{1}{A_{k-1}^{\frac{1}{2}}A_k^{\frac{1}{2}}}\,\frac{a_{sk,ij}^{k-1}}{(k-1)^{\frac{k-1}{2}}} \\ &+ \frac{1}{2A_{k-1}^{\frac{1}{2}}A_k^{\frac{3}{2}}}\,\frac{a_{sk}^{k}\,a_{k,ij}^{k-1}}{k^{\frac{k}{2}}(k-1)^{\frac{k-2}{2}}} \\ &+ \frac{1}{2A_{k-1}^{\frac{3}{2}}A_k^{\frac{1}{2}}}\,\frac{a_{sk}^{k}\,a_{k-1,ij}^{k-2}}{k^{\frac{k}{2}}(k-2)^{\frac{k-2}{2}}}\end{aligned} \tag{3.2}$$

where a_{sk}, $a_{sk,ij}$, $a_{k,ij}$ and $a_{k-1,ij}$ are the Hilbert-Schmidt norms of matrices corresponding to the minors A_{sk}, $\frac{\partial A_{sk}}{\partial a_{ij}}$, $\frac{\partial A_k}{\partial a_{ij}}$ and $\frac{\partial A_{k-1}}{\partial a_{ij}}$, respectively. In similar way, in the case $k = n$ (2.6) yields

$$\left|\frac{\partial l_{nn}}{\partial a_{ij}}\right| \le \frac{1}{2A_{n-1}^{\frac{1}{2}} A_n^{\frac{1}{2}}} \frac{a_{n,ij}^{n-1}}{(n-1)^{\frac{n-2}{2}}} + \frac{A_n^{\frac{1}{2}}}{2A_{n-1}^{\frac{3}{2}}} \frac{a_{n-1,ij}^{n-2}}{(n-2)^{\frac{n-2}{2}}}. \tag{3.3}$$

Since for $3 \le k \le n$ we have

$$\left.\begin{aligned} a_{sk} &\le \sqrt{k}\,\alpha_{sk} \\ a_{sk,ij} &\le \sqrt{k-1}\,\alpha_{sk,ij} \\ a_{k,ij} &\le \sqrt{k-1}\,\alpha_{k,ij} \\ a_{k-1,ij} &\le \sqrt{k-2}\,\alpha_{k-1,ij} \end{aligned}\right\}$$

where α_{sk}, $a_{sk,ij}$, $\alpha_{k,ij}$ and $\alpha_{k-1,ij}$ are the norms of the matrices corresponding to the minors A_{sk}, $\frac{\partial A_{sk}}{\partial a_{ij}}$, $\frac{\partial A_k}{\partial a_{ij}}$ and $\frac{\partial A_{k-1}}{\partial a_{ij}}$, respectively, we obtain from (3.2) that in the case $3 \le k < n$ we have

$$\left|\frac{\partial l_{sk}}{\partial a_{ij}}\right| \le \frac{\alpha_{sk,ij}^{k-1}}{A_{k-1}^{\frac{1}{2}} A_k^{\frac{1}{2}}} + \frac{\alpha_{sk}^{k}\,\alpha_{k,ij}^{k-1}}{2A_{k-1}^{\frac{1}{2}} A_k^{\frac{3}{2}}} + \frac{\alpha_{sk}^{k}\,\alpha_{k-1,ij}^{k-2}}{2A_{k-1}^{\frac{3}{2}} A_k^{\frac{1}{2}}} \tag{3.4}$$

and in the case $k = n$ it follows from (3.3)

$$\left|\frac{\partial l_{nn}}{\partial a_{ij}}\right| \le \frac{\alpha_{n,ij}^{n-1}}{2A_{n-1}^{\frac{1}{2}} A_n^{\frac{1}{2}}} + \frac{A_n^{\frac{1}{2}}\alpha_{n-1,ij}^{n-2}}{2A_{n-1}^{\frac{3}{2}}}. \tag{3.5}$$

Notice that all the norms α_{sk}, $\alpha_{sk,ij}$, $\alpha_{k,ij}$ and $\alpha_{k-1,ij}$ do not exceed the usual norm of A which happens to be equal to M. On the other hand, using the inequalities $A_k \ge m^k$ where $k = 1, \ldots, n$ (3.4) implies for $k < n$

$$\left|\frac{\partial l_{sk}}{\partial a_{ij}}\right| \le \frac{M^{k-1}}{m^{k-\frac{1}{2}}} + \frac{M^{2k-1}}{2m^{2k-\frac{1}{2}}} + \frac{M^{2k-2}}{2m^{2k-\frac{3}{2}}}, \tag{3.6}$$

and (3.5) implies for $k = n$

$$\left|\frac{\partial l_{nn}}{\partial a_{ij}}\right| \le \frac{M^{n-1}}{2m^{n-\frac{1}{2}}} + \frac{M^{\frac{3}{2}n-2}}{2m^{\frac{3}{2}n-\frac{3}{2}}}. \tag{3.7}$$

It follows from (3.6) and (3.7) that for all $k \le n$ we have

$$\left|\frac{\partial l_{sk}}{\partial a_{ij}}\right| \le 2\frac{M^{2n-3}}{m^{2n-\frac{5}{2}}}. \tag{3.8}$$

Summarizing inequalities (3.8), first for fixed k over i and j, and then for $k = 1, \ldots, n$ we obtain

$$\|L'(A)\| \le 2\sqrt{\sum_{k=1}^{n}(n+1-k)k^2\,\frac{M^{2n-3}}{m^{2n-\frac{5}{2}}}}. \tag{3.9}$$

Since

$$\sum_{k=1}^{n}(n+1-k)\,k^2 = \frac{n(n+1)^2(n+2)}{12}$$

estimate (3.1) follows from (3.9) ∎

It is not hard to show that the estimates obtained in Theorems 2.5 and 3.1 can be essentially improved. For both cases, it is possible to sharpen the estimates for $\|L'(A)\|$ by expressing them in terms of $A_1, \ldots, A_n$, $\|A\|$ and M, but the obtained formula is rather cumbersome and awkward. Trying to simplify the estimates for $\|L'(A)\|$ we derived (2.12) and (3.1) based on the idea of equal contributions of different entries of the matrix $L(A)$. The enormous difference between estimates (2.12) and (3.1), i.e. the exponential and quadratic growths, should be given few words of explanation. In the case of estimate (2.12) we deal with *a priori* information about A, i.e. we only use the characteristics of A that can be simply calculated from the coefficients of the matrix A (for example, $\|A\|, A_1, \ldots, A_n$), but in the case of estimate (3.1) we deal with *a posteriori* information requiring the knowledge of the bounds m and M which are not easy to calculate.

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