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Regular Potential Approximation for δ-Perturbation Supported by Curve of the Laplace-Beltrami Operator on the Sphere

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Abstract. Operator extension theory model for δ-perturbation supported by curve of the Laplace-Beltrami operator on the sphere is described. The sequence of operators with regular potentials converging to the model operator in norm resolvent sense is constructed.

Keywords. Laplace-Beltrami operator, δ-perturbation

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1. Introduction

Nanoscience is a new area of research in solid state physics. Modern technology allows to create the structures of submicron sizes which have unique properties: quantum Hall effect [16, 27], Aharonov-Bohm effect in quantum rings [4], quantization of conductance in quantum wires [9], etc. Besides theoretical interest, nanostructures are interesting from the standpoint of their practical use. Clearly that such structures have many advantages over existing electronic devices: compactness, low energy consumption, high-speed performance and others. Moreover, last decades concepts of quantum computer (based, particularly, on nanosrtuctures) are actively developed [10].

Last years curved nanostructures are investigated intensively. Recently methods of creating of curved 2D quantum layers and nano-objects of different forms are developed [28]. A number of physical works is related with physical properties of curved nanostructures, which are closely related with spectral properties of the corresponding Hamiltonian. For example, studies of spherical

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nanostructures have shown that they have interesting spectral [5, 29] and optical [18, 24] properties. Dependence of the absorption spectrum from optical properties of nanoparticle was discussed in [1]. In [19] theoretical model for description of optical properties of spherical nano-shell was developed.

Interesting way to construct the model of curved nanostructure is to consider the operator in curved space (e.g. Riemannian manifold) [6–8]. The spectra of the Hamiltonians for spaces of different geometries are investigated in [3,13,26]. The theory of self-adjoint operators perturbed by potential supported by a set of zero measure (point, curve, etc.) gives us an efficient instrument to construct models [2, 21]. The case of curve is often named a problem of leaky quantum graph [11, 12, 17, 22]. For justification of this model it is possible to construct the approximation of the model operator by the corresponding operator with smooth short-range potential. For $\mathbb{R}^2$ and $\mathbb{R}^3$ the corresponding approximations are constructed in [23, 25]. In the present paper we describe the corresponding model operator for the Laplace-Beltrami operator on the 2D sphere imbedded into $\mathbb{R}^3$ and construct the approximation of this operator (in norm resolvent sense) by the corresponding operator with short-range potential. We incorporate here the ideas of the proofs from the corresponding approximation problem for point-like perturbation in curved space [14] and for perturbation supported by curve in $\mathbb{R}^2$ [23].

Namely, let $R(\lambda), R_0(\lambda)$ be the resolvents of perturbed and unperturbed Hamiltonians, respectively, $R_\varepsilon(\lambda)$ denotes the resolvent of the Hamiltonian with a short-range potential (detailed description see in Section 3). The aim of this article is to prove the following theorem.

Theorem 1.1. *For large enough* $|\lambda|$ $(\lambda \neq 0)$ *resolvent* $R_\varepsilon(\lambda)$ *converges to resolvent* $R(\lambda)$ *when* $\varepsilon \to 0$ *in the space* $B\big(L_2(\mathbb{R}^2), H^1(\mathbb{R}^2)\big)$.

2. Model description

We consider the 2D unit sphere $S^2 \subset \mathbb{R}^3$. Let Ω be the domain in S^2 with smooth boundary $\partial\Omega$. For simplicity we consider the domain in semisphere S^2_+ restricted by the plane orthogonal to polar axis. We suppose that its orthogonal projection into the plane is a star-like domain. In standard spherical coordinates $(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$ the Laplace-Beltrami operator has the form

$$\Delta_{BL} = \left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right). \quad (1)$$

Let L^{in} denotes the operator Δ_{BL} on Ω, i.e., the closure of the operator acts in accordance with (1) and is defined on the set of smooth functions on Ω. Also we define the operator L^{ex} as the Laplace-Beltrami operator on the domain $S^2 \setminus \Omega$. Let $L = L^{in} \oplus L^{ex}$, i.e., $D(L) = \{(f_1(x), f_2(x)) : f_1(x) \in H^2(\Omega),$ $f_2(x) \in H^2(S^2 \setminus \Omega)\}$ and $L(f_1(x), f_2(x)) = \big(L^{in} f_1(x), L^{ex} f_2(x)\big)$. Here $H^2(\Omega)$ is the corresponding Sobolev space. The operator L is self-adjoint. To construct a model of perturbation of L supported by curve $\partial\Omega$ we use the so-called 'restriction-extension' procedure [21, 22].

Namely, let us consider the restriction of L onto the set

$$D = \{(f_1(x), f_2(x)) \in C^\infty(S^2) : f_1(x) = f_2(x) = 0 \quad \forall x \in \partial\Omega\}.$$

To get the domain of its self-adjoint extensions one should choose the elements from $D(\tilde{L}^*)$ which satisfy the following condition:

$$\langle \tilde{L}^* f(x) | g(x) \rangle - \langle f(x) | \tilde{L}^* g(x) \rangle = 0.$$

Here $\langle \cdot | \cdot \rangle$ marks the inner product in $L^2(\Omega^{in} \oplus \Omega^{ex})$. In more details, if $f(x) = (f_1(x), f_2(x))$, $g(x) = (g_1(x), g_2(x))$, then

$$\iint_\Omega (-g_1 \Delta_{BL} f_1 + f_1 \Delta_{BL} g_1)\, dS + \iint_{S^2 \setminus \Omega} (-g_2 \Delta_{BL} f_2 + f_2 \Delta_{BL} g_2)\, dS = 0. \quad (2)$$

We consider the first of these two integrals. Further, we introduce the standard polar coordinates (r, φ) on the plane $\mathbb{R}^2$, which are related with the spherical coordinates on the sphere by the expressions

$$r = \sin\theta, \quad \varphi = \phi.$$

Let Ω' denotes the orthogonal projection of Ω.

By replacing the variables in expression (1), one obtains that the Laplace-Beltrami operator in new coordinates has the form

$$\Delta'_{BL} = (1 - r^2)\frac{\partial^2}{\partial r^2} + \frac{1 - 2r^2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \varphi^2}.$$

The coefficients of the first fundamental form for the sphere are

$$E = r^2, \quad F = 0, \quad G = \frac{1}{1 - r^2}.$$

Hence, the first integral from (2) takes the form

$$\iint_\Omega (-g_1 \Delta_{BL} f_1 + f_1 \Delta_{BL} g_1)\, dS = \iint_{\Omega'} (-g_1 \tilde{\Delta}_{BL} f_1 + f_1 \tilde{\Delta}_{BL} g_1)\, dr d\varphi,$$

where

$$\tilde{\Delta}_{BL} = \frac{r}{\sqrt{1 - r^2}} \Delta'_{BL}.$$

Using integration by parts, we obtain

$$\iint_{\Omega'} (-g_1 \tilde{\Delta}_{BL} f_1 + f_1 \tilde{\Delta}_{BL} g_1)\, dr d\varphi$$
$$= \int_0^{2\pi} r\sqrt{1-r^2} \left(\frac{\partial f}{\partial r}\Big|_{ex} - \frac{\partial f}{\partial r}\Big|_{in} + \frac{\partial g}{\partial r}\Big|_{ex} - \frac{\partial g}{\partial r}\Big|_{in} \right) d\varphi.$$

By conventional way one obtains that self-adjoint extension can be described by boundary conditions on $\partial\Omega' = \{(r,\varphi) : r = r(\varphi)\}$ for the function from the operator domain and its derivatives:

$$f|_{ex} = f|_{in}, \quad \frac{\partial f}{\partial r}\Big|_{ex} - \frac{\partial f}{\partial r}\Big|_{in} = \alpha(r(\varphi),\varphi) f|_{in}, \tag{3}$$

where α is some real smooth function.

3. Approximation

Let us fix the extension. It means that we fix smooth function $\alpha(x)$ on $\partial\Omega$. Let L^α be the extension which is defined by condition (3):

$$D(L^\alpha) = \left\{ f\colon f \in H^1(S^2), f \in H^2(\Omega^{in,ex}), f|_{ex} = f|_{in}, \frac{\partial f}{\partial r}\Big|_{ex} - \frac{\partial f}{\partial r}\Big|_{in} = \alpha(r(\varphi),\varphi) f|_{in} \right\},$$

where H^s is the Sobolev space.

For simplicity we introduce the coordinates (t,φ), where $t = \frac{r}{1-r}$. Then the unit disc maps onto $\mathbb{R}^2$ and Ω' maps onto some star-like domain $\tilde{\Omega}$. Let $t = P(\varphi)$ determines $\partial\tilde{\Omega}$. We assume that $P(\varphi)$ is continuously differentiable function and $P(\varphi) \neq 0$ for all φ. Next, we use coordinates (τ,φ), where $\tau = \frac{t}{P(\varphi)}$. Clearly, that the equation of $\partial\tilde{\Omega}$ has the form $\tau = 1$ in this coordinates.

Further we introduce short-range potential $A_\varepsilon(x)$ as follows. Let $\rho(u)$ be fixed infinitely smooth function such that

$$\rho(u) \geq 0 \quad \forall u \in \mathbb{R}, \quad \mathrm{supp}\rho \subset [-1,1], \quad \int_{-\infty}^{+\infty} \rho(u)\, du = 1.$$

Then

$$A_\varepsilon(\tau,\varphi) = \varepsilon^{-1} \rho\left(\frac{\tau-1}{\varepsilon}\right) \alpha(1,\varphi).$$

As a first step of the proof of Theorem 1.1 we prove the following lemmas.

Lemma 3.1. *Let* $\varepsilon > 0$ *and*

$$\begin{array}{rcl} A & = & \min_{\varphi\in[0,2\pi]} \left\{ P(\varphi)\big((P'(\varphi))^2 + (P(\varphi))^2\big)^{-\frac{1}{2}} \right\} > 0, \\ B & = & \max_{\varphi\in[0,2\pi]} \left\{ \big((P'(\varphi))^2 + (P(\varphi))^2\big)^{\frac{1}{2}} \right\}. \end{array} \qquad (4)$$

Then for all $f \in H^1(\mathbb{R}^2)$ *the following inequalities hold*

$$\|Tf\|^2_{L^2(\Gamma_{\tau_0})} \le A^{-1}\varepsilon\|\nabla f\|^2 + A^{-1}\varepsilon^{-1}\|f\|^2,$$

where Γ_{τ_0} *is the line determined by the equation* $\tau = \tau_0$ $(\Gamma_1 = \partial\tilde{\Omega})$, T *is the operator which maps function from* $H^1(\mathbb{R}^2)$ *to its trace in* $L_2(\Gamma_{\tau_0})$.

Proof. First of all, we note that the set of infinitely smooth functions with compact supports is dense in $H^1(\mathbb{R}^2)$ and T is bounded operator. Therefore, we need to prove this statement for functions from $H^1(\mathbb{R}^2)$. Clearly, that $2|pq| \le \varepsilon|p|^2 + \varepsilon^{-1}|q|^2$. Thus, for all φ we have

$$\begin{aligned} |f(\tau P(\varphi),\varphi)|^2 &= -2\mathrm{Re}\int_{\tau P(\varphi)}^{\infty} \frac{\partial f}{\partial r}(r,\varphi)\overline{f(r,\varphi)}\,dr \\ &\le \varepsilon\int_{\tau P(\varphi)}^{\infty}\Big|\frac{\partial f}{\partial r}(r,\varphi)\Big|^2 dr + \varepsilon^{-1}\int_{\tau P(\varphi)}^{\infty}\Big|f(r,\varphi)\Big|^2 dr \\ &\le \varepsilon\int_{\tau P(\varphi)}^{\infty}\frac{r}{\tau P(\varphi)}\Big|\frac{\partial f}{\partial r}(r,\varphi)\Big|^2 dr + \varepsilon^{-1}\int_{\tau P(\varphi)}^{\infty}\frac{r}{\tau P(\varphi)}\Big|f(r,\varphi)\Big|^2 dr. \end{aligned}$$

Multiplying the both parts of these inequalities by $\tau P(\varphi)$, using the inequality

$$P(\varphi) \ge A\big((P'(\varphi))^2 + (P(\varphi))^2\big)^{\frac{1}{2}}$$

and integrating over φ, we get

$$A\int_{\Gamma_{\tau_0}}\Big|f(x)\Big|^2 dS_x \le \varepsilon\int_{\Omega^{ex}_{\tau_0}}\Big|\frac{\partial f}{\partial r}(x)\Big|^2 dx + \varepsilon^{-1}\int_{\Omega^{ex}_{\tau_0}}\Big|f(x)\Big|^2 dx \le \varepsilon||\nabla f||^2 + \varepsilon^{-1}||f||^2.$$

Here $\Omega^{in}_{\tau_0}$, $\Omega^{ex}_{\tau_0}$ denote the domains corresponding to the curve $\tau = \tau_0$. □

Lemma 3.2. *Let* $\tau > 0$ *and* f *belongs to the Schwartz' class* $S(\mathbb{R}^2)$. *Then*

$$||f(\tau,\cdot)||_{L_2(\Gamma_1)} \le A^{-\frac{1}{2}}\tau^{-\frac{1}{2}}||f||_{H_1}.$$

Proof. Statement of the lemma is simple consequence of the previous lemma. Namely, if $\varepsilon = 1$ then $\tau||f(\tau,\cdot)||^2_{L_2(\Gamma_1)} \le A^{-1}||f||^2_{H^1}$. □

Lemma 3.3. *Let $f \in S(\mathbb{R}^2)$. Then*

$$||f(\tau,\cdot) - f(\tau',\cdot)||_{L_2(\Gamma_1)} \leq \big(B|\tau-\tau'|(\min\{\tau,\tau'\})^{-1}\big)^{\frac{1}{2}}||f||_{H^1}.$$

Proof. Clearly, that it is enough to prove this lemma for the case $0 < \tau' < \tau$. Due to the Schwartz' inequality for all φ we have

$$\begin{aligned}|f(\tau,\varphi) - f(\tau',\varphi)|^2 &= \Big|\int_{\tau' P}^{\tau P} \frac{\partial f}{\partial r}(r,\varphi)\,dr\Big|^2 \\ &\leq P(\tau-\tau')\int_{\tau' P}^{\tau P}\Big|\frac{\partial f}{\partial r}(r,\varphi)\Big|^2 dr \\ &\leq (\tau-\tau')(\tau')^{-1}\int_{\tau' P}^{\tau P}\Big|\frac{\partial f}{\partial r}(r,\varphi)\Big|^2 r\,dr.\end{aligned}$$

Multiplying by $\big((P'(\varphi))^2 + (P(\varphi))^2\big)^{-\frac{1}{2}}$, taking into account (4) and integrating over φ, we obtain

$$\begin{aligned}||f(\tau,\cdot) - f(\tau',\cdot)||^2_{L_2(\Gamma_1)} &\leq B(\tau-\tau')(\tau')^{-1}\int_{\tilde\Omega^{in}_{\tau}\setminus\tilde\Omega^{in}_{\tau'}}\Big|\frac{\partial f}{\partial r}(x)\Big|^2 dx \\ &\leq B(\tau-\tau')(\tau')^{-1}||f||_{H_1}.\end{aligned}$$

□

Definition 3.4. Define F_Γ as the following transformation on $L_2(\Gamma_\tau)$

$$(F_{\Gamma_\tau} f)(\xi) = (2\pi)^{-1}\int_{\Gamma_\tau} e^{-\imath\xi\cdot x} f(x)\,dS_x, \quad \xi \in \mathbb{R}^2.$$

Definition 3.5. Define $H^s(\mathbb{R}^2)$ as the following space:

$$H^s(\mathbb{R}^2) = \big\{f(x) : (1+x^2)^{\frac{s}{2}} f(x) \in L_2(\mathbb{R}^2)\big\}$$

with the norm $\|f\|_{H^s(\mathbb{R}^2)} = \big\|(1+|\cdot|^2)^{\frac{s}{2}} f\big\|$.

Lemma 3.6. *Let $s > 2^{-1}$. Then for any $f \in L_2(\mathbb{R}^2)$ there exists a constant $C = C(\tau,s)$ such that*

$$\|F_{\Gamma_\tau} f\|_{H^{-s}(\mathbb{R}^2)} \leq C\|f\|_{L_2(\Gamma_\tau)}$$

The proof can be obtained by simple modifications of the corresponding statement in [20].

Lemma 3.7. *Let r and r' be positive. Then for any $f \in L_2(\Gamma_1)$ one has*

$$\|(F_\Gamma f)(r\cdot) - (F_\Gamma f)(r'\cdot)\|_{H^{-1}(\mathbb{R}^2)} \leq \big(B|r-r'|(\min\{r,r'\})^{-1}\big)^{\frac{1}{2}}\|f\|_{L_2(\Gamma_1)}.$$

Proof. Let $f \in S(\mathbb{R}^2)$, $u \in L_2(\Gamma_1)$. We consider the integral

$$\int_{\mathbb{R}^2} d\xi f(\xi)\overline{\big((F_\Gamma u)(r\xi) - (F_\Gamma u)(r'\xi)\big)} = \int_{\Gamma_1} dx\big((F^* f)(rx) - (F^* f)(r'x)\big)\overline{u(x)},$$

where F^* is the inverse Fourier transform. By Schwartz' inequality and Lemma 3.3 we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} d\xi f(\xi)\overline{\big((F_\Gamma u)(r\xi) - (F_\Gamma u)(r'\xi)\big)}\right| \\
&\le \|(F^* f)(r\cdot) - (F^* f)(r'\cdot)\|_{L_2(\Gamma_1)}\|u(x)\|_{L_2(\Gamma_1)} \\
&\le \big(B|r - r'|(\min\{r, r'\})^{-1}\big)^{\frac{1}{2}}\|F^* f\|_{H^1}\|u\|_{L_2(\Gamma_1)} \\
&\le \big(B|r - r'|(\min\{r, r'\})^{-1}\big)^{\frac{1}{2}}\|(1 + |\cdot|^2)f(\cdot)\| \cdot \|u\|_{L_2(\Gamma 1)} \\
&\le \big(B|r - r'|(\min\{r, r'\})^{-1}\big)^{\frac{1}{2}}\|f\|_{H^1(\mathbb{R}^2)}\|u\|_{L_2(\Gamma_1)}.
\end{aligned}$$

The statement of the lemma is a consequence of this inequality because $S(\mathbb{R}^2)$ is dense in $H^1(\mathbb{R}^2)$. □

The Green function $G_{LB}(x, y; k)$ of the operator $-\Delta_{LB}$ is well known [15]:

$$G_{LB}(x, y; k) = \frac{1}{\cos\left(\frac{\pi}{2}\sqrt{\frac{1}{4} + k}\right)}\mathcal{P}_{-\frac{1}{2}+\sqrt{\frac{1}{4}+k}}\left(-\cos\rho(x, y)\right), \quad x, y \in \mathbb{R}^2, \qquad (5)$$

where $\mathcal{P}_\nu(x)$ is the Legendre function, k is the square root of the spectral parameter λ, $\rho(x, y)$ is the geodesic distance between x and y. We can rewrite the expression for the function (5) using the variables (t, φ) as

$$\begin{aligned}
\widehat{G}_{LB}(t_1, \varphi_1, t_2, \varphi_2; k) =& \frac{t_2}{(t_2 + 1)^2\sqrt{2t_2 + 1}} \cdot \frac{1}{\cos\left(\frac{\pi}{2}\sqrt{\frac{1}{4} + k}\right)} \\
&\times \mathcal{P}_{-\frac{1}{2}+\sqrt{\frac{1}{4}+k}}\left(-\frac{\sqrt{(2t_1+1)(2t_2+1)}+t_1 t_2\cos(\varphi_1-\varphi_2)}{(t_1+1)(t_2+1)}\right).
\end{aligned}$$

Definition 3.8. Define M as the following operator from $L_2(\Gamma_1)$ to $H^1(\mathbb{R}^2)$:

$$Mf(x) = \int_{\Gamma_1} \widehat{G}_{LB}(x, y; k)\alpha(y)f(y)\,dS_y, \quad x \in \mathbb{R}^2.$$

If Im $k > 0$, then M is bounded operator.

Lemma 3.9. *Let ε, s, λ be such that $0<\varepsilon<2^{-1}, 2^{-1}<s<1, \lambda\in\mathbb{C}\backslash[0,+\infty)$. Then there exists a constant $C_1 = C_1(s)$ (which does not depend on ε and λ) such that*

$$\|R_0(\lambda)A_\varepsilon\|_{B(H^1(\mathbb{R}^2))} \le C_1\left(\sup_{\xi\in\mathbb{R}^2}\big((1+|\xi|^2)^{1+s}|V(\xi,\lambda)|^2\big)\right)^{\frac{1}{2}},$$

where $V(\xi,\lambda)$ is bounded function.

Proof. For any $u\in S(\mathbb{R}^2)$ we have

$$\begin{aligned}(FR_0(\lambda)A_\varepsilon u)(\xi) &= (2\pi)^{-1}\int_{\mathbb{R}^2} dx\, e^{-\imath\xi x}\int_{\mathbb{R}^2} dy\,\frac{1}{\varepsilon}\rho\left(\frac{\tau-1}{\varepsilon}\right)\alpha(1,\varphi)u(y)\widehat{G}_{LB}(x,y;k)\\ &= (2\pi)^{-1}\int_{\mathbb{R}^2} dy\, e^{-\imath\xi y}\frac{1}{\varepsilon}\rho\left(\frac{\tau-1}{\varepsilon}\right)\alpha(1,\varphi)u(y)V(\xi,\lambda),\end{aligned}$$

where $V(\xi,\lambda) = e^{\imath\xi y}\int_{\mathbb{R}^2} dx e^{-\imath\xi x}\widehat{G}_{LB}(x,y;k)$.

Here, the branch of the square root $\sqrt{\lambda}=k$ is chosen in such a way that $\operatorname{Im} k\ge 0$. Thus, we have

$$\begin{aligned}&\|R_0(\lambda)A_\varepsilon u\|^2_{H^1(\mathbb{R}^2)}\\ &= \int_{\mathbb{R}^2} d\xi(1+|\xi|^2)\left|\int_{1-\varepsilon}^{1+\varepsilon} d\tau\frac{\tau}{\varepsilon}\rho\left(\frac{\tau-1}{\varepsilon}\right)V(\xi,\lambda)\big(F_\Gamma(\beta(1,\cdot)u(\tau,\cdot))\big)(\tau\xi)\right|^2,\end{aligned}$$

where $\beta(1,\varphi)=\alpha(1,\varphi)P^2(\varphi)\big((P'(\varphi))^2+(P(\varphi))^2\big)^{-\frac{1}{2}}$. Schwartz' inequality and Fubini's theorem lead to the following inequality

$$\begin{aligned}\|R_0(\lambda)A_\varepsilon u\|^2_{H^1(\mathbb{R}^2)} &\le \int_{\mathbb{R}^2} d\xi(1+|\xi|^2)|V(\xi,\lambda)|^2\int_{1-\varepsilon}^{1+\varepsilon} d\tau\frac{\tau^2}{\varepsilon}\rho\left(\frac{\tau-1}{\varepsilon}\right)\\ &\quad\times\int_{1-\varepsilon}^{1+\varepsilon} d\tau\frac{1}{\varepsilon}\rho\left(\frac{\tau-1}{\varepsilon}\right)\left|\big(F_\Gamma(\beta(1,\cdot)u(\tau,\cdot))\big)(\tau\xi)\right|^2\\ &\le \sup_{\xi\in\mathbb{R}^2}\big((1+|\xi|^2)^{1+s}|V(\xi,\lambda)|^2\big)(1+\varepsilon)^2\int_{1-\varepsilon}^{1+\varepsilon} d\tau\frac{1}{\varepsilon}\rho\left(\frac{\tau-1}{\varepsilon}\right)\\ &\quad\times\int_{\mathbb{R}^2} d\xi(1+|\xi|^2)^{-s}\left|\big(F_\Gamma(\beta(1,\cdot)u(\tau,\cdot))\big)(\tau\xi)\right|^2.\end{aligned}\tag{6}$$

By replacing of the variables $\eta=\tau\xi$, one obtains

$$\begin{aligned}&\int_{\mathbb{R}^2} d\xi(1+|\xi|^2)^{-s}\left|\big(F_\Gamma(\beta(1,\cdot)u(\tau,\cdot))\big)(\tau\xi)\right|^2\\ &= \int_{\mathbb{R}^2} d\eta(1+\tau^{-2}|\eta|^2)^{-s}\tau^{-2}\left|\big(F_\Gamma(\beta(1,\cdot)u(\tau,\cdot))\big)(\eta)\right|^2.\end{aligned}$$

Using the evident inequality $(1+\tau^{-2}|\eta|^2)^{-s} \le \max\{\tau^{2s},1\}(1+|\eta|^2)^{-s}$ (for $s>0$ and $\tau>0$), one obtains by Lemmas 3.2 and 3.6

$$\begin{aligned}&\int_{\mathbb{R}^2} d\xi (1+|\xi|^2)^{-s} \left|\big(F_\Gamma(\beta(1,\cdot)u(\tau,\cdot))\big)(\tau\xi)\right|^2 \\ &\le \tau^{-2}\max\{\tau^{2s},1\}\|F_\Gamma(\beta(1,\cdot)u(\tau,\cdot))\|^2_{H^{-s}(\mathbb{R}^2)} \\ &\le \tau^{-2}\max\{\tau^{2s},1\}C^2\|\beta(1,\cdot)u(\tau,\cdot)\|^2_{L_\Gamma} \\ &\le A^{-1}\tau^{-3}\max\{\tau^{2s},1\}C^2(\max_{x\in\Gamma}|\beta(x)|)^2\|u\|^2_{H_1},\end{aligned} \tag{7}$$

where C is given in Lemma 3.6. Since $0<\varepsilon\le 2^{-1}$, one obtains

$$\begin{aligned}\|R_0(\lambda)A_\varepsilon u\|^2_{H^1(\mathbb{R}^2)} &\le \sup_{\xi\in\mathbb{R}^2}\big((1+|\xi|^2)^{1+s}|V(\xi,\lambda)|^2\big)C^2\big(\max_{x\in\Gamma}|\beta(x)|\big)^2 \\ &\quad\times \|u\|^2_{H^1}(1+\varepsilon^2)\int_{1-\varepsilon}^{1+\varepsilon} d\tau\frac{1}{\varepsilon}\,\rho\left(\frac{\tau-1}{\varepsilon}\right)\frac{1}{A}\frac{1}{\tau^4}\max\{\tau^{2s},1\} \\ &\le C_1^2(s)\sup_{\xi\in\mathbb{R}^2}\big((1+|\xi|^2)^{1+s}|V(\xi,\lambda)|^2\big)\|u\|^2_{H^1}.\end{aligned}$$

where $C_1(s)$ does not depend on ε $(0<\varepsilon\le 2^{-1})$. This inequality leads to the statement of the lemma, because $R_0(\lambda)$ is bounded operator from $L_2(\mathbb{R}^2)$ to $H^1(\mathbb{R}^2)$ and $S(\mathbb{R}^2)$ is dense in $H^1(\mathbb{R}^2)$. □

Lemma 3.10. *Let ε, s, λ be such that $0<\varepsilon\le 2^{-1}, 2^{-1}<s<1, \lambda\in\mathbb{C}\setminus[0,\infty)$. Then there exist a constant $C_2 = C_2(s,\lambda)$ (which does not depend on ε) such that*

$$\|R_0(\lambda)A_\varepsilon + MT\|_{B(H^1(\mathbb{R}^2))} \le C_2\varepsilon^{\frac{1}{2}}.$$

Proof. As in the proof of the previous lemma, we obtain for any function u from the Schwartz class

$$(FMTu)(\xi) = -V(\xi,\lambda)\big(F_\Gamma(\beta(1,\cdot)u(1,\cdot))\big)(\xi).$$

Thus, we have

$$\begin{aligned}&\big(F(R_0(\lambda)A_\varepsilon + MT)u\big)(\xi) \\ &= V(\xi,\lambda)\int_{1-\varepsilon}^{1+\varepsilon} d\tau(\tau-1)\frac{1}{\varepsilon}\,\rho\left(\frac{\tau-1}{\varepsilon}\right)\big(F_\Gamma(\beta(1,\cdot)u(\tau,\cdot))\big)(\tau\xi) \\ &\quad + V(\xi,\lambda)\int_{1-\varepsilon}^{1+\varepsilon} d\tau\frac{1}{\varepsilon}\,\rho\left(\frac{\tau-1}{\varepsilon}\right)\big(F_\Gamma(\beta(1,\cdot)(u(\tau,\cdot)-u(1,\cdot)))\big)(\tau\xi) \\ &\quad + V(\xi,\lambda)\int_{1-\varepsilon}^{1+\varepsilon} d\tau\frac{1}{\varepsilon}\,\rho\left(\frac{\tau-1}{\varepsilon}\right)\Big(\big(F_\Gamma(\beta(1,\cdot)u(1,\cdot))\big)(\tau\xi) - \big(F_\Gamma(\beta(1,\cdot)u(1,\cdot))\big)(\xi)\Big) \\ &= J_0(\xi) + J_1(\xi) + J_2(\xi).\end{aligned}$$

It is necessary to estimate $H^1(\mathbb{R}^2)$ norms of J_0, J_1 and J_2. By the way which we have used to derive (6), we obtain

$$\int_{\mathbb{R}^2} d\xi (1+|\xi|^2)|J_0(x)|^2 \le \varepsilon^2 b_0 \|u\|^2_{H^1}. \tag{8}$$

where b_0 does not depend on ε $(0<\varepsilon\le 2^{-1})$.

$$\int_{\mathbb{R}^2} d\xi (1+|\xi|^2)|J_1(\xi)|^2 \le \sup_{\xi\in\mathbb{R}^2}\left((1+|\xi|^2)^{1+s}|V(\xi,\lambda)|^2\right)\int_{1-\varepsilon}^{1+\varepsilon} d\tau \frac{1}{\varepsilon}\rho\left(\frac{\tau-1}{\varepsilon}\right)$$
$$\times \int_{\mathbb{R}^2} d\xi (1+|\xi|^2)^{-s}\left|\left(F_\Gamma(\beta(1,\cdot)(u(\tau,\cdot)-u(1,\cdot)))\right)(\tau\xi)\right|^2$$

The correlation (7) and Lemma 3.3 give us

$$\int_{\mathbb{R}^2} d\xi (1+|\xi|^2)^{-s}\left|\left(F_\Gamma(\beta(1,\cdot)(u(\tau,\cdot)-u(1,\cdot)))\right)(\tau\xi)\right|^2$$
$$\le \tau^{-2}\max\{\tau^{2s},1\}C^2\|\beta(1,\cdot)(u(\tau,\cdot)-u(1,\cdot))\|^2_{L_2(\Gamma)}$$
$$\le \tau^{-2}\max\{\tau^{2s},1\}C^2\left(\max_{x\in\Gamma}|\beta(x)|\right)^2|\tau-1|B\left(\min\{\tau,1\}\right)^{-1}\|u\|^2_{H^1}$$
$$\le \varepsilon B(1+\varepsilon)^{2s}\frac{1}{(1-\varepsilon)^4}\frac{1}{A}C^2\left(\max_{x\in\Gamma}|\beta(x)|\right)^2\|u\|^2_{H^1},$$

because $\tau\in[1-\varepsilon,1+\varepsilon]$. Hence, we obtain

$$\int_{\mathbb{R}^2} d\xi (1+|\xi|^2)|J_1(\xi)|^2 \le \varepsilon b_1 \|u\|^2_{H^1}. \tag{9}$$

where b_1 does not depend on ε $(0<\varepsilon\le 2^{-1})$.

To estimate the norm of J_2, we take into account that

$$|J_2(\xi)|^2 \le |V(\xi,\lambda)|^2$$
$$\times\int_{1-\varepsilon}^{1+\varepsilon} d\tau\frac{1}{\varepsilon}\rho\left(\frac{\tau-1}{\varepsilon}\right)\left|\left(F_\Gamma(\beta(1,\cdot)u(1,\cdot))\right)(\tau\xi)-\left(F_\Gamma(\beta(1,\cdot)u(1,\cdot))\right)(\xi)\right|^2.$$

Using Fubini's theorem, one comes to the following inequality:

$$\int_{\mathbb{R}^2} d\xi (1+|\xi|^2)|J_2(\xi)|^2$$
$$\le \sup_{\xi\in\mathbb{R}^2}\left((1+|\xi|^2)^2|V(\xi,\lambda)|^2\right)\int_{1-\varepsilon}^{1+\varepsilon} d\tau\frac{1}{\varepsilon}\rho\left(\frac{\tau-1}{\varepsilon}\right)$$
$$\times\int_{\mathbb{R}^2} d\xi(1+|\xi|^2)^{-1}\left|\left(F_\Gamma(\beta(1,\cdot)u(1,\cdot))\right)(\tau\xi)-\left(F_\Gamma(\beta(1,\cdot)u(1,\cdot))\right)(\xi)\right|^2$$
$$\le \sup_{\xi\in\mathbb{R}^2}\left((1+|\xi|^2)^2|V(\xi,\lambda)|^2\right)\int_{1-\varepsilon}^{1+\varepsilon} d\tau\frac{1}{\varepsilon}\rho\left(\frac{\tau-1}{\varepsilon}\right)$$
$$\times\left\|\left(F_\Gamma(\beta(1,\cdot)u(1,\cdot))\right)(\tau\cdot)-\left(F_\Gamma(\beta(1,\cdot)u(1,\cdot))\right)(\cdot)\right\|^2_{H^{-1}(\mathbb{R}^2)}.$$

It follows from Lemmas 3.2 and 3.7 that

$$\begin{aligned}
&\|\big(F_\Gamma(\beta(1,\cdot)u(1,\cdot))\big)(\tau\cdot) - \big(F_\Gamma(\beta(1,\cdot)u(1,\cdot))\big)(\cdot)\|^2_{H^{-1}(\mathbb{R}^2)}\\
&\le |\tau-1|B(\min\{\tau,1\})^{-1}\|\beta(1,\cdot)u(1,\cdot)\|^2_{L_2(\Gamma)}\\
&\le A^{-1}(\max_{x\in\Gamma}|\beta(x)|)^2|\tau-1|B(\min\{\tau,1\})^{-1}\|u\|^2_{H^1}\\
&\le \varepsilon(1-\varepsilon)^{-1}A^{-1}(\max_{x\in\Gamma}|\beta(x)|)^2B\|u\|^2_{H^1},
\end{aligned}$$

for $\tau\in[1-\varepsilon,1+\varepsilon]$. Therefore, we get

$$\int_{\mathbb{R}^2} d\xi(1+|\xi|^2)|J_2(\xi)|^2 \le \varepsilon b_2\|u\|^2_{H^1}, \tag{10}$$

where b_2 does not depend on ε $(0<\varepsilon\le 2^{-1})$. Combining (8), (9) and (10) and taking into account that MT is bounded operator in $H^1(\mathbb{R}^2)$ and that $S(\mathbb{R}^2)$ is dense in $H^1(\mathbb{R}^2)$, one comes to the statement of the lemma. □

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. By the closed graph theorem $R_\varepsilon(\lambda)$ and $R(\lambda)$ are bounded operators from $L^2(\mathbb{R}^2)$ to $H^2(\mathbb{R}^2)$ and $H^1(\mathbb{R}^2)$, respectively. The two resolvent identities hold:

$$R_0(\lambda)-R_\varepsilon(\lambda)=R_0(\lambda)A_\varepsilon R_\varepsilon(\lambda), \quad R(\lambda)-R_0(\lambda)=MTR(\lambda).$$

Hence $R_\varepsilon(\lambda)-R(\lambda)=-R_0(\lambda)A_\varepsilon(R_\varepsilon(\lambda))-(R_0(\lambda)A_\varepsilon+MT)R(\lambda)$.

Let the regular value λ $(\lambda\in\mathbb{C}\backslash[0,\infty))$ be sufficiently far form the origin and such that

$$C_1\left(\sup_{\xi\in\mathbb{R}^2}(1+|\xi|^2)^{1+s}|V(\xi,\lambda)|^2\right)^{\frac12}<\frac12.$$

This is possible because $\frac12<s<1$. Then, for any $u\in L_2(\mathbb{R}^2)$ we get by Lemmas 3.9 and 3.10

$$\begin{aligned}
&\|R_\varepsilon(\lambda)u-R(\lambda)u\|_{H^1}\\
&\le \|R_0(\lambda)A_\varepsilon(R_\varepsilon(\lambda)u-R(\lambda)u)\|_{H^1}+\|(R_0(\lambda)A_\varepsilon+MT)R(\lambda)u\|_{H^1}\\
&\le 2^{-1}\|R_\varepsilon(\lambda)u-R(\lambda)u\|_{H^1}+\varepsilon^{\frac12}C_2\|R(\lambda)u\|_{H^1}.
\end{aligned}$$

Consequently,

$$\|R_\varepsilon(\lambda)u-R(\lambda)u\|_{H^1}\le 2\varepsilon^{\frac12}C_2\|R(\lambda)u\|_{H^1}\le 2\varepsilon^{\frac12}C_2\|R(\lambda)\|_{B(L^2(\mathbb{R}^2),H^1(\mathbb{R}^2))}\|u\|.$$

This inequality gives us the required result. □

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