

Optimal Control Problem of Positive Solutions to Second Order Impulsive Differential Equations

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Abstract. In this paper, we consider optimal control problem of second order impulsive differential equations. We show the existence and uniqueness of positive solutions to our problem for each given control functions. Also, we consider the control problem of positive solutions to our equations. Then, we prove the existence of an optimal control that minimizes the nonlinear cost functional. Moreover we give an example of the main results.

Keywords. Positive solutions, second order impulsive differential equations, fixed point theorem, optimal control problem.

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1. Introduction

In this paper, we consider the optimal control problem for the following second order impulsive differential equations:

$$(P) \begin{cases} -x''(t) = f(t, x(t)) + u(t), & t \in (0, T) \setminus \{t_1, t_2, \dots, t_m\}, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), & k = 1, 2, \dots, m, \\ \Delta x'|_{t=t_k} = \bar{I}_k(x(t_k)), & k = 1, 2, \dots, m, \\ x(0) = a, \\ x'(0) = b, \end{cases} \quad (1)$$

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where $T > 0$ is an arbitrary finite positive real number, f is a given function in $C[[0, T] \times \mathbb{R}, \mathbb{R}]$, u is a given function on $[0, T]$, $0 < t_1 < t_2 < \dots < t_m < T$, $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, $\Delta x'|_{t=t_k} = x'(t_k^+) - x'(t_k^-)$, $x(t_k^+)$ and $x'(t_k^+)$ ($x(t_k^-)$ and $x'(t_k^-)$) denote the right limit (left limit) of $x(t)$ and $x'(t)$ at $t = t_k$, respectively. Also, I_k and \bar{I}_k are given functions in $C[\mathbb{R}, \mathbb{R}]$, $k = 1, 2, \dots, m$. Furthermore, $a > 0$ and $b \geq 0$ are given constants.

In this paper, we show the existence and uniqueness of positive solutions to (P) by using a fixed point theorem of generalized concave operators. Also, we consider the optimal control problem (OP) of (P) as follows:

Problem (OP). Find an optimal control $u^* \in \mathcal{U}_M$ such that

$$\pi(u^*) = \inf_{u \in \mathcal{U}_M} \pi(u).$$

Here, \mathcal{U}_M is a control space defined by

$$\mathcal{U}_M := \{u \in L^2(0, T) \mid -M \leq u(t) \leq 0, \forall t \in [0, T]\}, \quad (2)$$

where M is a fixed positive number. Also, $\pi(u)$ is the cost functional defined by

$$\pi(u) := \frac{1}{2} \int_0^T |(x - x_d)(t)|^2 dt + \frac{1}{2} |x(T)|^2 + \frac{1}{2} \int_0^T |u(t)|^2 dt, \quad (3)$$

where $u \in \mathcal{U}_M$ is the control, a function x is a unique positive solution to the state problem (P) with the source control term u , and x_d is the given desired target profiles in $L^2(0, T)$.

It is well known that the study of concave operators and convex operators has been discussed by many authors, since it provided important theoretical foundation in the area of application [3, 5, 11, 14, 22]. In fact, Krasnoselskii [11] introduced the definition of h -concave operators and showed that an increasing, h -concave operator has at most one positive fixed point. Also, Guo [5] widened the conditions and removed the hypotheses of continuation for operators, and then extended the results of fixed points, eigenvectors for α -concave ($(-\alpha)$ -convex) operators. The authors in [14] introduced the concept of locally u_0 -concave operators and obtained some results about the existence and uniqueness of the fixed points. In [24], the authors studied nonlinear operator equations $x = Ax + x_0$, where A is a monotone generalized concave operator without the compactness and continuity conditions.

Recently, the theory of nonlinear operators have been used extensively in many fields, especially, in the solutions of differential equations. For the related works, we refer to the series of paper by Guo (cf. [3–6]), [1, 2, 7, 8, 10, 12, 13, 15, 19, 23, 25] etc. In particular, second-order impulsive differential equations have been studied with much of the attention given to positive solutions (cf. [6, 9, 26]). For

instance, Zhai, Yang and Zhang [24] showed existence and uniqueness results of second order differential equations by using a fixed point theorem generalized concave operators.

Also, there is a vast literature on optimal problems to impulsive differential equations. For instance, we refer to [16–18, 20, 21]. But, there is no result of optimal control problem of positive solutions to the impulsive differential equations.

This present paper aims to focus on the positive solutions to (P), and then, to consider the optimal control problem (OP) for (P). The main novelties found in this paper are the following:

- (i) to prove the existence and uniqueness of positive solutions to (P) for each given control functions;
- (ii) to show the existence of an optimal control to (OP).

The plan of this paper is as follows. In Section 2, we briefly recall the fundamentals of a fixed point theorem of generalized concave operators. In Section 3, the main theorems, denoted by Theorems 3.2 and 3.3 are to be stated. In Section 4, we prove Theorem 3.2 concerned with the existence and uniqueness of positive solutions to (P). In Section 5, we consider the control problem (OP). Then, we prove Theorem 3.3 concerned with the existence of an optimal control to (OP). In final Section 6, we give an example of the main results.

Notations and basic assumptions. Firstly, we mention the notations that are used throughout this paper.

Let $J := [0, T]$ and let the set $D := \{t_1, t_2, \dots, t_m\}$ be a partition on $(0, T)$ such that $0 < t_1 < t_2 < \dots < t_m < T$. For the sake of convenience, we put $J_0 := [0, t_1]$, $J_1 := (t_1, t_2]$, \dots , $J_{m-1} := (t_{m-1}, t_m]$, $J_m := (t_m, T]$ and $J' := J \setminus D$.

We define

$$PC[J, \mathbb{R}] := \{x \mid x : J \rightarrow \mathbb{R}, x(t) \text{ is continuous at } t \neq t_k \text{ and} \\ \text{left continuous at } t = t_k, x(t_k^+) \text{ exists, } k = 1, 2, \dots, m\}.$$

Then, we easily see that $PC[J, \mathbb{R}]$ is a Banach space with the norm $|x|_{PC} := \sup_{t \in J} |x(t)|$. Also, we see that

$$PC^1[J, \mathbb{R}] := \{x \in PC[J, \mathbb{R}] \mid x'(t) \text{ is continuous at } t \neq t_k \text{ and} \\ \text{left continuous at } t = t_k, x'(t_k^+) \text{ exists, } k = 1, 2, \dots, m\}$$

is a Banach space with the norm $|x|_{PC^1} := |x|_{PC} + |x'|_{PC}$.

We put $H := L^2(J)$ with the usual real Hilbert structure, and denote by $|\cdot|_H$ the norm in H , for simplicity.

Next, let us give some assumptions on data. Throughout this paper, we assume the following conditions (H1)–(H5).

- (H1) $f \in C[J \times \mathbb{R}, \mathbb{R}]$ such that $f(t, 0) \leq 0$ and $f(t, \frac{1}{2}) < 0$ for all $t \in J$. Also, $f(t, x)$ is decreasing in $x \in [0, \infty)$ for each $t \in J$.
- (H2) $I_k \in C[\mathbb{R}, \mathbb{R}]$ and $\bar{I}_k \in C[\mathbb{R}, \mathbb{R}]$ such that $I_k(0) \geq 0$ and $\bar{I}_k(0) \geq 0$ for $k = 1, 2, \dots, m$. Also, $I_k(x)$ and $\bar{I}_k(x)$ are increasing in $x \in [0, \infty)$ for $k = 1, 2, \dots, m$.
- (H3) For all $\lambda \in (0, 1)$ and $x \geq 0$, there exist $\alpha_1(\lambda), \alpha_2(\lambda), \alpha_3(\lambda) \in (\lambda, 1]$ such that

$$f(t, \lambda x) \leq \alpha_1(\lambda)f(t, x), \quad I_k(\lambda x) \geq \alpha_2(\lambda)I_k(x), \quad \bar{I}_k(\lambda x) \geq \alpha_3(\lambda)\bar{I}_k(x)$$

for $k = 1, 2, \dots, m$.

- (H4) There is a constant $C_f > 0$ such that

$$|f(t, x) - f(t, y)| \leq C_f|x - y| \quad \text{for all } t \in J \text{ and } x, y \in [0, \infty).$$

Also, for each $k = 1, 2, \dots, m$, there exist positive constants $C_k > 0$ and $\bar{C}_k > 0$ such that

$$|I_k(x) - I_k(y)| \leq C_k|x - y|, \quad |\bar{I}_k(x) - \bar{I}_k(y)| \leq \bar{C}_k|x - y|$$

for all $x, y \in [0, \infty)$.

- (H5) x_d is the given desired target profile in $L^2(J)$.

Finally, throughout this paper, N_i and $N'_i, i = 1, 2, 3, \dots$, denote positive (or nonnegative) constants depending only on its argument(s).

2. Preliminary

In this section, we recall the fundamentals of a fixed point theorem of generalized concave operators.

Throughout this section, let E be a real Banach space with the norm $|\cdot|_E$ which is partially ordered by a cone $P \subset E$, i.e., $x \leq y$ if and only if $y - x \in P$. By θ we denote the zero element of E . Recall that a non-empty closed convex set $P \subset E$ is called a cone if it satisfies (i) $x \in P, \lambda \geq 0 \Rightarrow \lambda x \in P$; (ii) $x \in P, -x \in P \Rightarrow x = \theta$. Moreover, P is called normal if there exists a constant $N_0 > 0$ such that, for all $x, y \in E, \theta \leq x \leq y$ implies $|x|_E \leq N_0|y|_E$; in this case N_0 is called the normality constant of P .

We say that an operator $A : E \rightarrow E$ is increasing (resp. decreasing) if $x \leq y$ implies $Ax \leq Ay$ (resp. $Ax \geq Ay$).

For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. Clearly, \sim is an equivalence relation. Given $h > \theta$ (i.e., $h \geq \theta$ and $h \neq \theta$), we denote by P_h the set $P_h := \{x \in E ; x \sim h\}$. Clearly, $P_h \subset P$ is convex and $\lambda P_h = P_h$ for all $\lambda > 0$. For other detailed properties of cones, we refer to the monograph by Guo and Lakshmikantham [7].

Here, we recall the following fixed point theorem of generalized concave operators which is established by Zhai, Yang and Zhang [24].

Proposition 2.1 ([24, Theorem 2.1 and Lemma 2.1]). *Let $h > \theta$ and P be a normal cone. Assume that:*

- (D₁) *An operator $A : P \rightarrow P$ is increasing and $Ah \in P_h$.*
- (D₂) *For any $x \in P$ and $\lambda \in (0, 1)$, there exists $\alpha(\lambda) \in (\lambda, 1]$ with respect to λ such that $A(\lambda x) \geq \alpha(\lambda)Ax$.*

Then:

- (i) *There are $u_0, v_0 \in P_h$ and $r \in (0, 1)$ such that $rv_0 \leq u_0 < v_0$ and $u_0 \leq Au_0 \leq Av_0 \leq v_0$.*
- (ii) *An operator equation $x = Ax$ has a unique solution in P_h .*

Remark 2.2. We say that an operator A is *generalized concave* if A satisfies condition (D₂) in Proposition 2.1.

Remark 2.3. Under more general assumptions, Zhai, Yang and Zhang [24] established the theory of a fixed point theorem, which improve and generalize relevant results in [5, 7, 13] of generalized concave operators. For the detailed statements, we refer to [24].

3. Main results

In this section, we state the main results of this paper. We begin by defining the notion of solutions for (P).

Definition 3.1. Let $u \in H$, $T > 0$, $a > 0$ and $b \geq 0$. Then, a function $x \in PC^1[J, \mathbb{R}] \cap C^2[J', \mathbb{R}]$ is called a solution to (P), or (P; u, a, b) when the data u, a and b are specified, on J if it satisfies (1).

Now, we mention our first main theorem in this paper, which is concerned with the existence-uniqueness of positive solutions to (P).

Theorem 3.2. *Assume the conditions (H1)–(H4). Let M be a fixed positive constant, and let \mathcal{U}_M be the control space defined in (2). Then, for each positive constants $T > 0$, $a > 0$, $b \geq 0$ and the control function $u \in \mathcal{U}_M$, there exists a unique positive solution to (P; u, a, b) on J .*

In next Section 4, we give the proof of Theorem 3.2 by the similar arguments as in Zhai, Yang and Zhang [24], namely by using a fixed point theorem of generalized concave operators. Next, let us mention the second main result in this paper, which is concerned with the existence of an optimal control to (OP).

Theorem 3.3. *Assume (H1)–(H5). Let $T > 0$, $a > 0$ and $b \geq 0$. Then, the problem (OP) has at least one optimal control $u^* \in \mathcal{U}_M$ such that*

$$\pi(u^*) = \inf_{u \in \mathcal{U}_M} \pi(u).$$

Here, \mathcal{U}_M is a control space defined by (2), and $\pi(\cdot)$ is the cost functional defined in (3).

In Section 5, we prove Theorem 3.3 by using the result of the well-posedness for (P).

Remark 3.4. Unfortunately, Theorem 3.3 do not cover the uniqueness of optimal controls, because of the nonlinearities of f , I_k and \bar{I}_k . So, the uniqueness question of optimal controls to (OP) is still open.

4. Solvability of (P)

In this section, we give the proof of Theorem 3.2 by applying a fixed point theorem (Proposition 2.1) of generalized concave operators. To do so, we give the key lemma, which is concerned with the characterization of solutions to (P).

Lemma 4.1 ([6, Lemma 1a]). *Let $f \in C[J \times \mathbb{R}, \mathbb{R}]$ and $u \in H$. Then, $x \in PC^1[J, \mathbb{R}] \cap C^2[J', \mathbb{R}]$ is a solution to (P; u, a, b) on J if and only if $x \in PC^1[J, \mathbb{R}]$ is a solution to the following integral equation:*

$$\begin{aligned}
 x(t) = & a + bt - \int_0^t (t - s)f(s, x(s)) ds - \int_0^t (t - s)u(s) ds \\
 & + \sum_{0 < t_k < t} I_k(x(t_k)) + \sum_{0 < t_k < t} (t - t_k)\bar{I}_k(x(t_k)), \quad \forall t \in J.
 \end{aligned}
 \tag{4}$$

By Lemma 4.1, we can show the solvability of (P). In fact, we define an operator $A : PC[J, \mathbb{R}] \rightarrow PC[J, \mathbb{R}]$ by

$$\begin{aligned}
 Ax(t) := & a + bt - \int_0^t (t - s)f(s, x(s)) ds - \int_0^t (t - s)u(s) ds \\
 & + \sum_{0 < t_k < t} I_k(x(t_k)) + \sum_{0 < t_k < t} (t - t_k)\bar{I}_k(x(t_k)), \quad \forall t \in J, \forall x \in PC[J, \mathbb{R}].
 \end{aligned}
 \tag{5}$$

Then, we easily see that the following lemma holds.

Lemma 4.2. *Let $f \in C[J \times \mathbb{R}, \mathbb{R}]$ and $u \in H$. Then, $x \in PC^1[J, \mathbb{R}] \cap C^2[J', \mathbb{R}]$ is a solution to (P; u, a, b) on J if and only if $x \in PC^1[J, \mathbb{R}]$ is a fixed point of the operator A , where $A : PC[J, \mathbb{R}] \rightarrow PC[J, \mathbb{R}]$ is the operator defined by (5).*

Taking account of Proposition 2.1 and Lemmas 4.1–4.2, we can show Theorem 3.2 concerning the existence-uniqueness of positive solutions to (P).

Proof of Theorem 3.2. We prove this theorem by the arguments similar to those in [24, Section 3]. More precisely, we apply a fixed point theorem of generalized concave operators. To do so, set

$$\tilde{P} := \{x \in PC[J, \mathbb{R}] \mid x(t) \geq 0, t \in J\}.$$

Clearly, \tilde{P} is a normal cone in $PC[J, \mathbb{R}]$ and the normality constant is 1.

Also, let $A : PC[J, \mathbb{R}] \rightarrow PC[J, \mathbb{R}]$ is the operator defined by (5). Then, we first show that $A : \tilde{P} \rightarrow \tilde{P}$ is increasing, generalized concave, since the positive solution of (P; u, a, b) on J is a fixed point of $A : \tilde{P} \rightarrow \tilde{P}$ (cf. Lemmas 4.1 and 4.2).

In order to show a fixed point of $A : \tilde{P} \rightarrow \tilde{P}$, let us check the conditions (D₁) and (D₂) in Proposition 2.1.

Now, we show that an operator $A : \tilde{P} \rightarrow \tilde{P}$ is increasing. Note from (H1), (H2) and $u \in \mathcal{U}_M$ that if $x \in \tilde{P}$, then:

$$f(t, x(t)) \leq f(t, 0) \leq 0, \quad I_k(x(t_k)) \geq I_k(0) \geq 0,$$

$$\bar{I}_k(x(t_k)) \geq \bar{I}_k(0) \geq 0, \quad u(t) \leq 0$$

for all $t \in J$ and $k = 1, 2, \dots, m$. Also, note that initial data $a > 0$ and $b \geq 0$ are non-negative. Therefore, we see from (5) that $Ax \geq 0$ for any $x \in \tilde{P}$. Moreover, by the similar proof of Lemma 4.1, we have $Ax \in PC[J, \mathbb{R}]$. Hence, we see that A is the self-mapping on \tilde{P} . Clearly, we see from (5), (H1) and (H2) that $A : \tilde{P} \rightarrow \tilde{P}$ is increasing.

Next, we show (D₂), namely, we prove that $A : \tilde{P} \rightarrow \tilde{P}$ is generalized concave. Put

$$\alpha(\lambda) := \min \{ \alpha_1(\lambda), \alpha_2(\lambda), \alpha_3(\lambda) \}, \quad \lambda \in (0, 1).$$

Then, we see from (H3) that $\alpha(\lambda) \in (\lambda, 1]$. Therefore, for any $x \in \tilde{P}$ and $\lambda \in (0, 1)$, we see from (5) and (H3) that

$$\begin{aligned} A(\lambda x)(t) &= a + bt - \int_0^t (t-s)f(s, \lambda x(s)) ds - \int_0^t (t-s)u(s) ds \\ &\quad + \sum_{0 < t_k < t} I_k(\lambda x(t_k)) + \sum_{0 < t_k < t} (t-t_k)\bar{I}_k(\lambda x(t_k)) \\ &\geq a + bt + \alpha_1(\lambda) \left[- \int_0^t (t-s)f(s, x(s)) ds \right] - \int_0^t (t-s)u(s) ds \\ &\quad + \alpha_2(\lambda) \sum_{0 < t_k < t} I_k(x(t_k)) + \alpha_3(\lambda) \sum_{0 < t_k < t} (t-t_k)\bar{I}_k(x(t_k)) \\ &\geq \alpha(\lambda) \left\{ a + bt - \int_0^t (t-s)f(s, x(s)) ds - \int_0^t (t-s)u(s) ds \right. \\ &\quad \left. + \sum_{0 < t_k < t} I_k(x(t_k)) + \sum_{0 < t_k < t} (t-t_k)\bar{I}_k(x(t_k)) \right\} \\ &= \alpha(\lambda)Ax(t), \quad \forall t \in J, \end{aligned}$$

which implies that $A(\lambda x) \geq \alpha(\lambda)Ax$, for all $x \in \tilde{P}$ and all $\lambda \in (0, 1)$. Thus, the condition (D₂) holds.

Here, we define a function h by

$$h(t) := \frac{1}{2} + \int_0^t (t-s) ds = \frac{1}{2} + \frac{t^2}{2}, \quad \forall t \in J. \tag{6}$$

Then, we easily see from (6) that $\frac{1}{2} \leq h(t) \leq \frac{1}{2} + \frac{T^2}{2}$, for all $t \in J$.

Now, we show $Ah \in \tilde{P}_h$. To do so, we set

$$r_1 := \min_{t \in [0, T]} \left[-f \left(t, \frac{1}{2} \right) \right], \quad r_2 := \max_{t \in [0, T]} \left[-f \left(t, \frac{1}{2} + \frac{T^2}{2} \right) \right].$$

Also, we put $r_3 := \min\{2a, r_1\}$. Then, from (H1) and $a > 0$, we observe $r_2 \geq r_1 \geq r_3 > 0$. Further, from (H1), (H2), $a > 0$, $b \geq 0$ and $u \in \mathcal{U}_M$, it follows that

$$\begin{aligned} Ah(t) &= a + bt - \int_0^t (t-s)f(s, h(s)) ds - \int_0^t (t-s)u(s) ds \\ &\quad + \sum_{0 < t_k < t} I_k(h(t_k)) + \sum_{0 < t_k < t} (t-t_k)\bar{I}_k(h(t_k)) \\ &\geq a - \int_0^t (t-s)f \left(s, \frac{1}{2} \right) ds \\ &\geq a + r_1 \int_0^t (t-s) ds \\ &\geq \min\{2a, r_1\}h(t) \\ &= r_3h(t), \quad \forall t \in J. \end{aligned}$$

Also, for any $t \in J$, we have :

$$\begin{aligned} Ah(t) &= a + bt - \int_0^t (t-s)f(s, h(s)) ds - \int_0^t (t-s)u(s) ds \\ &\quad + \sum_{0 < t_k < t} I_k(h(t_k)) + \sum_{0 < t_k < t} (t-t_k)\bar{I}_k(h(t_k)) \\ &\leq a + bT - \int_0^t (t-s)f \left(s, \frac{1}{2} + \frac{T^2}{2} \right) ds + M \int_0^t (t-s) ds \\ &\quad + \sum_{k=1}^m I_k \left(\frac{1}{2} + \frac{T^2}{2} \right) + \sum_{k=1}^m \bar{I}_k \left(\frac{1}{2} + \frac{T^2}{2} \right) \\ &\leq 2 \left[a + bT + r_2 + M + \sum_{k=1}^m I_k \left(\frac{1}{2} + \frac{T^2}{2} \right) + \sum_{k=1}^m \bar{I}_k \left(\frac{1}{2} + \frac{T^2}{2} \right) \right] h(t), \end{aligned}$$

Thus, we observe that

$$r_3h \leq Ah \leq 2 \left[a + bT + r_2 + M + \sum_{k=1}^m I_k \left(\frac{1}{2} + \frac{T^2}{2} \right) + \sum_{k=1}^m \bar{I}_k \left(\frac{1}{2} + \frac{T^2}{2} \right) \right] h,$$

which implies that $Ah \in \tilde{P}_h$.

By the arguments as above, we see that the operator $A : \tilde{P} \rightarrow \tilde{P}$ defined by (5) satisfies the conditions (D₁) and (D₂) in Proposition 2.1. Therefore, by applying Proposition 2.1, we conclude that an operator equation $x = Ax$ has a unique solution in \tilde{P}_h , hence, that there exists a unique positive solution to (P; u, a, b) on J , where h is the function defined by (6). □

5. Optimal control problem (OP)

In this section, we prove Theorem 3.3, which is concerned with the existence of an optimal control to (OP). Throughout this section, we assume all the conditions of Theorem 3.3.

At first, we give the key lemma in order to show the result of continuous dependence of positive solutions to (P).

Lemma 5.1. *Let $\{u_n\} \subset H$, and let $Q : H \rightarrow C[J, \mathbb{R}]$ be an operator given by*

$$(Qz)(t) := \int_0^t (t - s)z(s) ds, \quad \forall z \in H, \forall t \in J. \tag{7}$$

Assume that $u_n \rightarrow u$ weakly in H as $n \rightarrow \infty$ for some $u \in H$. Then

$$Qu_n \rightarrow Qu \quad \text{in } C[J, \mathbb{R}] \text{ as } n \rightarrow \infty.$$

Proof. Since $u_n \rightarrow u$ weakly in H as $n \rightarrow \infty$, we easily see that

$$Qu_n(t) = \int_0^t (t - s)u_n(s) ds \rightarrow \int_0^t (t - s)u(s) ds = Qu(t) \quad \text{as } n \rightarrow \infty, \forall t \in J.$$

Also, we observe from Hölder’s inequality that:

$$\begin{aligned} |(Qu_n)(t) - (Qu_n)(\tau)| &= \left| \int_\tau^t (t - s)u_n(s) ds + \int_0^\tau (t - \tau)u_n(s) ds \right| \\ &\leq 2\sqrt{T}|t - \tau||u_n|_H, \quad \text{for any } \tau, t \in J \text{ with } \tau \leq t, \end{aligned}$$

which implies that $\{Qu_n\} \subset C[J, \mathbb{R}]$ is equicontinuous since $u_n \rightarrow u$ weakly in H as $n \rightarrow \infty$. Thus, we infer from Ascoli-Arzelà’s theorem that Lemma 5.1 holds. □

Taking account of Lemma 5.1, we can show the following proposition concerning the result of continuous dependence of positive solutions to (P).

Proposition 5.2. *Assume the same conditions as in Theorem 3.3. Let $\{u_n\} \subset \mathcal{U}_M$ and $u \in \mathcal{U}_M$. Assume $u_n \rightarrow u$ weakly in H as $n \rightarrow \infty$. Then, the unique positive solution x_n of (P; u_n, a, b) on J converges to one x of (P; u, a, b) on J in the sense that*

$$x_n \rightarrow x \quad \text{in } PC[J, \mathbb{R}] \text{ as } n \rightarrow \infty. \tag{8}$$

Proof. Note from Lemma 4.1 (cf. (4)) that x_n is a solution of $(P; u_n, a, b)$ on J if and only if

$$x_n(t) = a + bt - \int_0^t (t-s)f(s, x_n(s)) ds - \int_0^t (t-s)u_n(s) ds + \sum_{0 < t_k < t} I_k(x_n(t_k)) + \sum_{0 < t_k < t} (t-t_k)\bar{I}_k(x_n(t_k)), \quad \forall t \in J.$$

Now, let $t \in J_0 = [0, t_1] \subset J$. Then, we obtain from (H4) that:

$$\begin{aligned} |x_n(t) - x(t)| &\leq \left| \int_0^t (t-s)f(s, x_n(s)) ds - \int_0^t (t-s)f(s, x(s)) ds \right| \\ &\quad + \left| \int_0^t (t-s)u_n(s) ds - \int_0^t (t-s)u(s) ds \right| \\ &\leq TC_f \int_0^t |x_n(s) - x(s)| ds + |(Qu_n)(t) - (Qu)(t)| \\ &\leq TC_f \int_0^t |x_n(s) - x(s)| ds + |Qu_n - Qu|_{C[J, \mathbb{R}]}, \quad \forall t \in J_0 \end{aligned} \tag{9}$$

for all $n = 1, 2, \dots$, where Q is a function defined in (7).

Applying a Gronwall-type inequality (e.g., [10, Proposition 0.4.1]) to (9), we obtain

$$\int_0^t |x_n(s) - x(s)| ds \leq Te^{T^2C_f} |Qu_n - Qu|_{C[J, \mathbb{R}]}, \quad \forall t \in J_0 \tag{10}$$

for all $n = 1, 2, \dots$. Therefore, it follows from (9) and (10) that

$$\begin{aligned} |x_n(t) - x(t)| &\leq T^2C_f e^{T^2C_f} |Qu_n - Qu|_{C[J, \mathbb{R}]} + |Qu_n - Qu|_{C[J, \mathbb{R}]} \\ &\equiv N_1 |Qu_n - Qu|_{C[J, \mathbb{R}]}, \quad \forall t \in J_0 = [0, t_1], \quad \forall n = 1, 2, \dots \end{aligned} \tag{11}$$

By (11) and the assumption (H4), we also have

$$\begin{aligned} |x_n(t_1^+) - x(t_1^+)| &= |x_n(t_1) + I_1(x_n(t_1)) - x(t_1) - I_1(x(t_1))| \\ &\leq |x_n(t_1) - x(t_1)| + |I_1(x_n(t_1)) - I_1(x(t_1))| \\ &\leq (1 + C_1)|x_n(t_1) - x(t_1)| \\ &\leq (1 + C_1)N_1 |Qu_n - Qu|_{C[J, \mathbb{R}]} \\ &\equiv N'_1 |Qu_n - Qu|_{C[J, \mathbb{R}]} \quad \forall n = 1, 2, \dots \end{aligned} \tag{12}$$

Next, we consider the time interval $J_1 = (t_1, t_2]$. Then, we see from (11)

and (H4) that:

$$\begin{aligned}
 |x_n(t) - x(t)| &\leq \left| \int_0^t (t-s)f(s, x_n(s)) ds - \int_0^t (t-s)f(s, x(s)) ds \right| \\
 &\quad + \left| \int_0^t (t-s)u_n(s) ds - \int_0^t (t-s)u(s) ds \right| \\
 &\quad + |I_1(x_n(t_1)) - I_1(x(t_1))| + |(t-t_1)(\bar{I}_1(x_n(t_1)) - \bar{I}_1(x(t_1))))| \\
 &\leq TC_f \int_0^t |x_n(s) - x(s)| ds + |(Qu_n)(t) - (Qu)(t)| \\
 &\quad + C_1|x_n(t_1) - x(t_1)| + T\bar{C}_1|x_n(t_1) - x(t_1)| \\
 &\leq TC_f \int_0^t |x_n(s) - x(s)| ds \\
 &\quad + (1 + C_1N_1 + T\bar{C}_1N_1) |Qu_n - Qu|_{C[J, \mathbb{R}]}
 \end{aligned}$$

for any $t \in J_1$ and $n = 1, 2, \dots$. By the same argument as before (cf. (10)–(11)), we can take some constant $N_2 > 0$ so that

$$|x_n(t) - x(t)| \leq N_2 |Qu_n - Qu|_{C[J, \mathbb{R}]}, \quad \forall t \in J_1 = (t_1, t_2], \forall n = 1, 2, \dots \quad (13)$$

Also, from (H4) and (13), we obtain that

$$\begin{aligned}
 |x_n(t_2^+) - x(t_2^+)| &\leq |x_n(t_2) - x(t_2)| + |I_2(x_n(t_2)) - I_2(x(t_2))| \\
 &\leq (1 + C_2)|x_n(t_2) - x(t_2)| \\
 &\leq N'_2 |Qu_n - Qu|_{C[J, \mathbb{R}]}, \quad \forall n = 1, 2, \dots
 \end{aligned} \quad (14)$$

for some positive constant $N'_2 > 0$. By repeating this procedure, we can take positive constants $N_k > 0$ and $N'_k > 0$ such that

$$|x_n(t) - x(t)| \leq N_k |Qu_n - Qu|_{C[J, \mathbb{R}]}, \quad \forall t \in J_{k-1}, k = 1, 2, \dots, m + 1, \quad (15)$$

$$|x_n(t_k^+) - x(t_k^+)| \leq N'_k |Qu_n - Qu|_{C[J, \mathbb{R}]}, \quad k = 1, 2, \dots, m \quad (16)$$

for all $n = 1, 2, \dots$

Here, put $N := \max\{N_1, N'_1, N_2, N'_2, \dots, N_m, N'_m, N_{m+1}\}$. Then, we infer from (15) and (16) that

$$|x_n - x|_{PC} \leq N |Qu_n - Qu|_{C[J, \mathbb{R}]}, \quad \forall n = 1, 2, \dots \quad (17)$$

Since $u_n \rightarrow u$ weakly in H as $n \rightarrow \infty$, we observe from Lemma 5.1 that

$$Qu_n \rightarrow Qu \quad \text{in } C[J, \mathbb{R}] \text{ as } n \rightarrow \infty. \quad (18)$$

Hence, we see from (17) and (18) that $x_n \rightarrow x$ in $PC[J, \mathbb{R}]$ as $n \rightarrow \infty$. Thus, the proof of Proposition 5.2 has been completed. \square

Now, we prove our main Theorem 3.3 in this paper, which is concerned with the existence of an optimal control to (OP).

Proof of Theorem 3.3. By the quite standard method, we can prove this theorem. In fact, let $\{u_n\} \subset \mathcal{U}_M$ be a minimizing sequence so that

$$\lim_{n \rightarrow \infty} \pi(u_n) = \inf_{u \in \mathcal{U}_M} \pi(u).$$

By the definition (3) of $\pi(\cdot)$, we see that $\{u_n\}$ is bounded in H . Hence, there is a subsequence $\{n_k\} \subset \{n\}$ and a function $u^* \in \mathcal{U}_M$ such that $n_k \rightarrow \infty$ and

$$u_{n_k} \rightarrow u^* \quad \text{weakly in } H \text{ as } k \rightarrow \infty. \tag{19}$$

For any $k \in \mathbb{N}$, let x_{n_k} be a unique positive solution to $(P; u_{n_k}, a, b)$ on J . Then, from (19) and Proposition 5.2, we observe that

$$x_{n_k} \rightarrow x \quad \text{in } PC[J, \mathbb{R}] \text{ as } k \rightarrow \infty, \tag{20}$$

where x is a unique positive solution to $(P; u, a, b)$ on J .

Hence, it follows from (19), (20) and the weak lower semicontinuity of H -norm that

$$\pi(u^*) \leq \lim_{k \rightarrow \infty} \pi(u_{n_k}) = \inf_{u \in \mathcal{U}_M} \pi(u),$$

which implies that $u^* \in \mathcal{U}_M$ is an optimal control to (OP). □

6. A simple example

In this final section, we give an example of the main results.

Example. Consider the following Cauchy problem of second order impulsive differential equation:

$$\left\{ \begin{array}{l} -x''(t) = -\sqrt{tx+4} + u(t), \quad t \in (0, 1), \quad t \neq \frac{1}{2}, \\ \Delta x|_{t=\frac{1}{2}} = x\left(\frac{1}{2}\right), \\ \Delta x'|_{t=\frac{1}{2}} = 2x\left(\frac{1}{2}\right), \\ x(0) = 1, \\ x'(0) = 0. \end{array} \right. \tag{21}$$

Then, Cauchy Problem (21) can be regarded as the form $(P; u, 1, 0)$ with $J = [0, 1]$, $t_1 = \frac{1}{2}$, $f(t, x) = -\sqrt{tx+4}$, $I_1(x) = x$, $\bar{I}_1(x) = 2x$, $a = 1$ and $b = 0$.

In addition, let $\alpha_1(\lambda) = \sqrt{\lambda}$, $\alpha_2(\lambda) = \alpha_3(\lambda) = \lambda$, $C_f = \frac{1}{4}$, $C_1 = 1$, $\overline{C_1} = 2$. Then, we easily see that (H1)–(H5) hold. Hence, we can apply Theorem 3.2 to (21). Namely, we can get a unique positive solution to (21). Also, by applying Theorem 3.3, we see that Problem (OP) for (21) has at least one optimal control for each desired target profile x_d in $L^2(J)$.

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