Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 18 (1999), No. 3, 687–699

# An Integral Operator Representation of Classical Periodic Pseudodifferential Operators

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**Abstract.** In this note we prove that every classical 1-periodic pseudodifferential operator $A$ of order $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$ can be represented in the form

$$(Au)(t) = \int_0^1 \Big[\kappa_\alpha^+(t-s)a_+(t,s) + \kappa_\alpha^-(t-s)a_-(t,s) + a(t,s)\Big]u(s)\,ds$$

where $a_\pm$ and $a$ are $C^\infty$-smooth 1-periodic functions and $\kappa_\alpha^\pm$ are 1-periodic functions or distributions with Fourier coefficients $\hat\kappa_\alpha^+(n) = |n|^\alpha$ and $\hat\kappa_\alpha^-(n) = |n|^\alpha \mathrm{sign}(n)$ $(0 \neq n \in \mathbb{Z})$ with respect to the trigonometric orthonormal basis $\{e^{in2\pi t}\}_{n\in\mathbb{Z}}$ of $L^2(0,1)$. Some explicit formulae for $\kappa_\alpha^\pm$ are given. The case of operators of order $\alpha \in \mathbb{N}_0$ is discussed, too.

**Keywords:** *Classical periodic pseudodifferential operators, periodic integral operators, asymptotic expansions*

**AMS subject classification:** Primary 47 G 30, secondary 47 G 10, 58 G 15

## 1. Periodic pseudodifferential operators

By $H^\lambda$ $(\lambda \in \mathbb{R})$ we denote the Sobolev space of 1-periodic functions or distributions $u$ having a finite norm

$$\|u\|_\lambda = \Big(\sum_{n\in\mathbb{Z}} \underline{n}^{2\lambda}|\hat u(n)|^2\Big)^{\frac{1}{2}}$$

where

$$\hat u(n) = \int_0^1 u(t)e^{-in2\pi t}dt = \langle u, e^{-in2\pi t}\rangle$$

are the Fourier coefficients of $u$ and $\underline{n} = \max\{1, |n|\}$. As usual, $\mathcal{L}(H^\lambda, H^\mu)$ denotes the space of linear bounded operators from $H^\lambda$ into $H^\mu$. Every operator $A \in \mathcal{L}(H^\lambda, H^\mu)$ is of the form

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ISSN 0232-2064 / $ 2.50 

$$u(t) = \sum_{n\in\mathbb{Z}} \hat{u}(n)e^{in2\pi t} \longmapsto (Au)(t) = \sum_{n\in\mathbb{Z}} \sigma(t,n)\hat{u}(n)e^{in2\pi t} \tag{1}$$

(and one writes $A = Op\,\sigma$) where

$$\sigma(t,n) = e^{-in2\pi t} A e^{in2\pi t}$$

is called the *symbol* of $A$. Indeed, the Fourier series

$$u(t) = \sum_{n\in\mathbb{Z}} \hat{u}(n)e^{in2\pi t}$$

of $u \in H^\lambda$ converges in $H^\lambda$, therefore the series

$$(Au)(t) = \sum_{n\in\mathbb{Z}} \hat{u}(n)Ae^{in2\pi t} = \sum_{n\in\mathbb{Z}} \hat{u}(n)\sigma(t,n)e^{in2\pi t}$$

converges in $H^\mu$. Clearly, $\sigma(t,n)$ is 1-periodic in $t$.

A complex-valued function

$$\sigma = \sigma(t,n) \qquad (t \in \mathbb{R}, n \in \mathbb{Z})$$

is called a *periodic symbol of degree* $\alpha$ ($\alpha \in \mathbb{R}$), denoted $\sigma \in \Sigma^\alpha$, if it is $C^\infty$-smooth and 1-periodic in $t$ and satisfies the inequalities

$$\left|\left(\frac{\partial}{\partial t}\right)^j \Delta_n^k \sigma(t,n)\right| \le c_{jk}\,\underline{n}^{\alpha-k} \qquad (j,k \in \mathbb{N}_0, t \in \mathbb{R}, n \in \mathbb{Z}). \tag{2}$$

Here $\mathbb{N}_0 = \{0\} \cup \mathbb{N} = \{0,1,2,\ldots\}$, and $\Delta$ is the (forward) difference operator:

$$(\Delta\psi)(n) = \psi(n+1) - \psi(n)$$

for $\psi : \mathbb{Z} \to \mathbb{C}$. An operator $A = Op\,\sigma$ of form (1) with $\sigma \in \Sigma^\alpha$ is called a *periodic pseudodifferential operator of order* $\alpha$, denoted $A \in Op\,\Sigma^\alpha$. This definition originates from [1, 2]. Equivalent definitions can be found in [2 - 4, 12]. It occurs (see, e.g., [12]) that $A \in \mathcal{L}(H^\lambda, H^{\lambda-\alpha})$ for any $\lambda \in \mathbb{R}$ if $A \in Op\,\Sigma^\alpha$.

Introduce a $C^\infty$-smooth function $h : \mathbb{R} \to \mathbb{R}$ satisfying

$$h(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } 0 \neq k \in \mathbb{Z} \end{cases}$$

$$h \in S(\mathbb{R}), \quad \text{i.e.} \quad \sup_{\xi\in\mathbb{R}} |\xi^j h^{(k)}(\xi)| < \infty \quad (j,k \in \mathbb{N}_0)$$

$$\forall\, k \in \mathbb{N} \;\; \exists\, h_k \in S(\mathbb{R}) \quad \text{such that} \quad h^{(k)}(\xi) = (\Delta^k h_k)(\xi) \quad (\xi \in \mathbb{R})$$

(see [12] for a construction of $h$). The formula

$$\sigma(t,\xi) = \sum_{n\in\mathbb{Z}} \sigma(t,n)h(\xi - n) \qquad (\xi \in \mathbb{R})$$

defines a prolongation $\sigma : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ of $\sigma \in \Sigma^\alpha$. It occurs that (2) implies the inequalities

$$\left| \left(\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial \xi}\right)^k \sigma(t,\xi) \right| \le c'_{jk}(1+|\xi|)^{\alpha-k} \qquad (j,k \in \mathbb{N}_0; t,\xi \in \mathbb{R}). \tag{3}$$

Indeed,

$$\Delta_\xi h(\xi - n) = h(\xi + 1 - n) - h(\xi - n) = -\overline{\Delta}_n h(\xi - n)$$

where $\overline{\Delta}$ is the backward difference operator, thus

$$\begin{aligned} \left(\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial \xi}\right)^k \sigma(t,\xi) &= \sum_{n\in\mathbf{Z}} \left(\frac{\partial}{\partial t}\right)^j \sigma(t,n) h^{(k)}(\xi - n) \\ &= \sum_{n\in\mathbf{Z}} \left(\frac{\partial}{\partial t}\right)^j \sigma(t,n) \Delta_\xi^k h_k(\xi - n) \\ &= (-1)^k \sum_{n\in\mathbf{Z}} \left(\frac{\partial}{\partial t}\right)^j \sigma(t,n) \overline{\Delta}_n^k h_k(\xi - n) \\ &= \sum_{n\in\mathbf{Z}} h_k(\xi - n) \left(\frac{\partial}{\partial t}\right)^j \Delta_n^j \sigma(t,n) \end{aligned}$$

(summation by parts on the last step). Since $h_k \in S(\mathbb{R})$ we have $|h_k(\xi - n)| \le c_r(1 + |\xi - n|)^{-r}$ with any $r > 0$; we take $r > |\alpha - k| + 1$. Due to (2), we obtain (3):

$$\left| \left(\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial \xi}\right)^k \sigma(t,\xi) \right| \le c_r c_{jk} \sum_{n\in\mathbf{Z}} (1 + |\xi - n|)^{-r} \underline{n}^{\alpha-k} \le c'_{jk}(1+|\xi|)^{\alpha-k}.$$

The converse is also true: if $\sigma : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ satisfies (3), then its restriction to $\mathbb{R} \times \mathbb{Z}$ satisfies (2). Thus we may assume that the symbol $\sigma \in \Sigma^\alpha$ is defined and $C^\infty$-smooth on $\mathbb{R} \times \mathbb{R}$, 1-periodic in $t$ and satisfies (3). Nevertheless, only the values on $\mathbb{R} \times \mathbb{Z}$ of $\sigma$ are used to define $A = Op\,\sigma \in Op\,\Sigma^\alpha$.

A symbol $\sigma \in \Sigma^\alpha$ is called *classical* or *polyhomogeneous*, denoted $\sigma \in \Sigma^\alpha_{cl}$, if it admits an asymptotic expansion

$$\sigma(t,\xi) \sim \sum_{j=0}^{\infty} \sigma_j(t,\xi), \qquad \text{i.e. } \sigma - \sum_{j=0}^{N-1} \sigma_j \in \Sigma^{\alpha-N} \ \ (N \in \mathbb{N}) \tag{4}$$

where $\sigma_j \in \Sigma^{\alpha-j}$ are positively homogeneous of degree $\alpha - j$ in $\xi$ for $|\xi| \ge 1$:

$$\sigma_j(t,\tau\xi) = \tau^{\alpha-j}\sigma_j(t,\xi) \qquad (|\xi| \ge 1, \tau \ge 1).$$

Clearly,

$$\sigma_j(t,\xi) = \begin{cases} \sigma_j(t,1)\xi^{\alpha-j} =: a_j^+(t)\xi^{\alpha-j} & \text{for } \xi \ge 1 \\ \sigma_j(t,-1)|\xi|^{\alpha-j} =: a_j^-(t)|\xi|^{\alpha-j} & \text{for } \xi \le -1 \end{cases} \tag{5}$$

with $a_j^\pm \in C_1^\infty(\mathbb{R})$ where $C_1^\infty(\mathbb{R})$ denotes the set of 1-periodic $C^\infty$-smooth functions on $\mathbb{R}$. The corresponding $A = Op\,\sigma$ is called a *classical periodic pseudodifferential operator* of order $\alpha$, denoted $A \in Op\,\Sigma_{cl}^\alpha$. It follows from (4) and (5) that

$$A \sim \sum_{j=0}^{\infty} A_j, \qquad \text{i.e. } A - \sum_{j=0}^{N-1} A_j \in Op\,\Sigma^{\alpha-N} \quad (N \in \mathbb{N})$$

where

$$\begin{aligned} A_j &= [a_j^+(t)P^+ + a_j^-(t)P^-]L^{\alpha-j} \sim Op\,\sigma_j \\ P^+u &= \sum_{n\geq 0} \hat{u}(n)e^{in2\pi t} \\ P^-u &= \sum_{n<0} \hat{u}(n)e^{in2\pi t} \\ L^\lambda u &= \sum_{n\in\mathbb{Z}} \underline{n}^\lambda \hat{u}(n)e^{in2\pi t} \quad (\lambda \in \mathbb{R}). \end{aligned}$$

Let us comment on the polyhomogenuity of a symbol. It occurs that a symbol $\sigma \in \Sigma^\alpha$ belongs to $\Sigma_{cl}^\alpha$ if and only if $|\xi|^{-\alpha}\sigma(t,\xi)$ behaves in a regular manner as $\xi \to \pm\infty$, or equivalently, $\sigma_*(t,\eta) = |\eta|^\alpha \sigma(t, \frac{1}{\eta})$ with $\eta = \frac{1}{\xi}$ behaves in a regular manner as $\eta \to \pm 0$. Namely, if $\sigma_*$ has $C^\infty$-smooth continuations to $\eta = +0$ and $\eta = -0$, then the Taylor expansions

$$\begin{aligned} \sigma_*(t,\eta) &= \sum_{j=0}^{N-1} a_j^+(t)\eta^j + \mathcal{O}(\eta^N) \qquad \left(\eta \to +0,\ a_j^+(t) = \tfrac{1}{j!}\left(\tfrac{\partial}{\partial\eta}\right)^j \sigma_*(t,\eta)|_{\eta=+0}\right) \\ \sigma_*(t,\eta) &= \sum_{j=0}^{N-1} a_j^-(t)\eta^j + \mathcal{O}(\eta^N) \qquad \left(\eta \to -0,\ a_j^-(t) = \tfrac{1}{j!}\left(\tfrac{\partial}{\partial\eta}\right)^j \sigma_*(t,\eta)|_{\eta=-0}\right) \end{aligned}$$

hold true for all $N \in \mathbb{N}$. Returning to $\xi = \frac{1}{\eta}$ and $\sigma(t,\xi) = |\xi|^\alpha \sigma_*(t,\xi^{-1})$ we have

$$\begin{aligned} \sigma(t,\xi) &= \sum_{j=0}^{N-1} a_j^+(t)|\xi|^{\alpha-j} + \mathcal{O}(\xi^{\alpha-N}) \qquad (\xi \to +\infty) \\ \sigma(t,\xi) &= \sum_{j=0}^{N-1} a_j^-(t)|\xi|^{\alpha-j} + \mathcal{O}(\xi^{\alpha-N}) \qquad (\xi \to -\infty) \end{aligned}$$

and it can be checked that by those $a_\pm$ the asymptotic expansion (4) - (5) is defined.

## 2. Integral operator representation of periodic pseudodifferential operators

Here we follow some ideas from [3, 5, 8 - 10]. For $A \in Op\Sigma^\alpha$ $(\alpha < -1)$ and $u \in H^0 = L^2(0,1)$ we have

$$\begin{aligned}(Au)(t) &= \sum_{n\in\mathbb{Z}} \sigma(t,n)\hat{u}(n)e^{in2\pi t} \\ &= \sum_{n\in\mathbb{Z}} \sigma(t,n)e^{in2\pi t}\int_0^1 u(s)e^{-in2\pi s}ds \\ &= \int_0^1 \sum_{n\in\mathbb{Z}} \sigma(t,n)e^{in2\pi(t-s)}u(s)\,ds \\ &= \int_0^1 \mathcal{K}(t,t-s)u(s)\,ds\end{aligned}$$

where the series

$$\mathcal{K}(t,s) = \sum_{n\in\mathbb{Z}} \sigma(t,n)e^{in2\pi s}$$

converges uniformly in $t,s \in \mathbb{R}$ due to the estimate $|\sigma(t,n)| \le c_{00}\underline{n}^\alpha$ (see (2)). Thus $\mathcal{K}(t,s)$ is continuous on $\mathbb{R}\times\mathbb{R}$. Moreover, $\mathcal{K}(t,s)$ is $C^\infty$-smooth for $s \in \mathbb{R}\setminus\mathbb{Z}$ (and then $\mathcal{K}(t,t-s)$ is $C^\infty$-smooth for $t-s \notin \mathbb{Z}$). Indeed, consider the product

$$\begin{aligned}(e^{-i2\pi s}-1)\mathcal{K}(t,s) &= \sum_{n\in\mathbb{Z}} \sigma(t,n)(e^{i(n-1)2\pi s} - e^{in2\pi s}) \\ &= \sum_{n\in\mathbb{Z}} \left[\sigma(t,n+1) - \sigma(t,n)\right]e^{in2\pi s} \\ &= \sum_{n\in\mathbb{Z}} \left[\Delta\sigma(t,n)\right]e^{in2\pi s}.\end{aligned}$$

Repeating the multiplications by $(e^{-i2\pi s}-1)$ we obtain

$$(e^{-i2\pi s}-1)^l\mathcal{K}(t,s) = \sum_{n\in\mathbb{Z}} \left[\Delta^l\sigma(t,n)\right]e^{in2\pi s} \qquad (l\in\mathbb{N}).$$

Now estimate (2) yields that $(e^{-i2\pi s}-1)^l\mathcal{K}(t,s)$ is $l$-times continuously differentiable on $\mathbb{R}\times\mathbb{R}$. Since $l$ is arbitrary, $\mathcal{K}(t,s)$ is infinitely smooth for $(t,s)$ satisfying $e^{-i2\pi s}-1 \ne 0$, i.e. for $s \in \mathbb{R}\setminus\mathbb{Z}$. Also the case $\alpha \in [-1,0)$ can be treated, but then $\mathcal{K}(t,t-s)$ is weakly singular for $t=s$.

For $u \in H^l$ $(l \in \mathbb{N}_0)$ integration by parts yields

$$\hat{u}(n) = \int_0^1 u(s)e^{-in2\pi s}ds = \frac{1}{(2\pi in)^l}\int_0^1 u^{(l)}(s)e^{-in2\pi s}ds$$

and

$$(Au)(t) = \int_0^1 \mathcal{K}_l(t, t-s) u^{(l)}(s)\, ds, \qquad \mathcal{K}_l(t,s) = \sum_{N \in \mathbb{Z}} \frac{\sigma(t,n)}{(2\pi i n)^l} e^{in2\pi s}.$$

Now already for $\sigma \in \Sigma^\alpha$ with $\alpha < l-1$, the series converges uniformly and defines a continuous kernel $\mathcal{K}_l$ on $\mathbb{R} \times \mathbb{R}$; for $s \in \mathbb{R} \setminus \mathbb{Z}$, $\mathcal{K}_l(t,s)$ is again $C^\infty$-smooth.

One can try to represent $\mathcal{K}$ and $\mathcal{K}_l$ in the form of products

$$\mathcal{K}(t, t-s) = a(t,s)\kappa(t-s) \qquad \text{and} \qquad \mathcal{K}_l(t,t-s) = a_l(t,s)\kappa_l(t-s),$$

respectively, where $a$ and $a_l$ are $C^\infty$-smooth on the whole $\mathbb{R} \times \mathbb{R}$ whereas $\kappa$ and $\kappa_l$ are $C^\infty$-smooth on $\mathbb{R} \setminus \mathbb{Z}$. With some specifications we shall succeed in the case of classical periodic pseudodifferential operators. For a general (non-classical) periodic pseudodifferential operator a similar representations does not exist.

We point out also the following inverse result from [11].

**Theorem 1.** *An integral operator defined by*

$$(Au)(t) = \int_0^1 \kappa(t-s) a(t,s) u(s)\, ds$$

*with a $C^\infty$-smooth 1-biperiodic function $a$ and 1-periodic function or distribution $\kappa$ belongs to $Op\,\Sigma^\alpha$ if $\kappa$ satisfies*

$$|\Delta^k \hat{\kappa}(n)| \le c_k \underline{n}^{\alpha-k} \qquad (k \in \mathbb{N}_0, n \in \mathbb{Z})$$

*or, equivalently, if the extended function $\hat{\kappa} : \mathbb{R} \to \mathbb{C}$ (defined by $\hat{\kappa}(\xi) = \sum_{n\in\mathbb{Z}} \hat{\kappa}(n) h(\xi - n)$ or in some other way) satisfies*

$$\left| \left(\frac{d}{d\xi}\right)^k \hat{\kappa}(\xi) \right| \le c_k' (1 + |\xi|)^{\alpha - k} \qquad (k \in \mathbb{N}_0, \xi \in \mathbb{R}).$$

*Thereby $A$ has asymtotic expansions $A \sim \sum_{j=0}^\infty A_j$ with*

$$(A_j u)(t) = a_j(t) \int_0^1 \kappa_j(t-s) u(s)\, ds = a_j(t) \sum_{n \in \mathbb{Z}} \hat{\kappa}_j(n) \hat{u}(n) e^{in2\pi t}$$

*where*

$$\left.\begin{aligned} \hat{\kappa}_j(n) &= \frac{1}{j!} \Delta^j \hat{\kappa}(n) \quad (n \in \mathbb{Z}) \\ a_j(t) &= \left(\frac{1}{2\pi i}\frac{\partial}{\partial s} - (j-1)\right)\left(\frac{1}{2\pi i}\frac{\partial}{\partial s} - (j-2)\right) \cdots \left(\frac{1}{2\pi i}\frac{\partial}{\partial s} - 1\right) \frac{1}{2\pi i}\frac{\partial}{\partial s} a(t,s)\Big|_{s=t} \end{aligned}\right\}$$

*respectively*

$$\left.\begin{aligned} \hat{\kappa}_j(n) &= \frac{1}{j!}\left(\frac{d}{d\xi}\right)^j \hat{\kappa}(\xi)\Big|_{\xi=n} \quad (n \in \mathbb{Z}) \\ a_j(t) &= \left(\frac{1}{2\pi i}\frac{\partial}{\partial s}\right)^j a(t,s)\Big|_{s=t}. \end{aligned}\right\}$$

## 3. Integral operator representation of classical periodic pseudodifferential operators

Here we first formulate and at the end prove the main results of the paper.

**Theorem 2.** *Every operator $A \in Op\,\Sigma^\alpha_{cl}$ with $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$ can be represented in the form*

$$(Au)(t) = \int_0^1 \Big[\kappa^+_\alpha(t-s)a_+(t,s) + \kappa^-_\alpha(t-s)a_-(t,s) + a(t,s)\Big]u(s)\,ds \tag{6}$$

*where $a_\pm, a \in C^\infty_1(\mathbb{R}\times\mathbb{R})$, i.e. $a_\pm, a$ are $C^\infty$-smooth and 1-periodic with respect to both arguments and $\kappa^\pm_\alpha$ are 1-periodic functions or distributions defined by their Fourier coefficients*

$$\left.\begin{aligned}\hat\kappa^+_\alpha(n) &= |n|^\alpha\\ \hat\kappa^-_\alpha(n) &= |n|^\alpha \mathrm{sign}(n)\end{aligned}\right\}\qquad (0 \neq n \in \mathbb{Z}). \tag{7}$$

*Conversely, every integral operator of form* (6) – (7) *with $a_\pm, a \in C^\infty_1(\mathbb{R}\times\mathbb{R})$ belongs to $Op\,\Sigma^\alpha_{cl}$.*

Note that (7) define $\kappa^\pm_\alpha$ uniquely up to a constant addend $\hat\kappa^\pm_\alpha(0)$. Changing $\hat\kappa^\pm_\alpha(0)$, only the coefficient $a \in C^\infty_1(\mathbb{R}\times\mathbb{R})$ changes in (6). In (6), the integral means the usual Lebesgue integral for $\alpha < 0$ and $u \in H^0 = L^2(0,1)$. For $\alpha \geq 0$ and $u \in H^\mu$ $(\mu > \alpha + \frac{1}{2})$ the integral can be understood as the dual product between $H^\mu$ and $H^{-\mu}$, since $\kappa^\pm_\alpha \in H^{-\mu}$. The case of $u \in H^\lambda$ with an arbitrary $\lambda \in \mathbb{R}$ can be understood through the approximation of $u$ by smooth functions, e.g.

$$Au = \lim_{N\to\infty} AP_N u, \qquad P_N u = \sum_{|n|\leq N} \hat u(n) e^{in2\pi t}.$$

Here $P_N u \to u$ in $H^\lambda$ and $AP_N u \to Au$ in $H^{\lambda-\alpha}$ for $u \in H^\lambda$ (recall that $A \in \mathcal{L}(H^\lambda, H^{\lambda-\alpha})$ for any $\lambda \in \mathbb{R}$ if $A \in Op\,\Sigma^\alpha$).

Now consider the case $\alpha \in \mathbb{N}_0$ excluded from Theorem 2.

**Theorem 3.** *Every operator $A \in Op\,\Sigma^\alpha_{cl}$ with $\alpha = m \in \mathbb{N}_0$ has the representation*

$$\begin{aligned}(Au)(t) = &\sum_{j=0}^m \Big[c^+_j(t)u^{(m-j)}(t) + c^-_j(t)(H_0 u^{(m-j)})(t)\Big]\\ &+ \int_0^1 \Big[\kappa^+_{-1}(t-s)a_+(t,s) + \kappa^-_{-1}(t-s)a_-(t,s) + a(t,s)\Big]u(s)\,ds\end{aligned} \tag{8}$$

*where $c^\pm_j \in C^\infty_1(\mathbb{R})$ and $a_\pm, a \in C^\infty_1(\mathbb{R}\times\mathbb{R})$, $H_0$ is the Hilbert transformation*

$$(H_0 u)(t) = \frac{1}{i}\,\text{p.v.}\int_0^1 \cot\pi(s-t)u(s)\,ds = \sum_{n\geq 1} \hat u(n)e^{in2\pi t} - \sum_{n\leq -1} \hat u(n) e^{in2\pi t},$$

*and*

$$\kappa^{+}_{-1}(t) = -2\log|\sin \pi t| \tag{9}$$

$$\kappa^{-}_{-1} \textit{ is the 1-periodic extension of } t \mapsto -2\pi i t \textit{ from } [0,1) \textit{ to } \mathbb{R} \tag{10}$$

(*these functions satisfy* (7) *with* $\alpha = -1$).

**Remark 1.** Using the periodic Dirac delta function and its derivatives one can also (8) represent as an integral operator.

**Remark 2.** Clearly, $\Sigma^{\alpha}_{cl} \subset \Sigma^{\alpha+1}_{cl}$, therefore we actually have different possible integral operator representations of an operator $A \in Op\,\Sigma^{\alpha}_{cl}$. For instance, $A \in Op\,\Sigma^{-m}_{cl}$ with an $m \in \mathbb{N}$ can be represented in the form (6) with $\alpha = -1$ and $\kappa^{\pm}_{-1}$ defined in (9) - (10); the order $-m$ of the operator can be discovered by properties of the coefficients $a_{\pm}$:

$$\Big(\frac{\partial}{\partial s}\Big)^j a_{\pm}(t,s)\Big|_{s=t} = 0 \qquad (t \in \mathbb{R}; j = 0,\dots,m-2).$$

Operators of type (6) have been examined in [6, 12]. They often appear solving boundary integral equations on closed curves (see, e.g., [5 - 7, 12]).

**Proof of Theorem 2.** Let $A \in Op\,\Sigma^{\alpha}_{cl}$, i.e. its symbol $\sigma(t,\xi)$ has the asymptotic expansion (4) with $\sigma_j$ of form (5). We regularize the functions $|\xi|^{\beta}$ in the neighbourhood of $\xi = 0$ putting

$$\left.\begin{aligned} &\phi_\beta(0) = 0 \\ &\phi_\beta(\xi) = \phi_0(\xi)|\xi|^{\beta} \quad (\xi \in \mathbb{R}\setminus\{0\}) \end{aligned}\right\}$$

where $\phi_0 \in C^{\infty}(\mathbb{R})$ satisfies

$$\phi_0(\xi) = \begin{cases} 1 & \text{for } |\xi| \geq 1 \\ 0 & \text{for } |\xi| \leq \frac{1}{2}. \end{cases}$$

Thus we have

$$\sigma(t,\xi) \sim \sum_{j=0}^{\infty} \Big[b_j^{+}(t)\phi_{\alpha-j}(\xi) + b_j^{-}(t)\phi_{\alpha-j}(\xi)\mathrm{sign}(\xi)\Big] \tag{11}$$

where

$$b_j^{+}(t) = \frac{1}{2}[a_j^{+}(t) + a_j^{-}(t)] \qquad \text{and} \qquad b_j^{-}(t) = \frac{1}{2}[a_j^{+}(t) - a_j^{-}(t)].$$

On the other hand, by Theorem 1 the integral operator defined in (6) is a (classical) periodic pseudodifferential operator with the symbol $\tilde{\sigma}$ having the asymptotical expansion

$$\begin{aligned} \tilde{\sigma}(t,\xi) \sim & \sum_{j=0}^{\infty} \frac{1}{j!}\Big(\frac{\partial}{\partial \xi}\Big)^j \phi_\alpha(\xi)\Big(\frac{1}{2\pi i}\frac{\partial}{\partial s}\Big)^j a_{+}(t,s)\Big|_{s=t} \\ & + \sum_{j=0}^{\infty} \frac{1}{j!}\Big(\frac{\partial}{\partial \xi}\Big)^j \phi_\alpha(\xi)\mathrm{sign}(\xi)\Big(\frac{1}{2\pi i}\frac{\partial}{\partial s}\Big)^j a_{-}(t,s)\Big|_{s=t}. \end{aligned}$$

Representation (6) of $A \in Op\,\Sigma_{cl}^{\alpha}$ takes place if $\sigma \sim \tilde{\sigma}$, i.e. $\sigma - \tilde{\sigma} \in \Sigma^{-\infty}$. For $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$ this means that $a_{\pm} \in C_1^{\infty}(\mathbb{R} \times \mathbb{R})$ satisfy

$$\frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!}\Big(\frac{1}{2\pi i}\frac{\partial}{\partial s}\Big)^j a_{\pm}(t,s)\Big|_{s=t} = b_j^{\pm}(t) \qquad (t \in \mathbb{R}, j \in \mathbb{N}_0).$$

Thus, to prove Theorem 2, we simply have to solve the following elementary problem: given $b_j \in C_1^{\infty}(\mathbb{R})$ $(j \in \mathbb{N}_0)$, construct $a \in C_1^{\infty}(\mathbb{R} \times \mathbb{R})$ such that

$$\Big(\frac{\partial}{\partial s}\Big)^j a(t,s)\Big|_{s=t} = b_j(t) \qquad (t \in \mathbb{R}, j \in \mathbb{N}_0).$$

A solution may be given by a regularization and periodization of the Taylor series:

$$a(t,s) = \sum_{l=0}^{\infty} \frac{b_l(t)}{l!}[\chi(s-t)]^l \psi_{N_l}(s-t).$$

Here $\chi \in C_1^{\infty}(\mathbb{R})$ satisfies $\chi(s) = s$ for $|s| \le \frac{1}{4}$, and $\psi_N \in C_1^{\infty}(\mathbb{R})$ $(N \in \mathbb{N})$ satisfies

$$\psi_N(t) = \begin{cases} 1 & \text{for } |t| \le \frac{1}{8N} \\ 0 & \text{for } \frac{1}{4N} \le |t| \le \frac{1}{2}. \end{cases}$$

More concretely, we define

$$\psi_N(t) = \sum_{j \in \mathbf{Z}} \psi(Nt+j) \quad \text{where } \psi \in C^{\infty}(\mathbb{R}) \text{ with } \psi(t) = \begin{cases} 1 & \text{for } |t| \le \frac{1}{8} \\ 0 & \text{for } |t| \ge \frac{1}{4}. \end{cases}$$

The numbers $N_l \ge 1$ should be chosen so that the series itself and the series after applying $\big(\frac{\partial}{\partial t}\big)^j\big(\frac{\partial}{\partial s}\big)^k$ $(j,k \in \mathbb{N}_0)$ will converge uniformly for $t,s \in \mathbb{R}$. A sufficient condition is given by $N_l \ge d_l$ where

$$d_l = \max_{0 \le n \le l}\max_{0 \le t \le 1} |b_l^{(n)}(t)|.$$

Indeed, applying $\big(\frac{\partial}{\partial t}\big)^j\big(\frac{\partial}{\partial s}\big)^k$ we obtain a finite number of series of the type

$$\sum_{l=p}^{\infty} \frac{b_l^{(n)}(t)}{(l-p)!}[\chi(s-t)]^{l-p}\psi_{N_l}^{(q)}(s-t) \qquad (n \le j;\, p,q \le j+k)$$

(notice that $\chi(s) = s$ for $s \in \operatorname{supp}\psi_{N_l} \cap [-\frac{1}{2},\frac{1}{2}]$). For $l \ge n$ the members of the last series can be estimated by

$$\frac{d_l}{(l-p)!}(4N_l)^{-(l-p)}c_q N_l^q \le \frac{c_q}{(l-p)!}4^{-(l-p)}N_l^{-l+p+q+1}$$

guaranteeing uniform convergence of the series ∎

**Proof of Theorem 3.** For $\alpha = m \in \mathbb{N}_0$, we present (11) as the sum

$$\sigma = \sigma_m + \sigma^{[m]}$$

with

$$\sigma_m(t,\xi) = \sum_{j=0}^{m} \Big[ b_j^+(t)\phi_{m-j}(\xi) + b_j^-(t)\phi_{m-j}(\xi)\mathrm{sign}(\xi) \Big]$$

and

$$\begin{aligned} \sigma^{[m]}(t,\xi) &= \sum_{j=m+1}^{\infty} \Big[ b_j^+(t)\phi_{m-j}(\xi) + b_j^-(t)\phi_{m-j}(\xi)\mathrm{sign}(\xi) \Big] \\ &= \sum_{j=0}^{\infty} \Big[ b_{j+m+1}^+(t)\phi_{-1-j}(\xi) + b_{j+m+1}^-(t)\phi_{-1-j}(\xi)\mathrm{sign}(\xi) \Big] \end{aligned}$$

and the representation (8) - (10) for $A \in Op\,\Sigma_{cl}^m$ follows immediately from Theorem 2. Thereby,

$$c_j^{\pm}(t) = (2\pi i)^{j-m} b_j^{\pm}(t) \qquad (0 \le j \le m)$$

and the theorem is proved ∎

**Remark 3.** As it can be seen from the proof, also some non-classical periodic pseudodifferential operators have an integral operator representation similar to (6). Namely, if $\sigma \in \Sigma^\alpha$ has an asymptotic expansion

$$\sigma(t,\xi) \sim \sum_{j=0}^{\infty} \Big[ a_j^+(t)\gamma^{(j)}(\xi) + a_j^-(t)\gamma^{(j)}(\xi)\mathrm{sign}(\xi) \Big]$$

where

$$\left.\begin{array}{l} a_j^{\pm} \in C_1^\infty(\mathbb{R}) \\ \gamma \in C^\infty(\mathbb{R}) \ \text{ with } \ |\gamma^{(j)}(\xi)| \le c_j(1+|\xi|)^{\alpha-j} \quad (\xi \in \mathbb{R}, j \in \mathbb{N}_0) \end{array}\right\},$$

then $A = Op\,\sigma$ can be represented in the form

$$(Au)(t) = \int_0^1 \Big[ \kappa_+(t-s)a_+(t,s) + \kappa_-(t-s)a_-(t,s) + a(t,s) \Big] u(s)\,ds$$

where

$$a_\pm, a \in C_1^\infty(\mathbb{R}\times\mathbb{R}) \qquad \text{and} \qquad \left\{ \begin{array}{l} \hat\kappa_+(n) = \gamma(n) \\ \hat\kappa_-(n) = \gamma(n)\mathrm{sign}(n) \end{array} \right. \quad (0 \ne n \in \mathbb{Z}).$$

## 4. Functions $\kappa_\alpha^\pm$

Here we present some formulae of functions $\kappa_\alpha^\pm$ satisfying (7). For $\alpha = -1$ these formulae are well-known (see (9) - (10)). Consider the case $-1 < \alpha < 0$. Introduce the function

$$\kappa_\alpha(t) = t^{|\alpha|-1} + \sum_{j=1}^{\infty}[(t+j)^{|\alpha|-1} - \gamma_j] \qquad (0 < t \leq 1, -1 < \alpha < 0) \qquad (12)$$

where

$$\gamma_j = \int_0^1 (t+j)^{|\alpha|-1} dt = \frac{1}{|\alpha|}[(j+1)^{|\alpha|} - j^{|\alpha|}].$$

Note that the series in (12) converges uniformly in $t \in [0,1]$, since $\gamma_j$ as the mean value of $(t+j)^{|\alpha|-1}$ in $[0,1]$ has a representation $\gamma_j = (t_j + j)^{|\alpha|-1}$ with a $t_j \in (0,1)$, and

$$\left.\begin{array}{l} (t+j)^{|\alpha|-1} - \gamma_j = (t+j)^{|\alpha|-1} - (t_j+j)^{|\alpha|-1} = (|\alpha|-1)(t_j'+j)^{|\alpha|-2}(t-t_j) \\ |(t+j)^{|\alpha|-1} - \gamma_j| \leq (1-|\alpha|)j^{|\alpha|-2} \end{array}\right\}$$

where $t_j' \in (t, t_j) \subset (0,1)$. Clearly, also the series obtained after differentiations converge uniformly. Thus, $\kappa_\alpha \in C^\infty(0,1]$. Moreover, $\kappa_\alpha$ is decreasing and $0 < \kappa_\alpha(t) < t^{|\alpha|-1}$ $(0 < t \leq 1)$.

Define

$$\kappa_\alpha^\pm(t) = \gamma_\alpha^\pm[\kappa_\alpha(t) \pm \kappa_\alpha(1-t)] \qquad (0 < t < 1) \qquad (13)$$

where

$$\left.\begin{array}{rl} \gamma_\alpha^\pm &= \dfrac{1}{c_\alpha^\pm} \\ c_\alpha^+ &= 2(2\pi)^\alpha \Gamma(|\alpha|) \cos \dfrac{|\alpha|\pi}{2} \\ c_\alpha^- &= -2i(2\pi)^\alpha \Gamma(|\alpha|) \sin \dfrac{|\alpha|\pi}{2} \end{array}\right\}$$

and

$$\Gamma(\beta) = \int_0^\infty t^{\beta-1} e^{-t} dt \qquad (0 < \beta < 1)$$

is the Euler function. We preserve the designations $\kappa_\alpha^\pm$ also for the 1-periodic extensions of those functions and assert that

$$\left.\begin{array}{l} \hat{\kappa}_\alpha^+(n) = |n|^\alpha \\ \hat{\kappa}_\alpha^-(n) = |n|^\alpha \mathrm{sign}(n) \end{array}\right\} \qquad (0 \neq n \in \mathbb{Z},\ -1 < \alpha < 0). \qquad (14)$$

To prove this we first find the Fourier coefficients of $\kappa_\alpha(t)$ $(0 < t \leq 1)$ which we also regard as extended up to a 1-periodic function. Clearly,

$$\left.\begin{array}{l} \hat{\kappa}_\alpha(0) = \displaystyle\int_0^1 t^{|\alpha|-1} dt = \frac{1}{|\alpha|} \\ \hat{\kappa}_\alpha(n) = \displaystyle\sum_{j=0}^{\infty} \int_0^1 (t+j)^{|\alpha|-1} e^{-in2\pi t} dt \quad (0 \neq n \in \mathbb{Z}) \end{array}\right\}.$$

With the changes of variables

$$\left.\begin{aligned} t+j &= s \\ 2\pi|n|s &= \tau \end{aligned}\right\}$$

we have

$$\begin{aligned} \hat{\kappa}_\alpha(n) &= \sum_{j=0}^{\infty} \int_j^{j+1} s^{|\alpha|-1} e^{-in2\pi s} ds \\ &= \int_0^\infty s^{|\alpha|-1} e^{-in2\pi s} ds \\ &= (2\pi|n|)^\alpha \int_0^\infty \tau^{|\alpha|-1} e^{-\mathrm{sign}(n) i\tau} d\tau. \end{aligned}$$

It is known that

$$\int_0^\infty \tau^{\beta-1} e^{\pm i\tau} d\tau = e^{\pm i\frac{\pi}{2}\beta}\Gamma(\beta) \qquad (0 < \beta < 1)$$

(see, e.g., [13: p. 69] for a proof). Thus

$$\hat{\kappa}_\alpha(n) = (2\pi)^\alpha \Gamma(|\alpha|) e^{-i\mathrm{sign}(n)\frac{\pi}{2}|\alpha|}|n|^\alpha \qquad (0 \neq n \in \mathbb{Z}).$$

The Fourier coefficient of functions $v$ and $w$ such that $w(t) = v(1-t)$ are related by $\hat{w}(n) = \hat{v}(-n)$. Therefore, the Fourier coefficients of $\kappa_\alpha^\pm$ defined by (13) are as follows:

$$\hat{\kappa}_\alpha^\pm(n) = \gamma_\alpha^\pm (2\pi)^\alpha \Gamma(|\alpha|)\Big(e^{-i\mathrm{sign}(n)\frac{\pi}{2}|\alpha|} \pm e^{i\mathrm{sign}(n)\frac{\pi}{2}|\alpha|}\Big)|n|^\alpha \qquad (0 \neq n \in \mathbb{Z}).$$

This results to (14).

Now we have formulae of $\kappa_\alpha^\pm$ satisfying (7) for $-1 \le \alpha < 0$. The following obvious remark makes possible to extend the result for other $\alpha \in \mathbb{R}$.

**Remark 4.** The formulae

$$\left.\begin{aligned} \kappa_{\alpha-1}^+(t) &= 2\pi i \int_0^t [\kappa_\alpha^-(s) - \hat{\kappa}_\alpha^-(0)]\, ds \\ \kappa_{\alpha-1}^-(t) &= 2\pi i \int_0^t [\kappa_\alpha^+(s) - \hat{\kappa}_\alpha^+(0)]\, ds \end{aligned}\right\} \qquad (\alpha < 0)$$

and

$$\left.\begin{aligned} \kappa_{\alpha+1}^+(t) &= \frac{1}{2\pi i}\frac{d}{dt}\kappa_\alpha^-(t) \\ \kappa_{\alpha+1}^-(t) &= \frac{1}{2\pi i}\frac{d}{dt}\kappa_\alpha^+(t) \end{aligned}\right\} \qquad (\alpha \in \mathbb{R})$$

hold where $\frac{d}{dt}$ means the periodic distribution derivative.

**Acknowledgement** The author thanks Professor M. S. Agranovich for a useful discussion.

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Received 22.09.1998