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Full $C^{1,\alpha}$-Regularity for Minimizers of Integral Functionals with $L \log L$-Growth

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Abstract. We consider the integral functional with nearly-linear growth $\int_\Omega |Du| \log(1+|Du|)dx$ where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ ($n \geq 2$, $N \geq 1$) and we prove that any local minimizer u has locally Hölder continuous gradient in the interior of Ω thus excluding the presence of singular sets in Ω. This functional has recently been considered by several authors in connection with variational models for problems from the theory of plasticity with logarithmic hardening. We also give extensions of this result to more general cases.

Keywords: *Integral functionals, minimizers, $L \log L$-growth, Hölder continuity*

AMS subject classification: 49 N 60, 35 J 50

1. Introduction

The aim of this paper is to study the regularity of minimizers of integral functionals of the form

$$\mathcal{F}(u,\Omega) = \int_\Omega f(Du)\,dx \tag{1.1}$$

where Ω is an open subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}^N$. This problem has been widely investigated if the integrand f satisfies the growth assumptions

$$|z|^p \leq f(z) \leq L(1+|z|^p) \tag{1.2}$$

with $1 < p$. In the last years much attention has been dedicated to functionals verifying more general growth assumptions such as

$$|z|^p \leq f(z) \leq L(1+|z|^q) \tag{1.3}$$

where the main point is that now $1 < p \leq q$. The previous conditions are referred to in the literature as (p,q)-growth conditions and they have been extensively studied by

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Marcellini [11 - 14]. In this paper we consider a sort of limit case of conditions (1.3); let us explain in which sense. If we consider the function

$$f(z) = |z|^p \log(1 + |z|), \tag{1.4}$$

then we observe that the conditions in (1.3) are verified with $q = p + \varepsilon$ for each $\varepsilon > 0$ (for $|z|$ large). On the other hand, the function

$$f(z) = |z| \log(1 + |z|) \tag{1.5}$$

does not verify the conditions in (1.3) because in this last case $p = 1$, for $|z|$ large. For this reason a function as in (1.5) is said to be of *nearly linear growth*, that is

$$\limsup_{|z| \to +\infty} \frac{f(z)}{|z|^p} = 0 \qquad \text{and} \qquad \limsup_{|z| \to +\infty} \frac{f(z)}{|z|} = \infty$$

whenever $p > 1$.

In this paper we study regularity for local minimizers for a class of functionals of type (1.1), whose energy density f is modelled by the function in (1.5). The study of such functionals, from the regularity viewpoint, has been faced by a few authors in the last years (see [5, 7, 8]). In particular, in the paper [7], the authors considered the model functional

$$\mathcal{F}(u, \Omega) = \int_\Omega |Du| \log(1 + |Du|)\, dx$$

and proved $C^{1,\alpha}$-partial regularity for local minimizers provided $n \le 4$ and everywhere (in the interior) regularity in the case $n = 2$ (using ideas from [4, 5]). The partial regularity result has been subsequently obtained in [3] for any dimension n. In this paper we take up the problem of full regularity and we prove that any local minimizer of the functional $\mathcal{F}(u, \Omega)$ is in $C^{1,\alpha}_{loc}(\Omega; \mathbb{R}^n)$, thus solving a problem posed by M. Fuchs and G. Seregin in [7]. We also give a generalization of this result to a wide class of functionals with so called $L \log L$*-growth* (see Section 4). It is worth noticing that this type of functionals has been considered very recently in some papers (see [5, 7, 8]) in the contest of the theory of plasticity with logarithmic hardening. It also plays a relevant role in the study of the so called Prandtl-Eyring fluids, as pointed out in [8].

Now we spend a few words about our technique. The first technical difficulty we meet is that the functional $\mathcal{F}(u, \Omega)$ has sub-quadratic growth, that is condition (1.3) is satisfied with p (= 1) and q that are strictly less than 2 and it is not possible to apply the known techniques developed for functionals with (p, q)-growth. So we take a different path, that is we follow a particular approximation technique and introduce the family of regularized functionals

$$\mathcal{F}_{\varepsilon,\sigma}(u, \Omega) = \int_\Omega f_{\varepsilon,\sigma}(Du)\, dx$$

where

$$f_{\varepsilon,\sigma}(z) = \sqrt{\varepsilon + |z|^2} \log(1 + \sqrt{\varepsilon + |z|^2}) + \sigma(1 + |z|^2)^{\frac{3}{2}} \tag{1.6}$$

with $\varepsilon, \tau > 0$ and q suitably close to 1. The effect of this regularization is twofold. First of all we observe that the function $f_{\varepsilon,\sigma}$ has now polynomial growth of the type in (1.2). The most important thing is anyway that now the integrand $f_{\varepsilon,\sigma}$ depends directly on the quantity $|Du|^2$ rather than $|Du|$, and this fact reveals useful when deriving estimates for regularity (see Theorem 3.1/ Step 1).

Then we consider the solution $v_{\varepsilon,\sigma} \in W^{1,1}_{loc}(B_R; \mathbb{R}^N)$, $B_R \subset\subset \mathbb{R}^n$, of the regularized Dirichlet problem

$$\min\left\{\int_{B_R} f_{\varepsilon,\sigma}(Dw)\,dx \,:\, w \in u_\varepsilon + W^{1,q}_0(B_R, \mathbb{R}^N)\right\}$$

where ε and σ are two real parameters suitably related to each other and $u_\varepsilon \subset W^{1,q}(B_R, \mathbb{R}^N)$ is a sequence of smooth functions such that $u_\varepsilon \to u$ strongly in $W^{1,1}(B_R, \mathbb{R}^N)$. For the family $Dv_{\varepsilon,\sigma}$ we derive uniform L^∞-bounds, and, after proving that actually $v_{\varepsilon,\sigma} \to u$, we have the estimate

$$\sup_{B_\rho} |Du| \leq c\left(\int_{B_R} |Du|\log(1 + |Du|)\,dx + 1\right)^\beta, \tag{1.8}$$

$\rho < \frac{R}{8}$, that is the local boundedness of Du. Once proven (1.8), then we are also able to prove that $Du \in C^{0,\alpha}_{loc}$ for some $0 < \alpha \leq 1$.

2. Preliminaries and statements

In the following $B(x, R)$ will denote the open ball of $\mathbb{R}^n$ of center x and radius R, $\{y \in \mathbb{R}^n : |x - y| < R\}$, while Ω will denote an open bounded subset of $\mathbb{R}^n$ with $n \geq 2$. When no confusion shall arise we will just put $B_R \equiv B(x, R)$, and all the balls considered will have the same center. We will deal with functionals with $L\log L$-growth in the sense that

$$\log(1 + |z|)|z| \leq f(z) \leq L\big(1 + \log(1 + |z|)|z|\big),$$

so it is convenient to say something about the function space $L\log L(\Omega; \mathbb{R}^N)$, i.e. the subset of $L^1(\Omega; \mathbb{R}^N)$ of those functions u for which

$$\Xi(u)(\Omega) := \int_\Omega |u|\log(1 + |u|)\,dx < +\infty.$$

These are Banach spaces if equipped with a suitable norm related to the quantity $\Xi(u)$, and for further references on their use in this contest we address the interested reader to [8] and the references quoted there. We will not use topological properties of $L\log L(\Omega; \mathbb{R}^N)$ while the only result we are going to utilize is the following simple consequence of De LaValleé Poussin's theorem that we state here for the reader's convenience:

Proposition. *Let* $\{u_n\}_n \subset L\log L(\Omega;\mathbb{R}^N)$ *such that* $\Xi(u_n)(A) \le c < +\infty$ *where* A *is a measurable subset of* Ω. *Then there exist a subsequence* $\{u_{n_k}\}_k \subset \{u_n\}_n$ *and a function* $u \in L\log L(A;\mathbb{R}^N)$ *such that* $u_{n_k} \rightharpoonup u$ *weakly in* $L^1(A;\mathbb{R}^N)$.

In the vectorial case $N \ge 1$, for sake of simplicity, we will keep our attention on the model case

$$\mathcal{F}(u,\Omega) = \int_\Omega |Du|\log(1+|Du|)\,dx = \int_\Omega f(Du)\,dx \tag{2.1}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ $(N \ge 1)$.

Due to non-standard growth conditions verified by the functional $\mathcal{F} \equiv \mathcal{F}(u,\Omega)$ we shall adopt the following natural definition of a minimizer:

Definition. A function $u \in W^{1,1}_{loc}(\Omega;\mathbb{R}^N)$ is called *local minimizer* of the functional $\mathcal{F}$ if

$$f(Du) \in L^1_{loc}(\Omega) \qquad \text{and} \qquad \mathcal{F}(u,\Omega) \le \mathcal{F}(u+\varphi,\Omega)$$

for any $\varphi \in W^{1,1}_0(\Omega;\mathbb{R}^N)$.

Our main result is the following

Theorem 2.1. *Let* $u \in W^{1,1}_{loc}(\Omega;\mathbb{R}^N)$ *be a local minimizer of the functional* $\mathcal{F}$. *Then* Du *is locally Hölder continuous in* Ω.

The proof of this theorem will be given at the end of Section 3. As a byproduct of the arguments developed in order to prove Theorem 2.1 we will derive a point-wise bound for the quantity (see Theorem 3.1) $\sup_{B_\rho}|Du|$ with $B_\rho \subset\subset \Omega$, in terms of the quantity $\int_{B_R} f(Du)\,dx$.

After the previous result, we concentrate our attention on the scalar case $N = 1$. Here it is now useless to restrict ourselves to the model case $f(z) = |z|\log(1+|z|)$, because this is already treated in Theorem 2.1. From the proof of Theorem 2.1 in Section 3 it is clear that a crucial role is plaied by the fact that the function f directly depends on the quantity $|Du|$. Indeed, in more general cases, where there is no such a direct dependence of f on the quantity $|Du|$, the result of Theorem 2.1 is false as shown by counterexamples valid even in the case of functionals with usual quadratic growth. The only thing that can be said in the case of general structures is $C^{1,\alpha}$-partial regularity of local minimizers, that is the $C^{1,\alpha}$-regularity outside a negligible closed subset of Ω, as shown in the papers [3] and [7]. Nevertheless, things change in the scalar case, $N = 1$. Here we are going to deal with functions f as in (2.1), not necessarily depending on the modulus of the gradient. More precisely, we will make the following assumptions:

$$\begin{gathered} f \in C^2(\mathbb{R}^n) \\ |z|\log(1+|z|) \le f(z) \le L\big(1+|z|\log(1+|z|)\big) \\ \nu^{-1}\frac{|\lambda|^2}{\sqrt{1+|z|^2}} \le \langle D^2 f(z)\lambda,\lambda\rangle \le L\frac{\log(1+|z|)}{|z|}|\lambda|^2 \end{gathered} \tag{2.2}$$

for any $\lambda, z \in \mathbb{R}^n$ where $L > 1$ and $\nu > 0$. We observe that, in particular, the function $f(z) = |z|\log(1+|z|)$ satisfies conditions (2.2) for suitable L and ν.

Under the previous assumptions we are going to prove the following

Theorem 2.2. *Let $u \in W^{1,1}_{loc}(\Omega)$ be a local minimizer of the functional $\mathcal{F}$ and let the conditions in (2.2) being satisfied by the energy density f. Then Du is locally Hölder continuous in Ω.*

The proof of this theorem will be given at the end of Section 4.

3. The vectorial case

Our aim here is to prove that any local minimizer u of $\mathcal{F}$ is locally of class $C^{1,\alpha}$, thus proving Theorem 2.1. We start introducing the approximating functionals

$$\mathcal{F}_{\varepsilon,\sigma}(u,\Omega) = \int_\Omega f_{\varepsilon,\sigma}(Du)\,dx$$

where

$$f_{\varepsilon,\sigma}(z) = \sqrt{\varepsilon + |z|^2}\log(1 + \sqrt{\varepsilon + |z|^2}) + \sigma(1 + |z|^2)^{\frac{q}{2}} \tag{3.1}$$

with $\sigma, \varepsilon > 0$ and

$$1 < q < \begin{cases} \min\{\frac{n}{n-2}, 2\} & \text{if } n > 2 \\ 2 & \text{if } n = 2. \end{cases} \tag{3.2}$$

We start proving the local boundedness of the gradient of local minimizers of $\mathcal{F}$. More precisely, we will prove the following

Theorem 3.1. *Let $u \in W^{1,1}_{loc}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}$. Then Du is locally bounded. Moreover, if $B_R \subset\subset \Omega$ and $B_\rho \subset\subset B_R$ is such that $\rho < \frac{R}{8}$, then there exist constants $c = c(n, N, R)$ and $\beta = \beta(n)$, but independent of u, such that*

$$\sup_{B_\rho} |Du| \le c\left(\int_{B_R} |Du|\log(1 + |Du|)\,dx + 1\right)^\beta .$$

Proof. Step 1: *Caccioppoli type estimates.* In the following the constants c and β will freely denote two positive quantities, not necessarily the same in any two occurrences, while only the relevant dependences will be highlighted. We start observing that

$$f_{\varepsilon,\sigma}(z) = g_{\varepsilon,\sigma}(|z|^2) \tag{3.3}$$

where

$$g_{\varepsilon,\sigma}(t) = \sqrt{\varepsilon + t}\log(1 + \sqrt{\varepsilon + t}) + \sigma(1 + t)^{\frac{q}{2}},$$

that is $f_{\varepsilon,\sigma}$ depends on the quantity $|z|^2$. Moreover, we observe that $f_{\varepsilon,\sigma}$ has polynomial growth of order q ($<$ 2) and that the ellipticity and growth conditions

$$\begin{aligned} &c^{-1}\left(\sigma(1 + |z|^2)^{\frac{q-2}{2}} + \frac{1}{1 + \sqrt{\varepsilon + |z|^2}}\right)|\lambda|^2 \\ &\le \langle D^2 f_{\varepsilon,\sigma}(z)\lambda, \lambda\rangle \\ &\le c\left(\sigma(1 + |z|^2)^{\frac{q-2}{2}} + \frac{\log(1 + \sqrt{\varepsilon + |z|^2})}{\sqrt{\varepsilon + |z|^2}}\right)|\lambda|^2 \end{aligned} \tag{3.4}$$

are satisfied for any $z, \lambda \in \mathbb{R}^{nN}$, with the constant $c < +\infty$ independent of ε and σ.

We now state the following

Claim. *Let $v \in W^{1,q}(B_R, \mathbb{R}^N)$ be a local minimizer of*

$$\mathcal{F}_{\varepsilon,\sigma}(w, B_R) = \int_{B_R} f_{\varepsilon,\sigma}(Dw)\, dx$$

where $B_R \subset\subset \Omega$. Then there exist constants $c \equiv c(n, N, R) < +\infty$ and $\beta \equiv \beta(n) < +\infty$ but independent of ε, σ and v such that, if $B_\rho \subset\subset B_R$ and $\rho < \frac{R}{8}$, then

$$\sup_{B_\rho} |Dv| \le c \left(\int_{B_R} (f_{\varepsilon,\sigma}(Dv) + 1)\, dx \right)^\beta .$$

In other words we want to derive a priori estimates for the L^∞-norm of v, independently of ε and σ. We devote Step 1 and Step 2 to the proof of the previous claim.

From now on we will omit the subscripts ε and σ, so that we will denote $f_{\varepsilon,\sigma}$ simply by f. We will then recover the full notation only in the last step of the proof.

We are under the assumptions considered in [1] and we may use the results stated there. So it follows that

$$v \in C^{1,\alpha}_{loc}(B_R, \mathbb{R}^N) \cap W^{2,q}_{loc}(B_R, \mathbb{R}^N) \tag{3.5}$$

for some $0 < \alpha \le 1$. Therefore, we have the Euler-Lagrange system of $\mathcal{F}_{\varepsilon,\sigma}$

$$\int_{B_R} Df(Dv) D\varphi\, dx = \int_{B_R} D_{z_i^\alpha} f(Dv) D_i \varphi^\alpha dx = 0$$

satisfied for any choice of $\varphi \in W^{1,q}_0(B_R, \mathbb{R}^N)$ and $\alpha \in \{1, \dots, N\}$. Let us set

$$\Delta_{h,s} F(x) = \frac{F(x + he_s) - F(x)}{h}$$

whenever F is an integrable function defined on Ω and where $x \in \Omega$ and $0 < |h| < \operatorname{dist}(x, \partial\Omega)$ and $\{e_s\}_{s \le n}$ is the standard basis of $\mathbb{R}^n$. That is, $\Delta_{h,s}$ is the standard difference quotient operator. For the general properties of the difference quotients applied in this contest see also [10: Chapter 8].

In the Euler system we choose the test function $\varphi^\alpha = \Delta_{-h,s}(\eta^2 H^\gamma D_s v^\alpha)$ where $\gamma \ge 0$, $\alpha \in \{1, \dots, N\}$, $s \in \{1, \dots, n\}$, $\eta \in C^\infty_0(B_R)$ is such that $0 \le \eta \le 1$, $\eta \equiv 1$ on B_ρ, $\rho < R_0$, $\eta \equiv 0$ outside B_{R_0}, $R_0 < R$ and $|D\eta| \le c(R_0 - \rho)^{-1}$. Furthermore, we have set $H := 1 + \varepsilon + |Dv|^2$. It is easy to see that, by (3.5) and choosing h small enough, φ is an admissible test function for the Euler system of $\mathcal{F}$. With this choice we have, using Einstein's convention on repeated indexes,

$$\begin{aligned} &\int_{B_R} \Delta_{h,s} D_{z_i^\alpha} f(Dv) \\ &\times \left[2\eta D_i \eta H^\gamma D_s v^\alpha + \eta^2 \gamma H^{\gamma-1} D_i(|Dv|^2) D_s v^\alpha + \eta^2 H^\gamma D_{si} v^\alpha \right] dx = 0. \end{aligned}$$

Now we recall that under our assumptions $D_{z_i^\alpha} f(Du) \in W^{1,2}_{loc}(B_R)$ and $D^2 u \in L^2_{loc}(B_R, \mathbb{R}^N)$ follow (see [1: Lemma 2.5]) so that we may let $h \to 0$ in the previous formula obtaining

$$\int_{B_R} D_{z_i^\alpha z_j^\beta} f(Dv) D_{js} v^\beta \times \left[2\eta D_i \eta H^\gamma D_s v^\alpha + \eta^2 \gamma H^{\gamma-1} D_i(|Dv|^2) D_s v^\alpha + \eta^2 H^\gamma D_{si} v^\alpha\right] dx = 0. \tag{3.6}$$

Now we use the Cauchy-Schwartz inequality in order to estimate

$$\begin{aligned} 2\int_{B_R} & D_{z_i^\alpha z_j^\beta} f(Dv) D_{js} v^\beta \eta D_i \eta H^\gamma D_s v^\alpha \, dx \\ &\le 2\int_{B_R} H^\gamma \left(\eta^2 D_{z_i^\alpha z_j^\beta} f(Dv) D_{js} v^\beta D_{is} v^\alpha\right)^{\frac12} \cdot \\ &\quad \times \left(D_{z_i^\alpha z_j^\beta} f(Dv) D_j \eta D_s v^\beta D_i \eta D_s v^\alpha\right)^{\frac12} dx \\ &\le \frac12 \int_{B_R} D_{z_i^\alpha z_j^\beta} f(Dv) D_{js} v^\beta D_{is} v^\alpha \eta^2 H^\gamma dx \\ &\quad + c\int_{B_R} D_{z_i^\alpha z_j^\beta} f(Dv) D_j \eta D_s v^\beta D_i \eta D_s v^\alpha H^\gamma dx \end{aligned} \tag{3.7}$$

where we used also the Young inequality in a standard way. Connecting (3.6) and (3.7) and adding up over s we obtain

$$\begin{aligned} \int_{B_R} & D_{z_i^\alpha z_j^\beta} f(Dv) D_{js} v^\beta D_{is} v^\alpha \eta^2 H^\gamma dx \\ &\quad + \gamma \int_{B_R} D_{z_i^\alpha z_j^\beta} f(Dv) D_{js} v^\beta \eta^2 H^{\gamma-1} D_i(|Dv|^2) D_s v^\alpha dx \\ &\le c\int_{B_R} D_{z_i^\alpha z_j^\beta} f(Dv) D_j \eta D_s v^\beta D_i \eta D_s v^\alpha H^\gamma dx \end{aligned} \tag{3.8}$$

for a suitable constant c independent of ε and σ.

Now we proceed estimating the three terms in (3.8). The first one can be easily estimated from below using the ellipticity condition in (3.4), obtaining

$$\int_{B_R} D_{z_i^\alpha z_j^\beta} f(Dv) D_{js} v^\beta D_{is} v^\alpha \eta^2 H^\gamma dx \ge c^{-1} \int_{B_R} \eta^2 H^{\gamma-\frac12} |D^2 v|^2 dx$$

where we dropped from below the non-affecting term $c^{-1}\sigma \int_{B_R} \eta^2 H^{\gamma+\frac{q}{2}-1} |D^2 v|^2 dx$. The third term in (3.8) can be also easily estimated from above, by roughly using the growth conditions in (3.4):

$$\begin{aligned} \int_{B_R} & D_{z_i^\alpha z_j^\beta} f(Dv) D_j \eta D_s v^\beta D_i \eta D_s v^\alpha H^\gamma dx \\ &\le c\int_{B_R} |D\eta|^2 H^{\gamma+\frac12} \left[\log(1+\sqrt{\varepsilon + |Dv|^2}) + \sigma H^{\frac{q-1}{2}}\right] dx. \end{aligned}$$

In order to estimate the second integral in (3.8), we keep into account the particular structure of $f_{\varepsilon,\sigma}$ (see (3.3)). Indeed, the identity

$$D_{z_i^\alpha z_j^\beta} f_{\varepsilon,\sigma}(z) = 4g''_{\varepsilon,\sigma}(|z|^2)z_i^\alpha z_j^\beta + 2g'_{\varepsilon,\sigma}(|z|^2)\delta_{ij}\delta^{\alpha\beta}$$

is easy to verify so that

$$\begin{aligned}
&\gamma \int_{B_R} D_{z_i^\alpha z_j^\beta} f(Dv) D_{js}v^\beta D_i(|Dv|^2) D_s v^\alpha \eta^2 H^{\gamma-1}\,dx \\
&= 4\gamma \int_{B_R} g''(|Dv|^2) D_i v^\alpha D_j v^\beta D_{js} v^\beta D_i(|Dv|^2) D_s v^\alpha \eta^2 H^{\gamma-1}\,dx \\
&\quad + 2\gamma \int_{B_R} g'(|Dv|^2)\delta_{ij}\delta^{\alpha\beta} D_{js}v^\beta D_i(|Dv|^2) D_s v^\alpha \eta^2 H^{\gamma-1}\,dx \\
&= \gamma \int_{B_R} \left[2g''(|Dv|^2) D_i v^\alpha D_s v^\alpha + g'(|Dv|^2)\delta_{is}\delta^{\alpha\beta}\right] \\
&\quad \times D_i(|Dv|^2) D_s(|Dv|^2)\eta^2 H^{\gamma-1}\,dx \\
&\geq \gamma \frac{1}{2} \int_{B^R} D_{z_i^\alpha z_s^\alpha} f(Dv) D_i(|Dv|^2) D_s(|Dv|^2)\eta^2 H^{\gamma-1}\,dx \\
&\geq c^{-1}\gamma \int_{B_R} \eta^2 H^{\gamma-\frac{3}{2}} |D(|Dv|^2)|^2\,dx
\end{aligned}$$

where once again we used the ellipticity condition (1) within the particular structure of f and we dropped the non-affecting term containing σ. Connecting the previous estimates we finally obtain

$$\begin{aligned}
&\gamma \int_{B_R} \eta^2 H^{\gamma-\frac{3}{2}} |D(|Dv|^2)|^2\,dx + \int_{B_R} \eta^2 H^{\gamma-\frac{1}{2}} |D^2 v|^2\,dx \\
&\qquad \leq c \int_{B_r} |D\eta|^2 H^{\gamma+\frac{1}{2}} \left[\log(1+\sqrt{\varepsilon+|Dv|^2}) + \sigma H^{\frac{q-1}{2}}\right] dx.
\end{aligned}$$

Observing that

$$\int_{B_R} \eta^2 H^{\gamma-\frac{3}{2}} |D(|Dv|^2)|^2\,dx \leq c \int_{B_R} \eta^2 H^{\gamma-\frac{1}{2}} |D^2 v|^2\,dx$$

we finally deduce that

$$\begin{aligned}
&(\gamma+1) \int_{B_R} \eta^2 H^{\gamma-\frac{3}{2}} |D(|Dv|^2)|^2\,dx \\
&\qquad \leq c \int_{B_R} |D\eta|^2 H^{\gamma+\frac{1}{2}} \left[\log(1+\sqrt{\varepsilon+|Dv|^2}) + \sigma H^{\frac{q-1}{2}}\right] dx.
\end{aligned}$$

Now we introduce the positive quantity χ in such a way that

$$\chi = \frac{2^*}{2} = \frac{n}{n-2} > 1 \text{ if } n > 2 \qquad \text{and} \qquad 1 < q < \chi \text{ if } n = 2.$$

By the Sobolev embedding Theorem, for $n \geq 2$, we may estimate

$$
\begin{aligned}
\left(\int_{B_R} \eta^{2\chi} H^{(2\gamma+1)\frac{\chi}{2}} dx\right)^{\frac{1}{\chi}} & \\
&\leq c\int_{B_R} |D(H^{\frac{2\gamma+1}{4}}\eta)|^2 dx \\
&\leq c\int_{B_R} |D\eta|^2 H^{\gamma+\frac{1}{2}} dx + c(\gamma+1)^2 \int_{B_R} \eta^2 H^{\gamma-\frac{3}{2}} |D(|Dv|^2)|^2 dx \\
&\leq c(\gamma+1)\int_{B_R} |D\eta|^2 H^{\gamma+\frac{1}{2}} \left[\log(1+\sqrt{\varepsilon+|Dv|^2}) + \sigma H^{\frac{q-1}{2}} + 1\right] dx
\end{aligned}
\tag{3.9}
$$

and, obviously,

$$
\left(\int_{B_R} \eta^{2\chi} H^{\frac{(2\gamma+1)\chi}{2}} dx\right)^{\frac{1}{\chi}} \leq c(\gamma+1)\int_{B_R} |D\eta|^2 H^{\frac{(2\gamma+1)1}{2+\mu}} dx \tag{3.10}
$$

where the constant c is independent of $\varepsilon, \sigma, \gamma$ and v while we have set $\mu := \frac{q-1}{2}$.

Step 2: *Iteration.* Our next aim is to iterate formula (3.10) using a modified version of Moser's iteration technique in the case of non-standard, sub-quadratic growth. First of all we define inductively the sequences of exponents

$$
\begin{cases} \gamma_1 = 0 \\ \gamma_{k+1} = (2\gamma_k + 1)\frac{\chi}{2} - (2\mu+1)\frac{1}{2} \quad (k \geq 1) \end{cases}
$$

and

$$
\alpha_k = (2\gamma_k + 1)\frac{1}{2} \quad (k \geq 1) \qquad \text{so that} \quad \begin{cases} \alpha_1 = \frac{1}{2} \\ \alpha_{k+1} = \alpha_k \chi - \mu \quad (k \geq 1). \end{cases}
$$

Moreover, we introduce a sequence of radii

$$
\rho_k = \rho + (R_0 - \rho)2^{-k}
$$

where $\rho < \frac{R_0}{2}$, and a sequence of *cut-off* functions η_k such that

$$
0 \leq \eta_k \leq 1, \ \eta_k \equiv 1 \text{ on } B_{\rho_{k+1}}, \ \eta_k \equiv 0 \text{ outside } B_{\rho_k}, \ |D\eta_k| \leq c2^k(R_0-\rho)^{-1}.
$$

Note that with this choice $\rho_0 = R_0$ and that $\rho_k \to \rho$. Finally, we define

$$
A_k = \left(\int_{B_{\rho_k}} H^{\alpha_k+\mu}\right)^{\frac{1}{\alpha_k+\mu}}. \tag{3.11}
$$

Writing (3.10) with this choice of η_k and γ_k, by (3.11) we easily obtain

$$
A_{k+1} \leq \left[\frac{c\,\alpha_k 2^k}{(R_0-\rho)^2}\right]^{\frac{1}{\alpha_k}} A_k^{\frac{(\alpha_k+\mu)\chi}{\alpha_{k+1}+\mu}}.
$$

By induction it is possible to derive the representation formula

$$\alpha_{k+1} = \frac{1}{2} + \left(\frac{1}{2} - \frac{q}{2\chi}\right)\sum_{i=1}^{k}\chi^i \qquad (k \geq 1).$$

We observe that, in view of (3.3) and the choice of χ, $\frac{1}{2} - \frac{q}{2\chi} > 0$ follows. Moreover, it is also easy to check that

$$\vartheta := \prod_{i=1}^{\infty}\frac{(\alpha_k+\mu)\chi}{\alpha_{k+1}+\mu} = \lim_{k\to+\infty}\prod_{i=1}^{k}\frac{(\alpha_i+\mu)\chi}{\alpha_{i+1}+\mu} = \frac{q}{2}\lim_{k\to\infty}\frac{\chi^k}{\alpha_{k+1}+\mu} = q\,\frac{\chi-1}{\chi-q} < \infty.$$

Iterating (3.11) gives (without loss of generality we may assume that $A_1 \geq 1$)

$$A_{k+1} \leq \prod_{i=1}^{k}\left[\frac{c\alpha_i 2^i}{(R_0-\rho)^2}\right]^{\frac{\vartheta}{\alpha_i}} A_1^{\vartheta} \leq c_2\left[\frac{1}{(R_0-\rho)^2}\right]^{\sum_{k=1}^{\infty}\frac{\vartheta}{\alpha_k}} A_1^{\vartheta} \leq \frac{c}{(R_0-\rho)^{\beta}}A_1^{\vartheta} \tag{3.12}$$

since (we recall that $\alpha_k \simeq \chi^k$)

$$c_2 \leq \exp\left(\theta\sum_{k=1}^{\infty}\frac{\log(c\alpha_k 2^k)}{\alpha_k}\right) < +\infty \qquad \text{and} \qquad \sum_{k=1}^{\infty}\frac{1}{\alpha_k} \leq \frac{4}{\chi-q} < +\infty.$$

We stress the fact that the constants appearing in (3.12) are still independent of ε,σ and v. If we let $k\to\infty$ in (3.12), we get

$$\begin{aligned}\sup_{B_\rho}(1+\varepsilon+|Dv|^2) &= \lim_{k\to\infty}\left(\int_{B_\rho} H^{\alpha_k+\mu}dx\right)^{\frac{1}{\alpha_k+\mu}}\\ &\leq \limsup_{k\to+\infty} A_k\\ &\leq \frac{c}{(R_0-\rho)^{\beta}}A_1^{\vartheta}\\ &\leq \frac{c}{(R_0-\rho)^{\beta}}\left(\int_{B_{R_0}} H^{\frac{q}{2}}dx\right)^{\beta}\end{aligned} \tag{3.13}$$

with β independent of ε and σ.

Now let us take R_0 in such a way that $4R_0 < R$ and consider $\eta \in C_0^{\infty}(B_{2R_0})$ such that $\eta \equiv 1$ on B_{R_0}, $\eta \equiv 0$ outside B_{2R_0} and $|D\eta| \leq cR_0^{-1}$. If we put $\gamma = 0$ and this function η in (3.9), we obtain

$$\begin{aligned}\left(\int_{B_{R_0}} H^{\frac{q}{2}}dx\right)^{\frac{1}{\chi}} &\leq c\left(\int_{B_{R_0}} H^{\frac{\chi}{2}}dx\right)^{\frac{1}{\chi}}\\ &\leq \frac{c}{R_0^2}\int_{B_R} H^{\frac{1}{2}}\left[\log(1+\sqrt{\varepsilon+|Dv|^2})+\sigma H^{\frac{q-1}{2}}+1\right]dx\\ &\leq c\int_{B_R}(f_{\varepsilon,\sigma}(Dv)+1)\,dx\end{aligned}$$

where $c = c(R_0)$ is independent of ε, σ and v. Combining this last estimate with (3.13) easily gives

$$\sup_{B_\rho} |Dv| \le c \left(\int_{B_R} (f_{\varepsilon,\sigma}(Dv) + 1)\, dx \right)^\beta \tag{3.14}$$

with $\rho < \frac{R}{8}$ and $c \equiv c(n, N, R)$ and $\beta \equiv \beta(n)$ but independent of σ,ε and v. The claim at the beginning of Step 1 is finally proved.

Step 3: *Approximation.* Now we want to apply a-priori estimate (3.14) to a sequence of approximating minimizers and recover the same estimate for the original local minimizer u of the functional $\mathcal{F}$.

In the following, ε and σ denote two sequences of positive real numbers such that $\varepsilon, \sigma \to 0$. We will sometimes pass to subsequences that will be still denoted by ε and σ. Let us define

$$u_\varepsilon(x) = (u * \varphi_\varepsilon)(x) = \int_\Omega u(y)\varphi_\varepsilon(y - x)dy$$

where $\{\varphi_\varepsilon\}_\varepsilon$ is a family of smooth and positive mollifiers such that $\operatorname{supp}\varphi_\varepsilon \subset B_\varepsilon(0)$ and $\int_{B_1(0)} \varphi_\varepsilon dx = 1$. The functionals $\mathcal{F}_{\varepsilon,\sigma}$ are coercive, convex and therefore weakly lower semicontinuous in the space $W^{1,q}(B_R, \mathbb{R}^N)$, and so Direct methods of the Calculus of Variations allow us to define

$$v_{\varepsilon,\sigma} \in u_\varepsilon + W_0^{1,q}(B_R, \mathbb{R}^N)$$

as the (unique) solution to the Dirichlet problem

$$\min \left\{ \int_{B_R} f_{\varepsilon,\sigma}(Dw)\, dx : \ w \in u_\varepsilon + W_0^{1,q}(B_R, \mathbb{R}^N) \right\}$$

where $B_R \subset\subset \Omega$ and we took $\varepsilon \le \min\{1, 2^{-1}\mathrm{dist}\,(B_R, \partial\Omega)\}$.

Now let us choose

$$\sigma = \sigma(\varepsilon) = \left(1 + \varepsilon^{-1} + \|Du_\varepsilon\|^{2q}_{L^q(B_R)}\right)^{-1}.$$

Moreover, we set

$$v_\varepsilon = v_{\varepsilon,\sigma(\varepsilon)} \qquad \text{and} \qquad f_\varepsilon = f_{\varepsilon,\sigma(\varepsilon)}.$$

Estimate (3.14) is of course valid for v_ε, and using the minimality of v_ε we get

$$\begin{aligned}
\sup_{B_\rho} |Dv_\varepsilon| &\le c \left(\int_{B_R} (f_\varepsilon(Dv_\varepsilon) + 1)\, dx \right)^\beta \\
&\le c \left(\int_{B_R} f_\varepsilon(Du_\varepsilon)\, dx + 1 \right)^\beta \\
&= c \left(\int_{B_R} \sqrt{\varepsilon + |Du_\varepsilon|^2} \log(1 + \sqrt{\varepsilon + |Du_\varepsilon|^2})\, dx \right.
\end{aligned}$$

$$
\begin{aligned}
&+ 1 + \sigma \int_{B_R} (1 + |Du_\varepsilon|^2)^{\frac{q}{2}} dx \Big)^\beta \\
&\leq c \left(\int_{B_{R+\epsilon}} \sqrt{\varepsilon + |Du|^2} \log(1 + \sqrt{\varepsilon + |Du|^2})\, dx + 1 + o(\varepsilon) \right)^\beta \qquad (3.15) \\
&\leq c \left(\int_{B_{R+\epsilon}} f(Du)\, dx + 1 + o(\varepsilon) \right)^\beta
\end{aligned}
$$

provided $\rho \leq \frac{R}{8}$. Indeed, if $h : \mathbb{R}^{nN} \to \mathbb{R}$ is a convex function such that $\int_{B_{R+\epsilon}} h(Du)\, dx < +\infty$, using the Jensen inequality it is possible to estimate

$$
\begin{aligned}
\int_{B_R} h(Du_\varepsilon)\, dx &= \int_{B_R} h\left(\int_{B(x,\varepsilon)} Du(y)\varphi_\varepsilon(x-y)dy \right) dx \\
&\leq \int_{B_R} \int_{B(x,\varepsilon)} h(Du(y))\varphi_\varepsilon(x-y)\, dydx \\
&= \int_{B_R} \int_{B_{R+\epsilon}} h(Du(y))\varphi_\varepsilon(x-y)\, dydx \\
&= \int_{B_{R+\epsilon}} \int_{B_R} h(Du(y))\varphi_\varepsilon(x-y)\, dxdy \\
&\leq \int_{B_{R+\epsilon}} h(Du)\, dx.
\end{aligned}
$$

Therefore, applying the previous argument to the convex function

$$
z \to \sqrt{\varepsilon + |z|^2} \log\left(1 + \sqrt{\varepsilon + |z|^2}\right)
$$

we obtain

$$
\begin{aligned}
&\int_{B_R} \sqrt{\varepsilon + |Du_\varepsilon|^2} \log\left(1 + \sqrt{\varepsilon + |Du_\varepsilon|^2}\right) dx \\
&\qquad \leq \int_{B_{R+\epsilon}} \sqrt{\varepsilon + |Du|^2} \log\left(1 + \sqrt{\varepsilon + |Du|^2}\right) dx,
\end{aligned}
$$

that is the inequality used to derive (3.15).

We remark that $\rho < \frac{R}{8}$ and that the constants c and β still do not depend on ε and σ. In a similar way, by using the minimality of v_ε, we deduce that

$$
\Xi(Dv_\varepsilon)(B_R) \leq \int_{B_R} f_\varepsilon(Dv_\varepsilon)\, dx \leq c \int_{B_{R+\epsilon}} f(Du)\, dx + 1 + o(\varepsilon), \qquad (3.16)
$$

therefore, by (3.15) and (3.16) and using Proposition 2.1, we may assume that

$$
v_\varepsilon \rightharpoonup w \in u + W_0^{1,1}(B_R, \mathbb{R}^N)
$$

weakly in $W^{1,1}(B_R, \mathbb{R}^N)$ and weakly* in $W^{1,\infty}(B_\rho, \mathbb{R}^N)$. Letting $\varepsilon \to 0$ in (3.15) we therefore easily obtain

$$\sup_{B_\rho} |Dw| \le c \left(\int_{B_R} f(Du)\, dx + 1 \right)^{\beta} .$$

Now we conclude the proof showing that actually

$$u \equiv w. \tag{3.17}$$

Using the minimality of v_ε and the Jensen inequality in the way exposed above we obtain

$$\begin{aligned}
\int_{B_R} f(Dv_\varepsilon)\, dx &\le \int_{B_R} f_\varepsilon(Dv_\varepsilon)\, dx \\
&\le \int_{B_R} f_\varepsilon(Du_\varepsilon)\, dx \\
&\le \int_{B_R} \sqrt{\varepsilon + |Du_\varepsilon|^2} \log\left(1 + \sqrt{\varepsilon + |Du_\varepsilon|^2}\right) dx + o(\varepsilon) \\
&\le \int_{B_{R+\varepsilon}} \sqrt{\varepsilon + |Du|^2} \log\left(1 + \sqrt{\varepsilon + |Du|^2}\right) dx + o(\varepsilon).
\end{aligned}$$

By the lower semicontinuity of the convex functional $v \to \int_{B_R} f(Dv)\, dx$ with respect to the weak convergence of the gradients in L^1 we have, letting $\varepsilon \to 0$,

$$\begin{aligned}
\int_{B_R} f(Dw)\, dx &\le \liminf_{\varepsilon \to 0} \int_{B_R} f(Dv_\varepsilon)\, dx \\
&\le \lim_{\varepsilon \to 0} \int_{B_{R+\varepsilon}} \sqrt{\varepsilon + |Du|^2} \log\left(1 + \sqrt{\varepsilon + |Du|^2}\right) dx + o(\varepsilon) \\
&= \int_{B_R} f(Du)\, dx.
\end{aligned}$$

Finally, the minimality of u implies

$$\int_{B_R} f(Dw)\, dx = \int_{B_R} f(Du)\, dx.$$

The strict convexity of f and the fact that $u - w \in W_0^{1,1}(B_R, \mathbb{R}^N)$ implies (3.17). This concludes the proof of Theorem 3.1 ∎

Using the previous result we are now able to prove $C^{1,\alpha}$-regularity of minimizers of $\mathcal{F}$. Indeed, we come to the proof of the main result of the paper:

Proof of Theorem 2.1. We keep the notation used in the proof of Theorem 3.1. Let us consider the C^2-convex function $\Phi : \mathbb{R} \to \mathbb{R}$ defined by

$$\Phi(t) = \left(\max\{(t - 4M^2), 0\} \right)^3 .$$

where $M > 0$ is such that $\sup_{B_\rho} |Dv_\varepsilon| \le M$ $(\rho < \frac{R}{8})$. We consider the functional

$$\widetilde{\mathcal{F}}_\varepsilon(w) = \int_{B_\rho} \tilde{f}_\varepsilon(Dw)\,dx \qquad \text{where } \tilde{f}_\varepsilon(z) = f_\varepsilon(z) + \Phi(|z|^2).$$

Easy computation show that there is a constant $\nu = \nu(M) < \infty$ (in particular, ν is independent of ε) such that

$$\begin{gathered} \nu^{-1}|z|^6 \le \tilde{f}_\varepsilon(z) \le \nu(1+|z|^6) \\ \nu^{-1}(1+|z|^4)|\lambda|^2 \le \langle D^2\tilde{f}_\varepsilon(z)\lambda,\lambda\rangle \le \nu(1+|z|^4)|\lambda|^2 \end{gathered} \tag{3.18}$$

for any $z, \lambda \in \mathbb{R}^{nN}$. Note anyway that the ratio between the highest and the lowest eigenvalue of the matrix $D^2\tilde{f}_\varepsilon(z)$ is bounded by the constant $\nu^2 \equiv \nu^2(M)$ depending on M but independent of ε. Moreover, we observe that $\tilde{f}_\varepsilon(z) = f_\varepsilon(z)$ whenever $|z| < 2M$. Then, for any $\varphi \in C_0^\infty(B_\rho, \mathbb{R}^N)$ we have

$$\int_{B_\rho} D\tilde{f}_\varepsilon(Dv_\varepsilon)D\varphi\,dx = \int_{B_\rho} Df_\varepsilon(Dv_\varepsilon)D\varphi\,dx = 0$$

since v_ε is a minimizer of $\mathcal{F}_\varepsilon$. Therefore, v_ε also solves the Euler system of $\widetilde{\mathcal{F}}_\varepsilon$ and so is also a local minimizer for $\widetilde{\mathcal{F}}_\varepsilon$ (recall that $\widetilde{\mathcal{F}}_\varepsilon$ is a strictly convex functional). Keeping into account the growth and ellipticity conditions (3.18) we have that the standard regularity theory for non-degenerate functionals having the structure as in (3.3) applies for v_ε (see [9] and the references quoted therein), and it follows that there exist $c \equiv c(M) < \infty$ and $\alpha \equiv \alpha(M) > 0$ such that

$$|Dv_\varepsilon(x) - Dv_\varepsilon(y)| \le c\,|x-y|^\alpha \tag{3.19}$$

whenever $x, y \in B_{\frac{\rho}{2}}$. Now using the Ascoli-Arzelà Theorem, we may let $\varepsilon \to 0$ in (3.19) obtaining $|Du(x) - Du(y)| \le c\,|x-y|^\alpha$ in $B_{\frac{\rho}{2}}$. From this inequality the assertion of the theorem easily follows by a standard covering argument ∎

Remark. The approximation method used in order to prove Theorem 3.1 is flexible enough to allow more applications. In particular, it can be applied to integral functionals with more general growth assumptions. Results in this direction are in preparation and will appear later.

4. The scalar case

In this section we turn our attention to the scalar case, i.e. we consider $u : \Omega \to \mathbb{R}$, and a more general class of integral functionals.

We shall consider variational integrals of the type

$$\mathcal{F}(u,\Omega) = \int_\Omega f(Du)\,dx$$

where $\Omega \subset \mathbb{R}^n$ and the energy density f satisfies the growth and convexity conditions stated in (2.2). So we are going to prove $C^{1,\alpha}$-regularity in the interior of Ω, for local minimizers of $\mathcal{F}$. The scheme of the proof is the same of Section 3, so we will follow it giving only the relevant modifications.

We start with the L^∞-bound:

Theorem 4.1. *Let $u \in W^{1,1}_{loc}(\Omega)$ be a local minimizer of the functional $\mathcal{F}$. Then Du is locally bounded. Moreover, if $B_R \subset\subset \Omega$ and $B_\rho \subset\subset B_R$ with $\rho < \frac{R}{8}$, there exist constants $c = c(n, L, \nu, R)$ and $\beta = \beta(n)$ such that*

$$\sup_{B_\rho} |Du| \leq c \left(\int_{B_R} f(Du)\,dx + 1 \right)^{\beta} . \tag{4.1}$$

Proof. Step 1: We consider here a simpler approximation. We shall put

$$f_\sigma(z) = f(z) + \sigma(1 + |z|^2)^{\frac{q}{2}}$$

with the same choice of q as in Theorem 3.1. Also, this time f_σ has polynomial growth of order q and the ellipticity and growth conditions

$$\begin{aligned} c^{-1} &\left[\sigma(1 + |z|^2)^{\frac{q-2}{2}} + \frac{1}{\sqrt{1 + |z|^2}} \right] |\lambda|^2 \\ &\leq \langle D^2 f_\sigma(z)\lambda, \lambda \rangle \\ &\leq c \left[\sigma(1 + |z|^2)^{\frac{q-2}{2}} + \frac{\log(1 + |z|)}{|z|} \right] |\lambda|^2 \end{aligned} \tag{4.2}$$

are verified whenever $z, \lambda \in \mathbb{R}^n$ and where c is independent of σ. We then consider the functionals

$$\mathcal{F}_\sigma(w) = \int_{B_R} f_\sigma(Dw)\,dx,$$

where $B_R \subset\subset \Omega$, and a local minimizer $v \in W^{1,q}(B_R)$ of $\mathcal{F}_\sigma$. Also, this time we will derive a priori estimates for the L^∞-norm of Dv. Moreover, let us set

$$H := 1 + |Dv|^2$$

and choose, for h small enough, as for the proof of Theorem 3.1, the test function $\varphi = \Delta_{-h,s}(\eta^2 H^\gamma D_s v)$, with $\gamma \geq 0$ for the Euler equation of $\mathcal{F}_\sigma$:

$$\int_{B_R} D_{z_i} f_\sigma(Dv) D_i\varphi\,dx = 0.$$

We remark that this choice is admissible because also in this case the function v enjoys the same regularity properties as described in (3.5) (see, for instance, [2] or [10: Chapter 8] for a simple proof). The same computation of Theorem 3.1/Step 1 give

$$\begin{aligned} &\int_{B_R} D_{z_i z_j} f_\sigma(Dv) D_{js} v \\ &\times \left[2\eta D_i\eta H^\gamma D_s v + \eta^2 \gamma H^{\gamma-1} D_i(|Dv|^2) D_s v + \eta^2 H^\gamma D_{is} v \right] dx = 0 \end{aligned}$$

so that, proceeding as in Theorem 3.1, we arrive at the estimate

$$\begin{aligned} &\int_{B_R} D_{z_i z_j} f_\sigma(Dv) D_{js} v D_{is} v \eta^2 H^\gamma dx \\ &\qquad + \gamma \int_{B_R} D_{z_i z_j} f_\sigma(Dv) D_{js} v D_i(|Dv|^2) D_s v \eta^2 H^{\gamma-1} dx \\ &\leq c \int_{B_R} D_{z_i z_j} f_\sigma(Dv) D_i\eta D_s v D_j\eta D_s v H^\gamma dx. \end{aligned} \tag{4.3}$$

That is the analogous of (3.8) in this case. The first and the last term in (4.3) can be estimated just as the corresponding terms in (3.8). Indeed,

$$\int_{B_R} D_{z_i z_j} f_\sigma(Dv) D_{js}v D_{is}v \eta^2 H^\gamma dx \geq c^{-1} \int_{B_R} \eta^2 H^{\gamma-\frac{1}{2}} |D^2 v|^2 dx$$

and

$$\int_{B_R} D_{z_i z_j} f_\sigma(Dv) D_j\eta D_s v D_i\eta D_s v H^\gamma dx$$
$$\leq c \int_{B_R} |D\eta|^2 H^{\gamma+\frac{1}{2}} \left[\log(1+|Dv|) + \sigma H^{\frac{q-1}{2}}\right] dx$$

while for the second one we have, in a simpler way,

$$\gamma \int_{B_R} D_{z_i z_j} f_\sigma(Dv) D_{js}v D_i(|Dv|^2) D_s v \eta^2 H^{\gamma-1} dx$$
$$= \frac{\gamma}{2} \int_{B_R} D_{z_i z_j} f_\sigma(Dv) D_i(|Dv|^2) D_j(|Dv|^2) \eta^2 H^{\gamma-1} dx$$
$$\geq c^{-1}\gamma \int_{B_R} \eta^2 H^{\gamma-\frac{3}{2}} |D(|Dv|^2)|^2 dx$$

as in Theorem 3.1. If we connect the previous estimates we finally arrive at

$$(\gamma+1) \int_{B_R} H^{\gamma-\frac{3}{2}} |D(|Dv|^2)|^2 \eta^2 dx$$
$$\leq c \int_{B_R} |D\eta|^2 H^{\gamma+\frac{1}{2}} \left[\log(1+|Dv|) + \sigma H^{\frac{q-1}{2}} + 1\right] dx$$

that is just (3.9) in the scalar case.

From this point on, the iteration works exactly as in Theorem 3.1 and the whole procedure, using growth condition $(2.2)_1$, leads us to get for Dv the a priori bound

$$\sup_{B_\rho} |Dv| \leq \frac{c}{R_0^\beta} \left(\int_{B_R} H^{\frac{1}{2}} \left[\log(1+|Dv|) + \sigma H^{\frac{q-1}{2}} + 1\right] dx \right)^\beta$$
$$\leq c \left(\int_{B_R} (f_\sigma(Dv) + 1) dx \right)^\beta$$

with $\rho < \frac{R}{8}$, $c \equiv c(n, L, \nu, R)$ and $\beta \equiv \beta(n)$ but independent of σ and v.

Step 2: *Approximation.* Also, this time we follow the proof of Theorem 3.1. We define $v_{\varepsilon,\sigma} \in u_\varepsilon + W_0^{1,q}(B_R)$ as the unique solution to

$$\min \left\{ \int_{B_R} f_\sigma(Dw)\, dx : \ w \in u_\varepsilon + W_0^{1,q}(B_R) \right\}$$

with $B_R \subset\subset \Omega$ (as usual, we denote by u_ε a sequence of smooth functions obtained mollifying u). Let us define now $\sigma = \sigma(\varepsilon)$, v_ε and f_ε in a similar fashion to Theorem 3.1/Step 3. By the minimality of v_ε and the Jensen inequality we obtain

$$\begin{aligned} \sup_{B_\rho} |Dv_\varepsilon| &\le c\left(\int_{B_R} (f_\varepsilon(Dv_\varepsilon)+1)\,dx\right)^\beta \\ &\le c\left(\int_{B_R} f_\varepsilon(Du_\varepsilon)\,dx + 1\right)^\beta \\ &= c\left(\int_{B_R} f(Du_\varepsilon)\,dx + 1 + \sigma(\varepsilon)\int_{B_R}(1+|Du_\varepsilon|^2)^{\frac{q}{2}}dx\right)^\beta \\ &\le c\left(\int_{B_{R+\varepsilon}} f(Du)\,dx + 1 + o(\varepsilon)\right)^\beta . \end{aligned} \tag{4.4}$$

Then we have also, using $(2.2)_1$,

$$\Xi(Dv_\varepsilon)(B_R) \le c\int_{B_{R+\varepsilon}} f(Du)\,dx + 1 + o(\varepsilon)$$

and, as before, $v_\varepsilon \rightharpoonup w \in u + W_0^{1,1}(B_R)$ weakly in $W^{1,1}(B_R)$ and weakly* in $W^{1,\infty}(B_\rho)$. From these facts and the proof of Theorem 3.1 easily

$$\sup_{B_\rho} |Dw| \le c\left(\int_{B_R} f(Du)\,dx + 1\right)^\beta$$

follows where c depends only of n, L, ν and R. Finally, it can be proved that $w = u$ exactly as in Theorem 3.1 ∎

Also, this time the upper bound for Dv implies $C^{1,\alpha}$-regularity in the interior of Ω. Indeed, we are now able to prove Theorem 2.2.

Proof of Theorem 2.2. We consider again the approximating sequence v_ε constructed in Step 3 of the proof of Theorem 4.1 and keep the notations introduced there. Under our hypotheses, the local minimizer v_ε of $\mathcal{F}_\varepsilon$ is as regular as in (3.5) and, in particular (see again [1: Lemma 2.5]), turns out to be $W^{2,q}_{loc}(\Omega)$, and so, differentiating the Euler equation of $\mathcal{F}_\varepsilon$ in a way similar to the one in Theorem 3.1/Step 1, we obtain the second variation equation for $\mathcal{F}_\varepsilon$

$$\int_{B_\rho} D_{z_i z_j} f_\varepsilon(Dv_\varepsilon) D_{is} v_\varepsilon D_j\varphi\,dx = 0,$$

satisfied for any $\varphi \in C_0^\infty(B_\rho)$ and $s = 1,\dots,n$. We pick $M < \infty$ in such a way that $\sup_{B_\rho} |Dv_\varepsilon| \le M$ $(\rho \le \frac{R}{8})$ are satisfied for any $\varepsilon > 0$ (see (4.4)). With this notation we observe that, if we set $a_{ij}^\varepsilon(x) = D_{z_i z_j} f(Dv_\varepsilon(x))$ for $x \in B_\rho$, then the ellipticity and growth assumptions

$$\frac{\nu^{-1}}{1+M}|\lambda|^2 \le a_{ij}^\varepsilon(x)\lambda_i\lambda_j \le L|\lambda|^2$$

are satisfied for each $\lambda \in \mathbb{R}^n$. Note that also this time the ratio between the highest and the lowest eigenvalue of the matrix a^ε_{ij} is independent of ε and only depends on ν, L and M.Moreover, the coefficients a^ε_{ij} are bounded (independently of ε) and measurable and the function $D_s v_\varepsilon$ is a weak solution of the elliptic equation $-\mathrm{div}\,(a^\varepsilon Du) = 0$, for each $s \in \{1, \dots, n\}$. So by the De Giorgi-Nash-Moser Theorem we have that there exist two constants $\tilde{c} < \infty$ and $\alpha \in (0,1)$ depending of n, M, L and ν but not on ε, such that

$$\left|D_s v_\varepsilon(x) - D_s v_\varepsilon(y)\right| \le \tilde{c}\,|x-y|^\alpha \tag{4.5}$$

whenever $s = 1, \dots, n$, $\varepsilon > 0$ and $x, y \in B_{\frac{\varrho}{2}}$. Letting $\varepsilon \to 0$ in (4.5) as done in the proof of Theorem 2.2, we obtain $\left|D_s u(x) - D_s u(y)\right| \le \tilde{c}\,|x-y|^\alpha$ for each $s \in \{1, \dots, n\}$. The assertion of Theorem 4.2 follows by a standard covering argument ∎

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