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Recursion Formulae for $\sum_{m=1}^{n} m^k$

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Abstract. Using elementary approach and mathematical induction, several recursion formulae for $S_k(n) = \sum_{m=1}^{n} m^k$ are presented which show that $S_{k+1}(n)$ could be obtained from $S_k(n)$. A method and a formula of calculating Bernoulli numbers are proposed.

Keywords: *Recursion formulas, sum of powers, mathematical induction, Bernoulli numbers*

AMS subject classification: Primary 11 B 37, secondary 11 B 68, 11 B 83

1. Introduction

By definition and geometric meanings of the definite integral, it is well-known that the area under the curve $y = x^k$ over the closed interval $[0,1]$ equals

$$\lim_{n\to\infty} \sum_{m=1}^{n} \frac{1}{n}\left(\frac{m}{n}\right)^k = \lim_{n\to\infty} \frac{1}{n^{k+1}}\left(\sum_{m=1}^{n} m^k\right).$$

To complete the solution of this and many similar problems, it is then necessary to find the sums

$$S_k(n) = \sum_{m=1}^{n} m^k. \tag{1}$$

For small integer $k > 0$, the sums always appear in many calculus courses. For example,

$$S_7(n) = \frac{1}{24} n^2(n+1)^2(3n^4 + 6n^3 - n^2 - 4n + 2)$$

and the like [6: p. 11]. Such sums are usually proved by induction or derived from simple geometric pictures. For arbitrary k, unfortunately, the standard closed forms involve Bernoulli numbers or Stirling numbers of the second kind [4: p. 119], which come from reasonably complicated recurrence relations.

H. J. Schultz [10] derived a procedure for finding $S_k(n)$, k a positive integer, that is easy to remember, arises naturally, and can be used with very little background.

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However, he only illustrated the method by finding $S_6(n)$. According to [10], if one wants to compute, in general,

$$S_k(n) = A_{k+1}n^{k+1} + \ldots + A_1 n + A_0, \tag{2}$$

a system of $k+1$ equations

$$\sum_{i=j+1}^{k+1} (-1)^{i-j+1}\binom{i}{j}A_i = 0 \qquad (0 \le j \le k)$$

must be solved.

Let B_n be the n-th Bernoulli number defined in [6: p. 648] and [9: p. 632] by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \qquad (|x| \le 2\pi). \tag{3}$$

Then A_1 obtained from the formula for $S_k(n)$ is the k-th Bernoulli number B_k (for details see [11: p. 320]). It is noted that the concept of Bernoulli polynomial is generalized in [8] by the second author.

There are many inequalities related to the sum $S_\alpha(n) = \sum_{m=1}^{n} m^\alpha$, where α is an arbitrary real number. For instance,

$$n^{\alpha+1} < (\alpha+1)S_\alpha(n) < (n+1)^{\alpha+1} - 1$$

$$(\alpha+1)[S_\alpha(n) - 1] < n^{\alpha+1} - 1 < (\alpha+1)S_\alpha(n-1)$$

$$(n+1)^{\alpha+1} - n^{\alpha+1} < (\alpha+1)\big[S_\alpha(n) - S_\alpha(n-1)\big] < n^{\alpha+1} - (n-1)^{\alpha+1}$$

for $\alpha > 0$, $\alpha < -1$ and $-1 < \alpha < 0$, respectively. The proofs of these inequalities could be found in [7: pp. 84 - 85].

In [5, 12, 13] the relationships between Bernoulli numbers and the sum (1) were also studied using the Euler-Maclaurin formula and other devices. It is worth noting that a fascinating account of the early history of the problem above and standard recursion formulas for $S_k(n)$ as originally stated by Pascal are given in [3].

In this article, we prove that $S_k(n)$ is a $(k+1)$-th degree polynomial for n with constant term 0 (that is, formula (2) is valid) and

$$S_{k+1}(n) = (k+1)\left(\frac{A_{k+1}}{k+2}n^{k+2} + \frac{A_k}{k+1}n^{k+1} + \ldots + \frac{A_2}{3}n^3 + \frac{A_1}{2}n^2\right) + b_1 n \tag{4}$$

where

$$b_1 = \begin{cases} 0 & \text{for even } k > 0 \\ 1 - (k+1)\sum_{i=1}^{k+1}\frac{A_i}{i+1} & \text{for odd } k > 0. \end{cases}$$

Formula (4) shows that we can use the coefficients A_i $(1 \le i \le k+1)$ in $S_k(n)$ to get the expression of $S_{k+1}(n)$. In fact, it also gives a method of computing Bernoulli numbers B_{k+1}. At last, other formulae for calculating Bernoulli numbers and $\sum_{m=1}^{n} m^k$ are given.

2. Lemmas

To obtain our main results, the following lemmas are necessary. Moreover, these lemmas also give some recursion formulae for $S_k(n)$.

Lemma 1. *For any integers $k \geq 0$ and $n > 0$, we have*

$$(1+n)^{k+1} = 1 + \sum_{i=0}^{k} \binom{k+1}{i} S_i(n). \tag{5}$$

Proof. Recalling the binomial expansion $(1+m)^{k+1} = \sum_{i=0}^{k+1} \binom{k+1}{i} m^i$ we obtain

$$\begin{aligned}(1+n)^{k+1} + S_{k+1}(n) - 1 &= \sum_{m=1}^{n} (1+m)^{k+1} \\ &= \sum_{m=1}^{n} \left(\sum_{i=0}^{k+1} \binom{k+1}{i} m^i \right) \\ &= \sum_{i=0}^{k+1} \binom{k+1}{i} \left(\sum_{m=1}^{n} m^i \right) \\ &= \sum_{i=0}^{k+1} \binom{k+1}{i} S_i(n).\end{aligned}$$

This is equivalent to

$$(1+n)^{k+1} = 1 + \sum_{i=0}^{k} \binom{k+1}{i} S_i(n).$$

The proof of Lemma 1 is completed ∎

Lemma 1 shows that $S_k(n)$ could be deduced from $S_0(n), S_1(n), \ldots, S_{k-1}(n)$. Using Lemma 1 we can get

Lemma 2. *For arbitrary integer $k > 0$,*

$$S_k(n) = \frac{1}{k+1} n^{k+1} + \frac{1}{2} n^k + \sum_{i=1}^{k-1} A_i n^i. \tag{6}$$

Proof. By mathematical induction on k, the result that $S_k(n)$ is a $(k+1)$-th degree polynomial with constant term 0 follows straightforwardly. Equating the coefficients on the two sides of (5), it is deduced easily that the coefficients of n^{k+1} and n^k in $S_k(n)$ are $\frac{1}{k+1}$ and $\frac{1}{2}$, respectively. This completes the proof of Lemma 2 ∎

Since $S_k(1) = 1$, formula (6) implies

$$\sum_{i=1}^{k-1} A_i = 12 - \frac{1}{k+1}. \tag{7}$$

For any integer $k > 0$, let $\langle k \rangle$ stand for the largest odd number less than k. Then

$$k - \langle k \rangle = \begin{cases} 1 & \text{for any even } k \\ 2 & \text{for any odd } k. \end{cases}$$

For example, $\langle 2 \rangle = 1$, $\langle 5 \rangle = 3$, and so forth.

Let $A_p^{(q)}$ denote the coefficient of n^p in $S_q(n)$. Then

Lemma 3. *For any integer $k > 1$,*

$$S_k(n) = \frac{1}{k+1} n^{k+1} + \frac{1}{2} n^k + \frac{1}{2} \sum_{i=1}^{\frac{\langle k \rangle + 1}{2}} \frac{1}{i} \binom{k}{2i-1} A_1^{(2i)} n^{k-2i+1}, \tag{9}$$

that is,

$$A_{k-2i+1}^{(k)} = \frac{1}{2i} \binom{k}{2i-1} A_1^{(2i)} \qquad \left(1 \le i \le \tfrac{\langle k \rangle + 1}{2}\right) \tag{10}$$

where $A_1^{(2i)}$ is the coefficient of the term n in $S_{2i}(n)$.

Proof. We will use mathematical induction on k. It is clear that formula (9) is true for $k = 2$. Suppose the result is true for $3, \dots, k-1$. From Lemma 2, we have

$$S_k(n) = \frac{1}{k+1} n^{k+1} + \frac{1}{2} n^k + \sum_{i=1}^{k-1} A_{k-i}^{(k)} n^{k-i}.$$

Equating the coefficients of n^{k-j} for $j = 1, 3, \dots, \langle k \rangle$ in (5) gives us

$$\begin{aligned} A_{k-j}^{(k)} = \frac{1}{k+1} \bigg[& \frac{1}{2} \binom{k+1}{k-j} - \frac{1}{k-j} \binom{k+1}{k-j-1} \\ & - \sum_{i=0}^{\frac{j-3}{2}} A_{k-j}^{(k-j+2i+1)} \binom{k+1}{k-j+2i+1} \bigg]. \end{aligned} \tag{11}$$

By the inductive assumption, we have

$$A_{k-j}^{(k-j+2i+1)} = A_1^{(2(i+1))} \frac{1}{k-j+2(i+1)} \binom{k-j+2(i+1)}{2(i+1)} \tag{12}$$

for $0 \le i \le \frac{j-3}{2}$. Combining (11) and (12) yields

$$\begin{aligned} A_{k-j}^{(k)} = \frac{1}{k+1} \bigg[& \frac{1}{2} \binom{k+1}{k-j} - \frac{1}{k-j} \binom{k+1}{k-j-1} \\ & - \sum_{i=1}^{\frac{j-1}{2}} A_1^{(2i)} \frac{1}{k-j+2i} \binom{k-j+2i}{2i} \binom{k+1}{k-j+2i-1} \bigg]. \end{aligned} \tag{13}$$

From (7) and the inductive assumption, it follows that

$$A_1^{(j+1)} = \frac{1}{2} - \frac{1}{j+2} - \sum_{i=1}^{\frac{j-1}{2}} A_{2i+1}^{(j+1)} \tag{14}$$

and

$$A_{j-2i}^{(j+1)} = A_1^{(2(i+1))} \frac{1}{j+2} \binom{j+2}{2(i+1)} \qquad (0 \le i \le \tfrac{j-3}{2}). \tag{15}$$

Substituting (15) into (14) produces

$$\begin{aligned} A_1^{(j+1)} \frac{1}{k+1}\binom{k+1}{j+1} &= \frac{1}{k+1}\left[\frac{1}{2}\binom{k+1}{j+1} - \frac{1}{j+2}\binom{k+1}{j+1}\right. \\ &\quad \left. - \sum_{i=1}^{\frac{j-1}{2}} A_1^{2i} \frac{1}{j+2}\binom{j+2}{2i}\binom{k+1}{j+1}\right] \\ &= \frac{1}{k+1}\left[\frac{1}{2}\binom{k+1}{k-j} - \frac{1}{k-j}\binom{k+1}{k-j-1}\right. \\ &\quad \left. - \sum_{i=1}^{\frac{j-1}{2}} A_1^{2i} \frac{1}{k-j+2i}\binom{k-j+2i}{2i}\binom{k+1}{k-j+2i-1}\right]. \end{aligned} \tag{16}$$

From (13) and (16),

$$A_{k-j}^{(k)} = \frac{1}{k+1}\binom{k+1}{j+1} A_1^{(j+1)} \qquad (j = 1, 3, \ldots, \langle k \rangle)$$

is obtained. Similarly, by mathematical induction, we can prove that

$$A_{k-i}^{(k)} = 0 \qquad (i = 2, 4, 6, \ldots, \langle k \rangle + 1).$$

The proof of Lemma 3 is completed ∎

Note Lemma 3 shows that the coefficients of the term n in $S_2(n), \ldots, S_{2i-2}(n)$ can be used to calculate $S_{2i-1}(n)$ and $S_{2i}(n)$.

3. Main results

Now we use Lemma 3 to prove

Main Theorem. *For any integer $k > 1$, let*

$$S_k(n) = \frac{1}{k+1} n^{k+1} + \frac{1}{2} n^k + \sum_{i=1}^{\frac{\langle k \rangle + 1}{2}} A_{k-2i+1} n^{k-2i+1}.$$

Then

$$S_{k+1}(n) = \frac{1}{k+2} n^{k+2} + \frac{1}{2} n^{k+1} + (k+1) \sum_{i=1}^{\frac{\langle k\rangle+1}{2}} \frac{A_{k-2i+1}}{k-2(i-1)} n^{k-2(i-1)} + b_1 n$$

where

$$b_1 = \begin{cases} 0 & \text{for even } k \\ \frac{1}{2} - \left[\frac{1}{k+2} + (k+1)\sum_{i=1}^{\frac{k-1}{2}} \frac{A_{k-2i+1}}{k-2i+2}\right] & \text{for odd } k. \end{cases} \tag{17}$$

Proof. From (10) we know that the coefficients of n^{k-j} $(j = 1, 3, \dots, \langle k\rangle)$ in $S_k(n)$ are

$$A_{k-j}^{(k)} = \frac{1}{k+1}\binom{k+1}{j+1} A_1^{(j+1)}.$$

Therefore

$$\begin{aligned} A_{k-j}^{(k)} \frac{k+1}{k-j+1} &= A_1^{(j+1)} \frac{1}{k+1}\binom{k+1}{j+1}\frac{k+1}{k-j+1} \\ &= A_1^{(j+1)} \frac{1}{k+2}\binom{k+2}{j+1} \\ &= A_{k-j+1}^{(k+1)} \end{aligned}$$

is the coefficient of n^{k+1-j} $(j = 1, 3, \dots, \langle k\rangle)$ in $S_{k+1}(n)$. If k is even, since $k - \langle k\rangle + 1 = (k+1) - \langle k+1\rangle$, then $b_1 = 0$ follows from (9). If k is odd, formula (17) follows from (7). This completes the proof ∎

Corollary. *Let A_i be the coefficients of the terms n^i $(1 \le i \le k+1)$ in $S_k(n)$ and let B_i $(i > 1)$ be the i-th Bernoulli numbers. Then*

$$B_{2j+1} = 0$$

$$B_{2j} = \frac{1}{2} - \left[\frac{1}{2j+1} + 2j\sum_{i=1}^{j-1} \frac{A_{2(j-i)}}{2(j-i)+1}\right]$$

for every integer $j \ge 1$,

Remark. By Lemmas 1 - 3 and Main Theorem, calculating directly we obtain

$$\begin{aligned} S_{10}(n) &= \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - n^7 + n^5 - \frac{1}{2}n^3 + \frac{5}{66}n \\ S_{11}(n) &= \frac{1}{12}n^{12} + \frac{1}{2}n^{11} + \frac{11}{12}n^{10} - \frac{11}{8}n^8 + \frac{11}{6}n^6 - \frac{11}{8}n^4 + \frac{5}{12}n^2 \\ S_{12}(n) &= \frac{1}{13}n^{13} + \frac{1}{2}n^{12} + n^{11} - \frac{11}{6}n^9 + \frac{22}{7}n^7 - \frac{33}{10}n^5 + \frac{5}{3}n^3 - \frac{691}{2730}n \\ S_{20}(n) &= \frac{1}{21}n^{21} + \frac{1}{2}n^{20} + \frac{5}{3}n^{19} - \frac{19}{2}n^{17} + \frac{1292}{21}n^{15} - 323n^{13} + \frac{41990}{33}n^{11} \\ &\quad - \frac{223193}{63}n^9 + 6460n^7 - \frac{68723}{10}n^5 + \frac{219335}{63}n^3 - \frac{174611}{330}n \\ S_{21}(n) &= \frac{1}{22}n^{22} + \frac{1}{2}n^{21} + \frac{7}{4}n^{20} - \frac{133}{12}n^{18} + \frac{323}{4}n^{16} - \frac{969}{2}n^{14} + \frac{146965}{66}n^{12} \\ &\quad - \frac{223193}{30}n^{10} + \frac{33915}{2}n^8 - \frac{481061}{20}n^6 + \frac{219335}{12}n^4 - \frac{1222277}{220}n^2. \end{aligned}$$

From here the Bernoulli numbers

$$B_{10} = 566, \qquad B_{12} = -\frac{691}{2730}, \qquad B_{20} = -\frac{174611}{330}$$

are obtained.

4. Another formulae for $\sum_{m=1}^{n} m^k$ and Bernoulli numbers

In this section, another formulae for computing Bernoulli numbers and $\sum_{m=1}^{n} m^k$ will be given, from which we can get the Bernoulli numbers more easily (see [1] and [2: pp. 246 - 265]).

Define functions B_n by

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n \qquad (|z| < 2\pi)$$

and write $B_n = B_n(0)$ for the Bernoulli numbers. Then formula (3) follows by putting $x = 0$. We can equate coefficients of z^n in

$$\sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n = \frac{z}{e^z - 1} \cdot e^{xz} = \left(\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \right) \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} z^n \right)$$

to get

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}. \tag{18}$$

Also, since

$$\frac{z\, e^{(x+1)z}}{e^z - 1} - \frac{z\, e^{xz}}{e^z - 1} = z\, e^{xz},$$

we have

$$\sum_{n=0}^{\infty} \frac{B_n(x+1) - B_n(x)}{n!} z^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} z^{n+1},$$

and by equating coefficients of z^n we get

$$B_n(x+1) - B_n(x) = n\, x^{n-1}. \tag{19}$$

So putting $x = 0$ we have

$$B_n = B_n(0) = B_n(1) \qquad (n \neq 1). \tag{20}$$

Thus for $n \geq 2$ we can put $x = 1$ in (18) and use (20) to obtain

$$B_n = B_n(1) = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

This is a much simpler recursion formula for computing Bernoulli numbers.

Result (19) can be used, taking $x = 1, 2, \ldots, k-1, k$ and adding, to give

$$\begin{aligned} B_n(k+1) - B_n(1) &= \sum_{i=0}^{k-1} \left[B_n(k+1-i) - B_n(k-i)\right] \\ &= n \cdot k^{n-1} + n(k-1)^{n-1} + \ldots + n \cdot 2^{n-1} + n \cdot 1^{n-1} \\ &= n \sum_{m=1}^{k} m^{k-1}, \end{aligned}$$

that is,

$$\sum_{m=1}^{k} m^{k-1} = \frac{B_n(k+1) - B_n}{n}.$$

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