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Blow-Up in Exterior Domains: Existence and Star-Shapedness

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Abstract. For some nonlinear elliptic equations in divergence form, we consider the solutions, defined in the exterior of a contractible bounded domain $\mathcal{G}$, which become infinite at the boundary. We prove the existence of such solutions and study the shape of their level sets when $\mathcal{G}$ is star-shaped.

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1. Introduction

This paper deals with solutions of the elliptic equation

$$\operatorname{div}\big(g(|\nabla u|)\nabla u\big) = f(u)\,k(|\nabla u|) \tag{1.1}$$

defined in a domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, satisfying the following boundary condition:

$$\lim_{x\to\partial\Omega} u(x) = +\infty. \tag{1.2}$$

Such solutions are called *"explosive solutions"*, or *"blow-up solutions"*, or even *"large solutions"*. An existence result for the special case $\Delta u = f(u)$ was obtained by Keller [11] and Osserman [18]. The behaviour of large solutions near $\partial\Omega$ was then studied by Bandle and Marcus [3], Véron [19], Lazer and McKenna [13], to which we also refer for further historical details. The research in this field is still blowing up in different directions, such as the extension of known results to more general equations, as well as to non-smooth domains.

In a recent paper [2] the problem of existence, uniqueness, asymptotic behaviour and convexity of solutions to problem (1.1) - (1.2) in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, has been considered. In the case $N = 1$, existence of such solutions may be obtained directly by integration. In the case $N > 1$, an existence result was achieved under certain monotonicity assumptions on the function k and in domains whose boundary has positive mean curvature.

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Under the following set of hypotheses:

$$f \in C^1((t_0,+\infty),\mathbb{R}^+),\ \ f(t_0^+)=0,\ \ f(+\infty)=+\infty,\ \ f'\geq 0,\ \ f'(t_0^+)<+\infty \tag{1.3}$$

$$g \in C^2([0,+\infty),\mathbb{R}^+), \quad G(t) := \big(g(t)\,t\big)' > 0 \quad \text{for all } t \geq 0 \tag{1.4}$$

$$k \in C^1([0,+\infty),\mathbb{R}^+), \tag{1.5}$$

we obtain two main results:

(1) We prove existence in smooth bounded domains, as well as in exterior domains, under natural assumptions.
(2) We prove that certain solutions in an exterior star-shaped domain have star-shaped level sets.

(1) We consider g and k asymptotically of power type: more precisely, we assume that there exist a $\tau > -1$ and a $q \leq \tau + 2$ such that $G(t) \sim t^\tau$ and $k(t) \sim t^q$ as $t \to +\infty$, i.e., for t larger than a suitable t_1 the following relations hold:

$$\begin{cases} C_1 < \dfrac{G(t)}{t^\tau} < C_2 \\ C_1 < \dfrac{k(t)}{t^q} < C_2 \end{cases} \tag{1.6}$$

where C_1 and C_2 are positive constants. Recall that g and G are the eigenvalues of the characteristic matrix associated to equation (1.1). By the definition of G, assumption $G(t) \sim t^\tau$, $\tau > -1$, implies $g(t) \sim t^\tau$ and $g'(t) \sim t^{\tau-1}$. This and the strict positivity of $g(t)$ and $G(t)$ near $t = 0$ imply that equation (1.1) is uniformly elliptic with respect to any solution. Condition $q \leq \tau + 2$ may be regarded as *natural* for existence of classical solutions.

Let $F(t) := \int_{t_0}^t f$ and $H(t) := \int_0^t G(|s|)\, s \,/\, k(|s|)\, ds$. Note that if (1.6) holds, then H is unbounded. By assuming (1.3) - (1.6) we show that condition

$$\int^{+\infty} \frac{1}{H^{-1}(F(t))}\, dt < +\infty \tag{1.7}$$

is necessary and sufficient in order to have existence of solutions exploding at the boundary. Condition (1.7) appeared in this form in [2], (C-1), and reduces to (2) of [11] in the special case $g = k \equiv 1$. Following [2], we refer to (1.7) as to the *generalized Keller condition*. We shall remark that (1.7) follows from (1.3) and (1.6) in the case $q = \tau + 2$. We point out that also the positivity of $k(0)$ is necessary for existence if Ω is bounded, and that f must be unbounded in order to solve the problem in arbitrarily small domains (see Subsection 2.4).

(2) Additional assumptions are usually required to get qualitative properties of solutions of elliptic equations. As for star-shapedness, we distinguish two cases. In the case $N = 2$ we show that if $t^2\, G(t) \,/\, k(t)$ is non-decreasing, then the least solution in the exterior of a star-shaped domain has star-shaped level sets. The result also applies to classical solutions of non-uniformly elliptic equations, including the minimal surface equation.

In the semilinear case ($G \equiv 1$) star-shapedness follows for every $N \geq 2$ and for every solution, not only the least one. Note that uniqueness does not follow trivially by the maximum principle since u is infinite at the boundary.

It is remarkable that if $G(t) \sim t^{\tau}$, $\tau > -1$, and $k(t) \sim t^{q}$, then the monotonicity of $t^2 G(t) / k(t)$ implies $q \leq \tau + 2$. This and (1.7) imply existence.

2. Existence

2.1 Method and main results. In the case $N = 1$ an exhaustive study of existence may be carried out by integration. To prove the existence of an explosive solution for $N \geq 2$, instead, one usually starts with a solution u_m taking a constant (and finite) value m at the boundary, and then one lets m tend to $+\infty$. By the comparison principle, u_m is increasing in m and therefore it has a limit u. This u is candidate to solve the problem.

In order that this procedure works, one has to perform the following three steps:

i) Prove the existence of a solution u_m satisfying $u_m|_{\partial\Omega} \equiv m$.
ii) Prove that the sequence $\{u_m\}$ is locally bounded in Ω.
iii) Find an interior bound for the gradient ∇u_m that forces u to be still a solution.

The main contribution of this section concerns the second step. The first and the last ones follow by classical results. After [3] and [19], several attempts have been done to extend results to equations with non-linearities depending on the gradient of the solution. In [2] equations of the form (1.1) are considered, and some special monotonicity assumptions on k are made in order to prove existence in domains whose boundary has positive mean curvature.

The present approach differs from the previous ones since we do not try to estimate the first-order term in equation (2.1), namely, the term $(N-1)\,g(|u'|)\,u'/r$, by means of the quantity $f(u)\,k(|u'|)$. We prefer to argue by contradiction when proving the existence of radially symmetric explosive solutions. This method has the advantage to extend the existence result known for $\Delta u = f(u)$ to uniformly elliptic equations of the form (1.1), at least in the case G and k are of power growth, without monotonicity assumptions on k.

A contradiction argument was also used for equations of prescribed mean curvature in [8]. In that case, it turns out that the negligible term in (2.1) is the one of the second order.

Once we have proved the existence of radially symmetric explosive solutions in spheres of arbitrarily small radius we use them, as usual, to get an interior bound for u_m via the comparison principle. In doing this we suppose Ω smooth; see [16] for existence in fractal domains.

We also show that every solution in an exterior domain approaches t_0 as $|x|$ tends to $+\infty$. This observation was made in [3] for the equation $\Delta u = f(u)$.

Finally, we prove the necessity of some conditions in order that an explosive solution exists: 1) assumption (1.7) is necessary; 2) the assumption that $k(0) > 0$ is necessary if Ω is bounded; 3) the function f must be unbounded if we want Ω to be arbitrarily small.

2.2 Existence in small spheres. We present an existence theorem for problem (1.1) - (1.2) in the sphere. To obtain the result, we look for a radially symmetric solution $u = u(|x|)$ and therefore we are led to an ordinary differential equation. It turns out that if the radius of the sphere is sufficiently small, then a solution exists. Later on we shall use the explosive solutions in small spheres compactly contained in a general domain Ω, and the comparison principle, to bound from above the solutions of equation (1.1) defined in all of Ω and continuous up to $\partial\Omega$.

Theorem 2.1. *Assume* (1.3) - (1.7). *There exists a radially symmetric explosive solution of problem* (1.1) - (1.2) *in every sphere of sufficiently small radius.*

Proof. First we prove that the solution $u(r)$ of the initial-value problem

$$\begin{cases} G(|u'|)\,u'' + \frac{N-1}{r}\,g(|u'|)\,u' = f(u)\,k(|u'|) \\ u(0) = u_0 > t_0 \\ u'(0) = 0 \end{cases} \tag{2.1}$$

becomes infinite at a certain (finite) R, then we prove that R tends to zero as $u_0 \to +\infty$.

From the equality $N\,g(0)\,u''(0) = f(u_0)\,k(0)$ and since $g(0), f(u_0), k(0) > 0$ we have $u''(0) > 0$. It is readily seen that $u'(r), u''(r) > 0$ for all $r > 0$. Indeed, if $u^{(k)}$ vanishes at a certain $\bar{r}$ for $k = 1$ or $k = 2$, then the equation implies $u^{(k+1)}(\bar{r}) > 0$, a contradiction.

Classical gradient estimates, which hold by virtue of the structure conditions (1.6), prevent u' from becoming infinite at a finite R if $u(R)$ is also finite (see, for instance, Theorem 14.1 of [6] and compare (1.6) with (14.9) there; alternatively, a direct proof may be obtained by contradiction). Hence we have only to exclude the case that u is entire, i.e., the case that $u(r)$ exists for all $r \in \mathbb{R}$. We shall make use of the following equality:

$$H(u'(r)) - H(u'(r_1)) + \int_{r_1}^{r} \frac{N-1}{s}\,\frac{g(|u'|)\,(u')^2}{k(|u'|)}\,ds = F(u(r)) - F(u(r_1)), \tag{2.2}$$

which is obtained by multiplying the equation by $u'/k(|u'|)$ and then integrating from an r_1 to r.

We proceed by contradiction. Assume that there exists an entire solution of (2.1). In this case, it is easy to check that $u \to +\infty$ and $u' \to +\infty$ as $r \to +\infty$. The core of the proof is based on the fact that conditions (1.6) lead to the following estimates:

$$\begin{cases} H(t) \sim \begin{cases} t^{\tau-q+2} & \text{if } q < \tau + 2 \\ \log t & \text{if } q = \tau + 2 \end{cases} \\ g(t)\,t^2/k(t) \sim t^{\tau-q+2}. \end{cases} \tag{2.3}$$

The last one implies that

$$\int_{r_1}^{r} \frac{1}{s}\,\frac{g(u')(u')^2}{k(u')}\,ds \sim \begin{cases} \int_{r_1}^{r} \frac{1}{s}(u')^{\tau-q+2}\,ds < C\,(u'(r))^{\tau-q+2}\log r & \text{if } q < \tau + 2 \\ \log r & \text{if } q = \tau + 2. \end{cases}$$

By substituting these estimates into (2.2) we find, for large r,

$$F(u) < \begin{cases} C\,(u')^{\tau-q+2}\log r & \text{if } q < \tau+2 \\ C\,\log(u'r) & \text{if } q = \tau+2, \end{cases}$$

which in turn imply

$$\int^{+\infty} \frac{u'\,dr}{\big(F(u(r))\big)^{\frac{1}{\tau-q+2}}} > \epsilon \int^{+\infty} \frac{dr}{(\log r)^{\frac{1}{\tau-q+2}}} = +\infty \quad \text{if } q < \tau+2,$$

$$\int^{+\infty} \frac{u'\,dr}{e^{\epsilon\,F(u(r))}} > \int^{+\infty} \frac{dr}{r} = +\infty \quad \text{if } q = \tau+2.$$

Since all integrals in the left-hand side of the above inequalities are finite by (1.7) (see also Remark 2.2) we have a contradiction. This shows that problem (2.1) cannot have an entire solution, hence $u(r) \to +\infty$ as r approaches a finite R.

Now we prove that $R = R(u_0)$ tends to 0 as u_0 tends to $+\infty$.

1) By uniqueness, u is strictly increasing with respect to u_0, hence R is non-increasing. Suppose that $\overline{R} := \lim R(u_0) > 0$.

2) Observe that u' is non-decreasing in u_0 (hint: the function $v(x) := u(|x|) + C$ is a supersolution of (1.1) for every positive C).

3) Observe, further, that $u'(r)$ becomes infinite at each $r > 0$ as $u_0 \to +\infty$: indeed, if $u'(r) < M$, since $u \to +\infty$ and since $f(+\infty) = +\infty$ the equation implies $u'' \to +\infty$ uniformly in the interval $[r/2, r]$, a contradiction.

4) Choose $0 < r_1 < r_2 < \overline{R}$. Assume, for simplicity, $q < \tau + 2$. Since $u' \to +\infty$ as $u_0 \to +\infty$, we deduce by (1.6) and (2.2) that $F(u(r)) - F(u(r_1)) < C\,(u'(r))^{\tau-q+2}$ for $r > r_1$ and large u_0. This implies that

$$\int_0^{+\infty} \frac{dt}{\big(F(t+u(r_2)) - F(u(r_1))\big)^{\frac{1}{\tau-q+2}}} = \int_{u(r_2)}^{+\infty} \frac{dt}{\big(F(t) - F(u(r_1))\big)^{\frac{1}{\tau-q+2}}} > \epsilon\,(\overline{R} - r_2),$$

where the integral is finite by (1.7). By 2) and 3) we see that $u(r_2) - u(r_1) = \int_{r_1}^{r_2} u'\,dr \nearrow +\infty$ as $u_0 \to +\infty$. By the Monotone Convergence Theorem and the convexity of F, this implies that the above integral vanishes as $u_0 \to +\infty$. We reach a contradiction that proves that $\overline{R}$ must be zero. The conclusion is similar in the case $q = \tau + 2$ ∎

Remark 2.2. If $q = \tau + 2$ in (1.6), then we see from (2.3) that $e^{\epsilon_1 t} < H^{-1}(t) < e^{\epsilon_2 t}$ for t large. Condition (1.7) then follows from (1.3) by observing that $F(u) > \sigma\,u$ with a positive σ.

2.3 Existence in general domains. The upper bound provided by Theorem 2.1 allows us to prove the existence of a solution to (1.1) - (1.2) when Ω is a bounded domain or *the exterior* of a contractible bounded domain. We give details for the last case. The following result is classical:

Lemma 2.3. *Assume* (1.3) - (1.6). *Let $\mathcal{G} \subset \mathbb{R}^N$ be a contractible bounded $C^{2,\alpha}$ domain, and let R be so large that $\mathcal{G} \subset\subset B(0,R)$. Define $\Omega_R = B(0,R) \setminus \overline{\mathcal{G}}$. For every $m > t_0$ there exists a (unique) solution $u_R \in C^{2,\alpha}(\Omega_R)$ of equation* (1.1) *satisfying $t_0 < u_R < m$ in Ω_R, $u_R \equiv m$ on $\partial\mathcal{G}$, $u_R(x) \equiv t_0$ for $|x| = R$.*

For reader's convenience we recall the *structure of the proof:* the result follows from the Leray-Schauder fixed point theorem after an *a priori* estimate of u, which follows directly from the maximum principle, and an *a priori* estimate of ∇u, which can be obtained via *barrier functions* by virtue of the assumptions $G(t) \sim t^\tau$ and $k(t) = O(t^{\tau+2})$, $\tau > -1$ (see [6: Theorems 14.1 and 15.1]) ∎

Note that since the equation is invariant under rotations, and by uniqueness, if $\mathcal{G}$ is a ball centered at the origin, then the corresponding u_R is radially symmetric.

Now let $R \to +\infty$. By the comparison principle, u_R is increasing in R and bounded from above by m. This and the structure conditions (1.6) provide a gradient bound in compact sets [14: Chapter IV, Théorème 3.1], which in turn implies a Hölder estimate for the gradient. Finally, the Schauder estimate and Arzelà's theorem guarantee the existence of a sequence u_{R_n} converging to a solution u_m of (1.1) in Ω, uniformly on compact sets together with its first and second derivatives:

Lemma 2.4 *Assume* (1.3) - (1.6), *let $\mathcal{G}$ be as above and define $\Omega = \mathbb{R}^N \setminus \overline{\mathcal{G}}$. For every $m > t_0$ there exists a solution $u_m \in C^{2,\alpha}(\Omega)$ of equation* (1.1) *satisfying $t_0 < u_m < m$ in Ω, $u_m|_{\partial\Omega} \equiv m$.*

The last step consists in letting $m \to +\infty$. By using the solutions in small spheres compactly contained in Ω, which exist by Theorem 2.1, as an upper bound, we obtain a solution to (1.1) satisfying (1.2). The procedure to prove convergence is the same as before:

Theorem 2.5. *Let Ω be a (the exterior of a contractible) bounded $C^{2,\alpha}$ domain. If* (1.3) - (1.7) *hold, then there exists a solution $u \in C^{2,\alpha}(\Omega)$, $u > t_0$, of problem* (1.1) - (1.2). *In the case Ω is an exterior domain, every solution to the same problem satisfies $u(x) \to t_0$ as $|x| \to +\infty$.*

To prove the *second claim* of the theorem we argue as follows. Assume that there exists such a solution u. For every $\rho > 0$ and for sufficiently large R there exists a radially symmetric solution v_R in the annulus $B(0,R) \setminus \overline{B}(0,\rho + 1/R)$ satisfying $v_R(\rho + 1/R) = v_R(R) = +\infty$. As $R \to +\infty$, v_R decreases and is bounded from below by the solution v in the exterior of $B(0,\rho)$ whose existence follows by the first claim. Hence v_R tends to a solution of the same equation in the exterior of $B(0,\rho)$ exploding at ρ, which we shall denote by $\overline{v}$ (since both v and $\overline{v}$ are infinite at ρ, we may not conclude trivially that $v = \overline{v}$).

Since $\overline{v}$ is radial and $k(0) > 0$, by the same argument as in the proof of Theorem 2.1 we can conclude that $\overline{v}(r)$ is convex in r. Moreover, $\overline{v}(r)$ must be decreasing for r close to ρ. If there were an r_0 at which $\overline{v}'(r_0) = 0$, then $\overline{v}$ would become infinite at a certain finite radius (see the proof of Theorem 2.1) but this is not possible since $\overline{v}$ is defined in the whole exterior of $B(0,\rho)$. Hence $\overline{v}'(r) < 0$ for all $r > \rho$, and passing to the limit in the equation in (2.1) one can easily see that $\overline{v}(r) \to t_0$ as $r \to +\infty$.

If we choose ρ so large that $\partial\Omega \subset B(0,\rho)$, then we have $u \leq v_R$ for every R, hence $u \leq \overline{v}$. The second claim follows and the proof is complete ∎

2.4 Necessary conditions. In this subsection we point out that some of the hypotheses of this paper are necessary for the existence of an explosive solution. Namely, we deal with the positivity of $k(0)$, the generalized Keller condition (1.7), and the unboundedness of the function f in the equation.

Proposition 2.6. *Assume* (1.4) *and let* $f(u)\,k(t)$ *be locally Lipschitz continuous in* $[t_0, +\infty) \times [0, +\infty)$ *and non-decreasing with respect to* u. *If there exists a solution* $u \geq t_0$ *of problem* (1.1) - (1.2) *in a bounded domain* Ω, *then* $k(0) > 0$.

Proof. If we had $k(0) = 0$, then every constant larger than t_0 would be a solution of (1.1); in particular, choosing a constant $C > \min u$ we contradict uniqueness in the set $\{ x \in \Omega \mid u(x) < C \}$ ∎

Theorem 2.7. *Assume* (1.3) - (1.6). *Hypothesis* (1.7) *is necessary for the existence of a solution to problem* (1.1) - (1.2).

Proof. Assume there exists such a solution, say v, in Ω. Let us first consider Ω bounded. Suppose, without loss of generality, that $0 \in \Omega$. Choose $u_0 > v(0)$. We claim that the solution u of (2.1) is explosive at an $R < +\infty$. As mentioned before, by (1.6) it suffices to exclude the case that u is entire. If this were the case, since v is infinite at $\partial\Omega$ and since $u(|x|)$ would be finite there, by comparison we would have $v(0) \geq u(0) = u_0$, a contradiction. By (2.2) with $r_1 = 0$ we have $H(u') < F(u)$, hence $u'/H^{-1}(F(u)) < 1$. By integration from 0 to R we find $\int_{u_0}^{+\infty} 1/H^{-1}(F(t)) < R$. Thus (1.7) is satisfied.

Consider now a contractible bounded domain $\mathcal{G}$ containing the origin and let $\Omega = \mathbb{R}^N \setminus \overline{\mathcal{G}}$. By Remark 2.2 we may assume that $q < \tau + 2$. Let R be the largest positive number such that $B(0, R) \subset \mathcal{G}$. If there exists an explosive solution v in Ω, then we may use this v as an upper bound for the solution u_m in the exterior of $B(0, R)$ constructed in Subsection 2.3. Letting $m \to +\infty$ we obtain an explosive solution $u = u(r)$ satisfying the equation in (2.1) and such that $u(R^+) = +\infty$. We observe now that the integral in (2.2) is a $o(H(u_1'))$ as $r_1 \to R^+$. This follows from the estimates (2.3) and the well-known property of an integral function to be an infinite of lower order with respect to the function under the sign of integral (see [13: Lemma 2.1] for a proof). Hence, (2.2) implies that $H(u'(r_1)) \sim F(u(r_1))$ as $r_1 \to R^+$. Taking into account estimate (2.3), inequality (1.7) follows ∎

Proposition 2.8. *Let* $f \in C^1((t_0, +\infty), \mathbb{R}^+)$, $f' \geq 0$, *and assume* (1.4) - (1.6). *If, for arbitrarily small* R, *there exists a solution of problem* (2.1) *in the interval* $[0, R)$ *satisfying* $u(R) = +\infty$, *then* $f(+\infty) = +\infty$.

Proof. Taking $r_1 = 0$ in (2.2) we get $H(u') < F(u) - F(u_0)$. By (1.4) - (1.6) the function H is increasing and unbounded, hence we obtain

$$\int_0^{+\infty} \frac{dt}{H^{-1}(F(t + u_0) - F(u_0))} < R.$$

By the mean value theorem we have $F(t + u_0) - F(u_0) = t\, f(\xi)$. If we assume, contrary to the claim, that $f \leq M$, then we find

$$\int_0^{+\infty} \frac{dt}{H^{-1}(M\, t)} < R,$$

hence R cannot be arbitrarily small ∎

3. Star-shapedness

We investigate the star-shapedness of the level sets of large solutions in exterior domains. For simplicity, we say that a domain $\mathcal{G}$ is *star-shaped* when it is star-shaped with respect to the origin, i.e., $\lambda x \in \mathcal{G}$ for all $x \in \mathcal{G}$ and all $\lambda \in [0,1]$. In the sequel we denote by $\mathcal{G}$ a star-shaped bounded domain in $\mathbb{R}^N$, and we let $\Omega = \mathbb{R}^N \setminus \overline{\mathcal{G}}$.

Star-shapedness of level sets, together with convexity, was considered by Díaz and Kawohl [4] and by Acker [1] for solutions of some elliptic equations in a bounded star-shaped ring with finite Dirichlet data. Longinetti [15] proved a maximum principle for star-shapedness for solutions of Poisson equation by considering the angle between the level surfaces and the radial direction. This result was generalized by the first author [5].

Additional assumptions and suitable estimates of the boundary behaviour are usually required to get uniqueness of large solutions and their qualitative properties such as radial symmetry [17] and convexity in a convex domain [9]. If $N = 1$, star-shapedness follows trivially from $u'' \geq 0$. In the case $N = 2$ we show that if $t^2\, G(t) \,/\, k(t)$ is non-decreasing, then the least solution in the exterior of a star-shaped domain has star-shaped level sets. In the semilinear case ($G \equiv 1$) the result holds for every $N \geq 2$ and for every solution, not only the least one.

It is remarkable that if $G(t) \sim t^\tau$, $\tau > -1$, and $k(t) \sim t^q$, then the monotonicity of $t^2\, G(t) \,/\, k(t)$ implies $q \leq \tau + 2$. This and (1.7) imply existence.

3.1 Two-dimensional case. The proof of the next theorem is based on the construction of a suitable differential equation in a *three-dimensional domain*. The idea of searching auxiliary equations in a domain of dimension larger than Ω was successfully used by Korevaar [12] and by Porru and the second author [10] to obtain convexity results. Of course, difficulty arises since one has to define a new elliptic operator Q able to provide the desired result via the maximum principle.

Theorem 3.1. *Assume* (1.4), $N = 2$, *and let* $f(\rho, u, t)$ *be of class* C^1 *in* $\mathbb{R}^+ \times \mathbb{R} \times [0, +\infty)$ *and satisfying* $f, f_\rho, f_u \geq 0$,

$$\frac{f(\rho,u,t)}{t^2\, G(t)} \quad \text{non-increasing in } t \text{ for } \rho > 0,\ u > t_0,\ t > 0. \tag{3.1}$$

If u_m *is a solution of* $\operatorname{div}(g(|\nabla u|)\, \nabla u) = f(|x|, u, |\nabla u|)$ *in the exterior domain* Ω *satisfying* $u_m > t_0$ *in* Ω, $u_m|_{\partial\Omega} \equiv m > t_0$, $u_m(x) \to t_0$ *as* $|x| \to +\infty$, *then for every* $c \in \mathbb{R}$ *the set* $\overline{\mathcal{G}} \cup \{\, x \mid u_m(x) > c \,\}$ *is star-shaped.*

Proof. We introduce polar coordinates (ρ, θ) such that $x = \rho\cos\theta$, $y = \rho\sin\theta$. The characteristic matrix of the operator $Qu = \operatorname{div}(g(|\nabla u|)\, \nabla u)$ with respect to the polar coordinates is the following:

$$\begin{pmatrix} g + \frac{1}{|\nabla u|}\, g'\, u_\rho^2 & \frac{1}{|\nabla u|}\, g'\, \frac{1}{\rho^2}\, u_\rho\, u_\theta \\ \frac{1}{|\nabla u|}\, g'\, \frac{1}{\rho^2}\, u_\rho\, u_\theta & \frac{1}{\rho^2}\, g + \frac{1}{|\nabla u|}\, g'\, \frac{1}{\rho^4}\, u_\theta^2 \end{pmatrix}.$$

The first-order part of Q is given by $g\, u_\rho/\rho - g'\, u_\rho\, u_\theta^2/(\rho^3\, |\nabla u|)$.

Let $S := \{(\rho_1,\rho_2,\theta) \mid (\rho_1,\theta),(\rho_2,\theta) \in \Omega,\ \rho_1 < \rho_2\} \subset \mathbb{R}^3$, and denote by $\mathcal{Q}$ the quasilinear operator in $C^2(S)$ having the following characteristic matrix:

$$\begin{pmatrix} \left(g(q_1) + \frac{g'(q_1)\,v_1^2}{q_1}\right)\frac{\rho_1^2}{G(q_1)} & 0 & \left(\frac{g'(q_1)\,v_1\,v_\theta}{\rho_1^2\,q_1}\right)\frac{\rho_1^2}{G(q_1)} \\ 0 & \left(g(q_2) + \frac{g'(q_2)\,v_2^2}{q_2}\right)\frac{\rho_2^2}{G(q_2)} & \left(\frac{g'(q_2)\,v_2\,v_\theta}{\rho_2^2\,q_2}\right)\frac{\rho_2^2}{G(q_2)} \\ \left(\frac{g'(q_1)\,v_1\,v_\theta}{\rho_1^2\,q_1}\right)\frac{\rho_1^2}{G(q_1)} & \left(\frac{g'(q_2)\,v_2\,v_\theta}{\rho_2^2\,q_2}\right)\frac{\rho_2^2}{G(q_2)} & 1 - \frac{g'(q_1)\,v_1^2}{q_1\,G(q_1)} - \frac{g'(q_2)\,v_2^2}{q_2\,G(q_2)} \end{pmatrix},$$

where $q_i = \sqrt{v_1^2 + v_2^2 + v_\theta^2/\rho_i^2}$ and $v_i = \partial v/\partial \rho_i$, $i = 1,2$, v being a generic element of $C^2(S)$. Furthermore, let the first-order part of $\mathcal{Q}$ be as follows:

$$\frac{\rho_1\,g(q_1)}{G(q_1)}\,v_1 + \frac{\rho_2\,g(q_2)}{G(q_2)}\,v_2 - \frac{g'(q_1)}{\rho_1\,q_1\,G(q_1)}\,v_1\,v_\theta^2 - \frac{g'(q_2)}{\rho_2\,q_2\,G(q_2)}\,v_2\,v_\theta^2.$$

Define $\overline{v}(\rho_1,\rho_2,\theta) := u_m(\rho_1,\theta)$ and $\underline{v}(\rho_1,\rho_2,\theta) := u_m(\rho_2,\theta)$. Since $\overline{v}_2 = \underline{v}_1 = 0$, we have $q_1(\overline{v}) = |\nabla u(\rho_1,\theta)|$ and $q_2(\underline{v}) = |\nabla u(\rho_2,\theta)|$. With this in mind, and taking into account that $u_\theta^2/\rho^2 = |\nabla u|^2 - u_\rho^2$, it is easy to check that $\mathcal{Q}$ is elliptic with respect to both $\overline{v}$ and $\underline{v}$. By the equation for u and by the assumptions on f one verifies that

$$\mathcal{Q}\overline{v} \le \tilde{\rho}^2\,\frac{f(\tilde{\rho},\overline{v},\tilde{q}(\overline{v}))}{G(\tilde{q}(\overline{v}))} \qquad \text{and} \qquad \mathcal{Q}\underline{v} \ge \tilde{\rho}^2\,\frac{f(\tilde{\rho},\underline{v},\tilde{q}(\underline{v}))}{G(\tilde{q}(\underline{v}))},$$

where $2\,\tilde{\rho} = \rho_1 + \rho_2$ and $\tilde{q}(v) = \sqrt{v_1^2 + v_2^2 + v_\theta^2/\tilde{\rho}^2}$. Let us give details for the first case. We have

$$\mathcal{Q}\overline{v}(\rho_1,\rho_2,\theta) = \rho_1^2\,\frac{Qu(\rho_1,\theta)}{G(|\nabla u|)} = \rho_1^2\,\frac{f(\rho_1,u,|\nabla u|)}{G(|\nabla u|)}.$$

If $u_\theta = 0$, then $\tilde{q} = |\nabla u|$ and, since $f, f_\rho \ge 0$, we are done. If, instead, $u_\theta \ne 0$, by (3.1) we have

$$\frac{\rho_1^2\,f(\tilde{\rho},u,|\nabla u|)}{u_\theta^2\,G(|\nabla u|)} = \frac{|\nabla u|^2}{|\nabla u|^2 - u_\rho^2}\,\frac{f(\tilde{\rho},u,|\nabla u|)}{|\nabla u|^2\,G(|\nabla u|)} \le \frac{\tilde{q}^2}{\tilde{q}^2 - u_\rho^2}\,\frac{f(\tilde{\rho},\overline{v},\tilde{q})}{\tilde{q}^2\,G(\tilde{q})},$$

because the function $\tilde{q}^2/(\tilde{q}^2 - u_\rho^2)$ is decreasing in $\tilde{q}$ and $\tilde{q} < |\nabla u|$. Since $\tilde{q}^2 - u_\rho^2 = u_\theta^2/\tilde{\rho}^2$ we conclude that $\mathcal{Q}\overline{v} \le \tilde{\rho}^2 f(\tilde{\rho},\overline{v},\tilde{q}(\overline{v}))\,/\,G(\tilde{q}(\overline{v}))$. The inequality for $\underline{v}$ is obtained similarly. Since $\overline{v} \ge \underline{v}$ on ∂S, and since both $\overline{v}$ and $\underline{v}$ approach t_0 as $\rho_1,\rho_2 \to +\infty$, by the comparison principle we conclude that $\overline{v} \ge \underline{v}$ in S. The claim follows ∎

Remarks. 1) Of course, the result continues to hold if Ω is a *star-shaped ring*, i.e., $\Omega = \mathcal{G}_0 \setminus \overline{\mathcal{G}}_1$ where $\mathcal{G}_1 \subset\subset \mathcal{G}_0$ are star-shaped bounded domains, provided $u_m > t_0$ in Ω, $u_m \equiv m$ on $\partial\mathcal{G}_1$ and $u_m \equiv t_0$ on $\partial\mathcal{G}_0$.

2) Since we do not require uniform ellipticity as $|\nabla u| \to +\infty$ nor strict positivity of f, the result is also applicable to the minimal surface equation, i.e., to the case when $g(t) = (1+t^2)^{-1/2}$ and $f \equiv 0$.

3) If Ω is a star-shaped ring of class $C^{2,\alpha}$, a solution u of the minimal surface equation satisfying $u \equiv m$ on $\partial\mathcal{G}_1$ and $u \equiv t_0$ on $\partial\mathcal{G}_0$ exists provided $|m - t_0|$ is small enough [7].

Star-shapedness is preserved when $m \to +\infty$. Hence, by comparing the theorem with the construction of an explosive solution developed in Section 2 we arrive at

Corollary 3.2. *Let the exterior domain Ω be of class $C^{2,\alpha}$ and assume (1.3) - (1.7), $N = 2$,*

$$\frac{k(t)}{t^2\, G(t)} \quad \textit{non-increasing in } t \textit{ for } t \in \mathbb{R}^+.$$

Denote by u_m the solution of equation (1.1) in Ω satisfying $u_m|_{\partial\Omega} \equiv m > t_0$, $u_m(x) \to t_0$ as $|x| \to +\infty$. If we let $u(x) := \lim_{m\to+\infty} u_m(x)$, then we obtain a solution (the least one) of problem (1.1) - (1.2) such that for every $c \in \mathbb{R}$ the set $\overline{\mathcal{G}} \cup \{\, x \mid u(x) > c \,\}$ is star-shaped.

3.2 Semilinear case. We show that *every* explosive solution of the equation $\Delta u = f(u)\, k(|\nabla u|)$ in the exterior of an N-dimensional star-shaped domain has star-shaped level sets, provided $k(t)/t^2$ is non-increasing for $t > 0$. The result follows by showing that the function $u(\lambda x)$ is a supersolution of the same equation. The method is derived from [4] and makes use of the fact that any of such solutions approaches t_0 at infinity (Theorem 2.5). We present the result in a more general form:

Theorem 3.3. *Let Ω be as before and $N \geq 2$. Assume that $f\colon \Omega\times(t_0,+\infty)\times\mathbb{R}^N \to \mathbb{R}^+$ is of class C^1 and satisfies $x \circ \nabla_x f(x,u,p) \geq 0$, $f_u(x,u,p) \geq 0$,*

$$\lambda^2 f(x,u,p) \leq f(x,u,\lambda p) \quad \textit{for all } x \in \Omega,\ u > t_0,\ p \in \mathbb{R}^N,\ \lambda \in (0,1). \tag{3.2}$$

If u is a solution of

$$\Delta u = f(x,u,\nabla u) \quad \textit{in } \Omega \tag{3.3}$$

satisfying (1.2) and such that $\lim_{|x|\to+\infty} u(x) = t_0$, then for every $c \in \mathbb{R}$ the set $\overline{\mathcal{G}} \cup \{\, x \mid u(x) > c \,\}$ is star-shaped.

Proof. Following [4], define $u_\lambda(x) := u(\lambda x)$ in $\Omega_\lambda = \{\, x \mid \lambda x \in \Omega \,\}$ for a $\lambda \in (0,1)$. Since $x \circ \nabla_x f(x,u,p) \geq 0$ we have:

$$\Delta u_\lambda = \lambda^2\, \Delta u(\lambda x) = \lambda^2\, f(\lambda x, u(\lambda x), \nabla u(\lambda x)) \leq \lambda^2\, f(x, u(\lambda x), \nabla u(\lambda x)).$$

This and (3.2) imply

$$\Delta u_\lambda \leq f\big(x, u(\lambda x), \lambda\, \nabla u(\lambda x)\big) = f\big(x, u_\lambda(x), \nabla u_\lambda(x)\big),$$

hence u_λ is a supersolution of (3.3). Furthermore we have $u_\lambda(x) \to +\infty$ as $x \to \partial\Omega_\lambda$ and $u_\lambda(x) \to t_0$ as $|x| \to +\infty$, hence by the comparison principle we deduce that $u_\lambda \geq u$ in Ω_λ. Since λ is arbitrary the claim follows ∎

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