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# Impulsive Controllability and Optimization Problems Lagrange's Method and Applications

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**Abstract.** In the present paper, a class of optimization problems to impulsive control of smooth dynamical processes is investigated. Necessary and sufficiently conditions for existence of optimal impulsive controllability of the initial-value problem for a dynamical system are obtained. The derived results are applied to the analysis of some classical problems from population dynamics.

**Keywords:** *Impulsive systems, impulsive controllability of smooth dynamical systems, population dynamics*

**AMS subject classification:** Primary 34 A 37, secondary 49 J 15, 92 D 25

## 1. Introduction

Impulsive differential systems are suitable mathematical models to simulate evolution of large classes of real processes: human intervention in the evolution of one or some populations, proceeding and controlling chemical reactions – catalyze reactions, etc. The common feature of these processes is the existence of short temporary perturbations during their evaluation. The continuation of these perturbations is insignificant compared to the duration of the whole process. That is why the perturbations occur "immediately" as impulses. Basic monographs on this subject are [3 - 5, 7].

In the present paper, we consider some problems for impulsive (discrete) controllability of the solutions of smooth dynamical systems. Rapid out-side actions over the evolutionary system result in research of its impulsive control. We shall introduce some examples from population dynamics.

Most of models of single species dynamics have been derived from a differential equation in the form

$$\dot{x} = xf(t,x) + g(t,x), \tag{1.1}$$

where the solution $x = x(t)$ is treated as population size (or biomass) in time $t > 0$, the function $f = f(t,x)$ is characterized as population change at the moment $t$, the function $g = g(t,x)$ describes the continuous influence of out-side factors. Various choices of the functions $f$ and $g$ lead us to various differential equation models. For instance:

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**(1)** If $f(t,x) = \frac{a}{b}(b-x)$, we obtain the *Verhulst differential equation*

$$\dot{x} = \frac{a}{b}x(b-x) + g(t,x) \tag{1.2}$$

where $a \in \mathbb{R}_+ = [0,\infty)$ is the reproductive potential of population and $b \in \mathbb{R}_+$ is the capacity of environment.

**(2)** If $f(t,x) = a - c\ln x$, we obtain the *Gompertz differential equation*

$$\dot{x} = x(a - c\ln x) + g(t,x) \tag{1.3}$$

where $c \in \mathbb{R}_+$ is the coefficient of interspecies competition.

**(3)** If $f(t,x) = f(x)$, then (1.1) is an evolutionary model of stationary population.

**(4)** If $g(t,x) = 0$, then (1.1) is an evolutionary model of isolated population.

Let $\eta = \eta(t; t_0, x_0)$ be a solution of the differential equation (1.1) with initial condition

$$\eta(t_0; t_0, x_0) = x_0 \qquad ((t_0, x_0) \in \mathbb{R}^2_+). \tag{1.4}$$

Let $\tau_1 < \ldots < \tau_p$ $(t_0 \leq \tau_1, p \in \mathbb{N})$ be moments of out-side perturbations on the evolution of the considered population system. For example, subtracting or adding some quantity of biomass and etc. Then

$$x(\tau_i + 0; t_0, x_0) = \Phi(\tau_i, x(\tau_i; t_0, x_0)) \qquad (i \in N_p = \{1, \ldots, p\}) \tag{1.5}$$

where $x(\tau_i + 0; t_0, x_0) = \lim_{t\to\tau_i+0} x(t; t_0, x_0)$, $\Phi = \Phi(t,x)$ is a map, which characterized the out-side action in the moments $\tau_1, \ldots, \tau_p$. For example, if $\Phi(\tau_i, x) = x - d_i$ $(d_i > 0, i \in N_p)$, then in each moment $\tau_i$, we subtract the quantity $d_i$ from the population biomass.

The system (1.1), (1.5) is called an *impulsive system.*

In general, impulsive systems describe out-side actions on the investigated population. Moreover, there is a possibility to control moments of perturbations and some parameters of impulsive out-side influence. That is why optimal choosing of impulsive moments $\tau_i$ and impulsive map $\Phi$ is very important. For instance, let us consider the Verhulst's model (1.2) and let $\Phi(\tau_i, x) = x - d_i$. There arise the following question: How shall we choose the numbers $\tau_i$ and $d_i$ so that the population output $d_1 + \ldots + d_p$ will be maximum on a $t$-interval $[0,T]$? On Figure 1, we present the time-portrait of Verhulst equation (1.2), where $a = 0.03$, $b = 100$, $t \in [0,100]$ and co-responding impulsive time-portrait, $p = 3$.

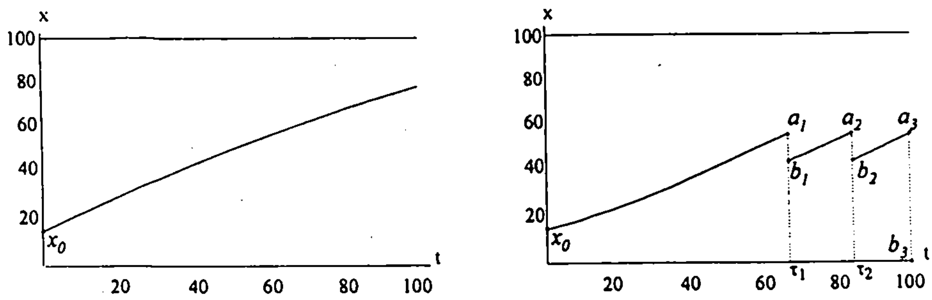


Figure 1

The present paper consists of the following three features:

(1) Mathematical definition of impulsive systems — Subsection 2.1.

(2) Statement of impulsive control problem — Subsection 2.2. In Section 3, we obtain some necessary conditions for the existence of solution of considered optimal impulsive control problems. The present method is based on investigation of classical Lagrange's function for a proper optimal problem. Moreover, we shall present some necessary and sufficiently conditions for existence and/or uniqueness of solution of impulsive problem.

(3) Applications in population dynamics — Section 4. In this section we shall apply obtained results to investigate some optimization problems related to Verhulst and Gompertz differential equations.

## 2. Impulsive systems. Impulsive control

Let $\langle\cdot,\cdot\rangle$ be the Euclidean scalar product in $\mathbb{R}^n$ and $\|\cdot\|$ the corresponding norm. Let $C^1(A,\mathbb{R}^n)$ be the space of all $C^1$-smooth maps form $A\subset\mathbb{R}^n$ into $\mathbb{R}^n$ with Ulam's $C^0$-topology. Let $h\in C^0(A,\mathbb{R}^n)$. We set $\|h\|_0=\sup_{x\in A}\|h(x)\|$. Let $a\in\mathbb{R}^n$, $h\in C^1(\mathbb{R}^n,\mathbb{R})$ and $x^{(m)}=(x^1,\ldots,x^m)$, where $m\le n$. We set

$$\nabla_{x^{(m)}}h(a)=(h'_{x^1},\ldots,h'_{x^m}),\qquad \nabla h(a)=\nabla_x h(a),\qquad x=x^{(n)}.$$

**2.1 Impulsive systems.** Let $p\in\mathbb{N}$; $f\in C^1(\mathbb{R}_+\times\mathbb{R}^n,\mathbb{R}^n)$, $\overline{\tau}=\{\tau_1,\ldots,\tau_p\}$ be a finite increasing sequence and $\Phi\in C^1(\overline{\tau}\times\mathbb{R}^n,\mathbb{R}^n)$. We say that the system

$$\dot{x}=f(t,x)\qquad (t\in\mathbb{R}_+\setminus\overline{\tau})\tag{2.1}$$

$$x(t+0)=\Phi(t,x)\qquad (t\in\overline{\tau})\tag{2.2}$$

is an *impulsive system of differential equations with fixed moments* (see [3 - 5]).

**Definition 1.** We say that the map

$$x : [t_0, T) \longrightarrow \mathbb{R}^n, \quad t \longrightarrow x(t; t_0, x_0)$$

is a *solution* of the impulsive system (2.1), (2.2) with initial condition

$$x(t_0; t_0, x_0) = x_0 \qquad ((t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n) \tag{2.3}$$

if the following assertions are valid:

1. If $t_0 < \tau_1$, then $x(t_0; t_0, x_0) = x_0$. If $t_0 = \tau_1$, then $x(t_0; t_0, x_0) = \Phi(t_0, x_0)$.
2. The map $x|([t_0, T) \setminus \overline{\tau})$ is $C^1$-smooth.
3. Equality (2.1) is valid, for each $t \in [t_0, T) \setminus \overline{\tau}$.
4. The equalities $x(t; t_0, x_0) = x(t - 0; t_0, x_0)$ and $x(t + 0; t_0, x_0) = \Phi(t, x(t; t_0, x_0))$ are valid, for each $t \in [t_0, T) \cap \overline{\tau}$.

The numbers $\overline{\tau} = \{\tau_1, \ldots, \tau_p\}$ are called *impulsive moments*. The map $\Phi$ is called *impulsive map*.

We shall consider the following non-impulsive initial-value problem:

$$\dot{x} = f(t, x) \qquad ((t, x) \in \mathbb{R}_+ \times \mathbb{R}^n) \tag{2.4}$$

$$x(t_0; t_0, x_0) = x_0 \qquad ((t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n). \tag{2.5}$$

Let us introduce the following hypotheses (H1):

**(H1)$_1$** $f \in C^1(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R})$, and there exists a number $M \in \mathbb{R}_+$ such that $\|f\|_0 \leq M$.

**(H1)$_2$** For each couple $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, the solution $\eta = \eta(t; t_0, x_0)$ of non-impulsive initial-value problem (2.4), (2.5) is defined in $[t_0, \infty)$.

**2.2 Impulsive control.** Let $h_i \in C^1(\mathbb{R}^n, \mathbb{R})$, $\nabla h_i(x) \neq 0$ if $h_i(x) = 0$, $D_i = \{x \in \mathbb{R}^n : h_i(x) < 0\}$, where $i \in N_2$, and let $\emptyset \neq D_2 \subset \overline{D}_2 \subset D_1$. Let $x_0 \in \overline{D}_1$ and $t_0, T \in \mathbb{R}_+$ with $t_0 < T$. Let $p \in \mathbb{N}$, and let $I_p$ denote the set of all ordered couples $(\overline{\tau}, \Phi)$ such that:

1. $\overline{\tau} = \{\tau_1, \ldots, \tau_p\} \subset [t_0, T]$, with $\tau_i \leq \tau_{i+1}$ $(i \in N_{p-1})$.
2. $\Phi \in C^1(\overline{\tau} \times \mathbb{R}^n, \mathbb{R}^n)$.
3. $\{x(t; t_0, x_0) : t \in [t_0, T]\} \subset \overline{D}_1$.
4. $x(T; t_0, x_0) \in \overline{D}_2$.

Let $(\overline{\tau}, \Phi) \in I_p$. We set

$$\begin{aligned} a_i &= x(\tau_i; t_0, x_0),\ b_i = \Phi(\tau_i, a_i) \qquad (i \in N_p) \\ a_0 &= b_0 = x_0,\ a_{p+1} = x(T; t_0, x_0) \\ \overline{a} &= (a_1, \ldots, a_p),\ \overline{b} = (b_1, \ldots, b_p). \end{aligned} \tag{2.6}$$

Let $\pi : I_p \longrightarrow R^{(2p+1)n}$, $\pi(\overline{\tau}, \Phi) = (\overline{a}, \overline{b}, a_{p+1})$ and $F : R^{(2p+1)n} \longrightarrow [0, 1]$ be a $C^1$-smooth bounded function. We set $\widehat{F} : I_p \longrightarrow [0, 1]$, $\widehat{F}(\overline{\tau}, \Phi) = F(\pi(\overline{\tau}, \Phi))$. Now, we shall formulate the problem of optimal impulsive control:

Does there exist a couple $(\overline{\tau}^0, \Phi^0) \in I_p$ such that

$$\widehat{F}(\overline{\tau}^0, \Phi^0) = \inf\{\widehat{F}(\overline{\tau}, \Phi) : (\overline{\tau}, \Phi) \in I_p\}? \tag{2.7}$$

## 3. Main results

In the present section, we shall derive some necessary conditions for existence of the solutions of optimal impulsive problem (2.7). For this purpose, we introduce the following hypothesis (H2):

**(H2)** One of the following two implications is valid:

$$\inf\Big\{\langle f(t,x),\nabla h_1(x)\rangle : (t,x)\in(t_0,T)\times\partial D_1\Big\}\geq 0 \tag{3.1}$$

or

$$\sup\Big\{\langle f(t,x),\nabla h_1(x)\rangle : (t,x)\in(t_0,T)\times\partial D_1\Big\}\leq 0, \tag{3.2}$$

i.e. the scalar product $\langle f(t,x),\nabla h_1(x)\rangle$ has one and the same sign on $(t_0,T)\times\partial D_1$.

**3.1 Lagrange's form of the optimal impulsive problem.** Let $g_i\in C^1(\mathbb{R}^n,R_0)$ be the characteristic function of the domain $D_i$, i.e.

$$g_i(x)=\begin{cases}0 & \text{if } x\in\overline{D}_i\\ \exp(-h_i^{-1}(x)) & \text{if } x\in\mathbb{R}^n\setminus\overline{D}_i\end{cases}\qquad (i\in 1,2).$$

We set

$$\overline{x}=(x_1,\dots,x_p)\ \ (x_i\in\mathbb{R}^n),\quad \overline{y}=(y_1,\dots,y_p)\ \ (y_i\in\mathbb{R}^n)\qquad (i\in N_p) \tag{3.3}$$
$$G:\mathbb{R}^{(p+1)n}\longrightarrow\mathbb{R}_+,\quad G(\overline{x},x_{p+1})=g_2(x_{p+1})+\sum_{i=1}^{p}g_1(x_i).$$

Let $\{\Psi_1(t,x),\dots,\Psi_n(t,x)\}$ be the independent first integrals of system (2.4) in the domain containing the closed set $[t_0,T]\times\overline{D}_1$. We consider the following extremum problem:

$$F(\overline{x},\overline{y},x_{p+1})\longrightarrow\min \tag{3.4}$$

under conditions

$$G(\overline{x},x_{p+1})=0 \tag{3.5}$$
$$\Psi_j(\theta_i,x_i)=\Psi_j(\theta_{i-1},y_{i-1}) \tag{3.6}$$

where $j\in N_n$, $i\in N_{p+1}$, $\theta_0=t_0$, $\overline{\theta}=\{\theta_1,\dots,\theta_p\}\in\mathbb{R}^p$, $\theta_{p+1}=T$ and $y_0=x_0$.

In the following theorem, we shall prove the equivalence of impulsive control problem (see the end of Subsection 2.2) and Lagrange's extremum problem (3.4) - (3.6).

**Theorem 1.** *Let hypotheses* (H1) *and* (H2) *are satisfied and let* $p\in\mathbb{N}$. *Then:*

1. *The next two statements are equivalent:*

(a) *There exists a couple* $(\overline{\tau}^0,\Phi^0)\in I_p$ *for which equality* (2.7) *is valid.*

(b) *The extremum problem* (3.4) - (3.6) *possesses a solution* $(\overline{\theta}^0,\overline{x}^0,\overline{y}^0,x_{p+1}^0)$.

2. *If* $\theta_i^0 < \theta_{i+1}^0$ $(i \in N_{p-1})$, *then* $\overline{\theta}^0 = \overline{\tau}^0$, $\overline{x}^0 = \overline{a}$, $\overline{y}^0 = \overline{b}$, $x_{p+1}^0 = a_{p+1}$ (*see* (2.6)).

**Proof.** Let us suppose that condition (3.1) is valid (the proof is similar if condition (3.2) holds). Let $(\overline{\theta}^0, \overline{x}^0, \overline{y}^0, x_{p+1}^0)$ be a solution of extremum problem (3.4) - (3.6) and $\theta_i^0 < \theta_{i+1}^0$ $(i \in N_{p-1})$. We set $\overline{\tau}^0 = \overline{\theta}^0$, $\Phi^0 \in C^\infty(\overline{\tau}^0 \times \mathbb{R}^n, \mathbb{R}^n)$, $\Phi^0(\tau_i^0, x_i^0) = y_i^0$ where $i \in N_p$, $\overline{x}^0 = (x_1^0, \ldots, x_p^0)$ and $\overline{y}^0 = (y_1^0, \ldots, y_p^0)$. We shall prove that the solution $x = x(t; t_0, x_0)$ of the impulsive initial problem

$$\begin{aligned} \dot{x} &= f(t,x) && (t \in [t_0, T] \setminus \overline{\tau}^0) \\ x(t+0) &= \Phi(t,x) && (t \in \overline{\tau}^0) \\ x(t_0; t_0, x_0) &= x_0 \end{aligned}$$

satisfies the inclusions $\{x(t; t_0, x_0) : t \in [t_0, T]\} \subset \overline{D}_1$ and $x(T; t_0, x_0) \in \overline{D}_2$.

Let us assume that there exists $t^* \in (t_0, T)$ such that $x(t^*; t_0, x_0) \in \mathbb{R}^n \setminus \overline{D}_1$. We set $t_* = \inf\{t \in (0, t^*] : x(t; t_0, x_0) \in \mathbb{R}^n \setminus \overline{D}_1\}$. Then, $t_* \notin \overline{\theta}^0$ or $t_* \in \overline{\theta}^0$.

Let $t_* \notin \overline{\theta}^0$. We choose an integer $i \in N_{p+1}$ such that $\theta_{i-1}^0 < t_* < \theta_i^0$. From condition (3.5), it follows that $x_{i-1}^0 = x(\theta_{i-1}^0; t_0, x_0) \in \overline{D}_1$ and $x(t^*; t_0, x_0) \in \mathbb{R}^n \setminus \overline{D}_1$. Therefore $x(t_*; t_0, x_0) \in \partial D_1$ and $t_* \in (\theta_{i-1}^0, t^*)$. From condition (2) and the obtained inclusion, it follows that

$$\begin{aligned} &\{x(t; t_0, x_0) : t \in (\theta_{i-1}^0, t_*]\} \subset \overline{D}_1 \\ &\{x(t; t_0, x_0) : t \in (t_*, \theta_i^0]\} \subset \mathbb{R}^n \setminus \overline{D}_1. \end{aligned}$$

Hence $x_i^0 = x(\theta_i^0; t_0, x_0) \in \mathbb{R}^n \setminus \overline{D}_1$. The inclusion $x_i^0 \in \mathbb{R}^n \setminus \overline{D}_1$ contradicts to condition (3.5).

Let $t_* = \overline{\theta}_i^0$, where $i \in N_p$. Then hypothesis (H2), inclusions $y_i^0 \in \partial D_1$ and $x(t^*; t_0, x_0) \in \mathbb{R}^n \setminus \overline{D}_1$ yield $x_i^0 \in \mathbb{R}^n \setminus \overline{D}_1$. The derived inclusion contradicts equality (3.5) and the definition of the function $G = G(\overline{x}, x_{p+1})$.

Therefore $\{x(t; t_0, x_0) : t \in [t_0, T]\} \subset \overline{D}_1$. From the definitions of the functions $g_2 = g_2(x_{p+1})$ and $G = G(\overline{x}, x_{p+1})$ and from equality (3.5) it follows that $x_{p+1}^0 = x(T; t_0, x_0) \in \overline{D}_2$. Hence $(\overline{\theta}^0, \Phi^0) \in I_p$. Equality (2.7) follows from the fact that $(\overline{\theta}^0, \overline{x}^0, \overline{y}^0, x_{p+1}^0)$ is a solution of extremum problem (3.4).

Let equality (2.7) be valid for a couple $(\overline{\tau}^0, \Phi^0) \in I_p$. Then from the definitions of the set $I_p$, function $G = G(\overline{x}, x_{p+1})$ and equality (2.7) it follows that $(\overline{\theta}^0, \overline{x}^0, \overline{y}^0, x_{p+1}^0)$ is a solution of problem (3.4) - (3.6), where $\overline{\theta}^0 = \overline{\tau}$, $\overline{x}^0 = \overline{a}$, $\overline{y}^0 = \overline{b}$ and $x_{p+1}^0 = a_{p+1}$ (see (2.6)) ∎

From Theorem 1 and closeness of the set of all points $(\overline{\theta}, \overline{x}, \overline{y}, x_{p+1})$, for which equalities (3.5) and (3.6) are valid, solvability of the impulsive control problem follows.

**Corollary 1.** *Let the following conditions are valid:*

1. *The hypotheses* (H1) *and* (H2) *are fulfilled,* $p \in \mathbb{N}$.

2. *The domain* $D_1$ *is bounded.*

*Then there exists a couple* $(\overline{\tau}^0, \Phi^0) \in I_p$ *satisfying equality* (2.7).

**3.2 Necessary conditions for existence of solutions of the optimal impulsive problem.** In the present subsection we shall prove the following theorem.

**Theorem 2.** *Let the following conditions hold:*

1. *Hypotheses* (H1) *and* (H2) *are valid,* $p \in N$.
2. *Equality* (2.7) *is valid for the couple* $(\overline{\tau}^0, \Phi^0) \in I_p$, $\overline{\tau}^0 = (\tau_1^0, \ldots, \tau_p^0)$.

*Then there exist numbers* $\xi$ *and* $\eta_i^j$ $(i \in N_{p+1}, j \in N_n)$ *such that:*

$$|\xi| + \sum_{i=1}^{p+1} \sum_{j=1}^{n} |\eta_i^j| \neq 0 \tag{3.7}$$

$$\nabla_{x_i} F(\overline{a}^0, \overline{b}^0, a_{p+1}^0) = \sum_{j=1}^{n} \eta_i^j \nabla_x \Psi_j(\tau_i^0, a_i^0) + \xi \nabla_{x_i} G(\overline{a}^0, a_{p+1}^0) \tag{3.8}$$

$$\nabla_{y_i} F(\overline{a}^0, \overline{b}^0, a_{p+1}^0) = -\sum_{j=1}^{n} \eta_{i+1}^j \nabla_x \Psi_j(\tau_i^0, a_i^0) \tag{3.9}$$

$$\sum_{j=1}^{n} \eta_i^j (\Psi_j)'_t(\tau_i^0, a_i^0) = \sum_{j=1}^{n} \eta_{i+1}^j (\Psi_j)'_t(\tau_i^0, b_i^0) \tag{3.10}$$

*where* $a_i^0 = x(\tau_i^0; t_0, x_0)$ *and* $b_i^0 = \Phi(\tau_i^0, a_i^0)$ *for* $i \in N_p$, $\overline{a}^0 = (a_1^0, \ldots, a_p^0)$, $\overline{b}^0 = (b_1^0, \ldots, b_p^0)$ *and* $a_{p+1}^0 = x(T; t_0, x_0)$.

**Proof.** From Theorem 1 it follows that the assertion of Theorem 2 is equivalent with the next statement. Equations (3.8) - (3.10) are necessary conditions for the existence of solution of optimal impulsive problem (3.4) - (3.6).

We construct the Lagrangian function of problem (3.4) - (3.6):

$$L(\overline{\theta}, \overline{x}, \overline{y}, x_{p+1}, \xi, \eta_i^j) = F(\overline{x}, \overline{y}, x_{p+1}) + \xi G(\overline{x}, x_{p+1}) + \sum_{i=1}^{p+1} \sum_{j=1}^{n} \eta_i^j \left[\Psi_j(\theta_i, x_i) - \Psi_j(\theta_{i-1}, y_{i-1})\right].$$

The derivatives of the Lagrangian function are:

$$L'_{\theta_i} = \sum_{j=1}^{n} \left[\eta_i^j (\Psi_j)'_t(\theta_i, x_i) - \eta_{i+1}^j (\Psi_j)'_t(\theta_i, y_i)\right] \tag{3.11}$$

$$L'_{x_i^k} = F'_{x_i^k}(\overline{x}, \overline{y}, x_{p+1}) + \xi G'_{x_i^k}(\overline{x}, x_{p+1}) + \sum_{j=1}^{n} \eta_i^j (\Psi_j)'_{x_i^k}(\theta_i, x_i) \tag{3.12}$$

$$L'_{y_i^k} = F'_{x_i^k}(\overline{x}, \overline{y}, x_{p+1}) - \sum_{j=1}^{n} \eta_{i+1}^j (\Psi_j)'_{x_i^k}(\theta_i, x_i) \tag{3.13}$$

where $k \in N_n$. Equalities (3.8) - (3.10) follow from Lagrange Theorem and (3.11) - (3.13), respectively ∎

**Corollary 2.** *Let the following conditions are satisfied:*

1. *The hypotheses* (H1) *and* (H2) *are valid,* $p \in N$.
2. *The equalities* $D_1 = D_2 = \mathbb{R}^n$ *are fulfilled.*
3. *Equality* (2.7) *is valid for a couple* $(\overline{\tau}^0, \Phi^0) \in I_p$.

*Then*

$$\left\langle \nabla_{x_i} F(\overline{a}^0, \overline{b}^0, a^0_{p+1}), f(\tau^0_i, a^0_i) \right\rangle = -\left\langle \nabla_{y_i} F(\overline{a}^0, \overline{b}^0, a^0_{p+1}), f(\tau^0_i, b^0_i) \right\rangle \tag{3.14}$$

*where* $i \in N_{p+1}$.

**Proof.** From equation (3.8) and formulae

$$(\Psi_j)'_t\,(t,x) = \sum_{k=1}^{n} f_k(t,x)(\Psi_j)'_{x^k}(t,x) \qquad (j \in N_n)$$

where $f(t,x) = (f_1(t,x), \dots, f_n(t,x))$ with $x = (x^1, \dots, x^n)$ there follows that

$$\begin{aligned} \eta^j_i (\Psi_j)'_t(\tau^0_i, a^0_i) &= \sum_{k=1}^{n} \eta^j_i f_k(\tau^0_i, a^0_i)(\Psi_j)'_{x^k_i}(\tau^0_i, a^0_i) \\ &= \left\langle \nabla_{x_i} F(\overline{a}^0, \overline{b}^0, a^0_{p+1}), f(\tau^0_i, a^0_i) \right\rangle. \end{aligned} \tag{3.15}$$

Similarly we can obtain the next formulae

$$\nabla_{y_i} F(\overline{a}^0, \overline{b}^0, a^0_{p+1}) = -\left\langle \nabla_{y_i} F(\overline{a}^0, \overline{b}^0, a^0_{p+1}), f(\tau^0_i, b^0_i) \right\rangle. \tag{3.16}$$

Formulas (3.15), (3.16) and (3.1) imply (3.14) ∎

The following result improves previous results in terms of sufficiency and necessity to optimal controllability of the considered problem. For simplification, we shall suppose that the domain of the problem under consideration is $\mathbb{R}^n$.

**Theorem 3.** *Let the following conditions hold:*

1. *The hypotheses* (H1) *and* (H2) *are valid,* $p \in N$.
2. $D_1 = D_2 = \mathbb{R}^n$.
3. *All first integrals* $\Psi_j(t,x)$ $(j \in N_n)$ *are convex functions.*

*Then equality* (2.7) *is valid for the couple* $(\overline{\tau}^0, \Phi^0) \in I_p$ $(\overline{\tau}^0 = (\tau^0_1, \dots, \tau^0_p))$ *if and only if there exist numbers* $\eta^j_i$ $(i \in N_{p+1}, j \in N_n)$ *such that* $\eta^j_i \geq 0$ *and*

$$\nabla_{x_i} F(\overline{a}^0, \overline{b}^0, a^0_{p+1}) = \sum_{j=1}^{n} \eta^j_i \nabla_x \Psi_j(\tau^0_i, a^0_i) \tag{3.17}$$

$$\nabla_{y_i} F(\overline{a}^0, \overline{b}^0, a^0_{p+1}) = -\sum_{j=1}^{n} \eta^j_{i+1} \nabla_x \Psi_j(\tau^0_i, a^0_i) \tag{3.18}$$

$$\sum_{j=1}^{n} \eta^j_i \,(\Psi_j)'_t(\tau^0_i, a^0_i) = \sum_{j=1}^{n} \eta^j_{i+1} \,(\Psi_j)'_t(\tau^0_i, b^0_i) \tag{3.19}$$

$$\Psi_j(\theta_i, x_i) \leq \Psi_j(\theta_{i-1}, y_{i-1}) \tag{3.20}$$

$$\eta^j_i \left(\Psi_j(\theta_i, x_i) - \Psi_j(\theta_{i-1}, y_{i-1})\right) = 0 \tag{3.21}$$

*where* $a_i^0 = x(\tau_i^0; t_0, x_0)$ *and* $b_i^0 = \Phi(\tau_i^0, a_i^0)$ *for* $i \in N_p$, $\overline{a}^0 = (a_1^0, \ldots, a_p^0)$, $\overline{b}^0 = (b_1^0, \ldots, b_p^0)$ *and* $a_{p+1}^0 = x(T; t_0, x_0)$.

**Proof.** The proof of Theorem 3 is similar to that of Theorem 2, using H. W. Kuhn's and A. W. Tacker's theorem. So, we omit any details ∎

## 4. Application in population dynamics

We say that hypotheses (H3) are valid if:

**(H3)$_1$** Hypotheses (H1) and (H2) hold, where $f \in C^1(\mathbb{R}^2, \mathbb{R})$.

**(H3)$_2$** $D_1 = D_2 = R$.

**(H3)$_3$** $\Psi = \Psi(t, x)$ is a first integral of differential equation (2.4).

We consider the optimal impulsive problem of the solution $\eta = \eta(t; t_0, x_0)$ of initial-value problem (2.4), (2.5) with optimizing function

$$F(\overline{a}, \overline{b}, a_{p+1}) = a_{p+1} + \sum_{i=1}^{p} (a_i - b_i). \tag{4.1}$$

**Theorem 4.** *Let the following conditions hold:*

1. *Hypotheses* (H3) *are valid.*
2. *Equality* (2.7) *is fulfilled for a couple* $(\overline{\tau}^0, \Phi^0) \in I_p$.
3. *The optimization function* $F$ *is defined by formula* (4.1).

*Then*

$$\Psi(\tau_i^0, b_i^0) = \Psi(\tau_{i-1}^0, a_{i-1}^0) \qquad (i \in N_{p+1}) \tag{4.2}$$

$$\Psi_t'(\tau_i^0, b_i^0) = \Psi_t'(\tau_i^0, a_i^0) \qquad (i \in N_p) \tag{4.3}$$

$$f(\tau_i^0, a_i^0) = f(\tau_i^0, b_i^0) \qquad (i \in N_p). \tag{4.4}$$

**Proof.** From Theorem 2, formulae (3.8) and (3.9) it follows that

$$-1 = \eta_i \Psi_x'(\tau_i^0, a_i^0), \quad 1 = -\eta_{i+1} \Psi_x'(\tau_i^0, a_i^0) \qquad (i \in N_p)$$

where $\{\eta_i : i \in N_{p+1}\} \subset \mathbb{R}$. Hence $\eta_i = \eta_{i+1}$ and formula (4.3) is true. Note that condition (3) of Theorem 4 and formulae (3.8) and (3.9) imply that $\eta_i \neq 0$ $(i \in N_{p+1})$. Equation (4.4) follows from condition (3) of Theorem 4 and Corollary 2 ∎

**Remark 1.** From formula (4.4) it follows that "impulsive jumps" under optimal controllability are realized on the level surface of the function $f = f(\tau_i, \cdot)$. We shall go into details on geometrical interpretation of Theorem 4 under the following additional assumptions:

1. $f(t, r_1) = f(t, r_2) = 0$, where $r_1 < r_2$.

2. For each $t \in [0,T]$, the function $f = f(t,\cdot)$ has a unique maximum.
3. $p = 1$ and $\tau_1 \in (0,T)$.

Note that Verhulst equation (1.2) for an isolated population satisfies assumptions (1) and (2).

Let $x_0 \in (r_1, r_2)$, $a_1 = \eta(\tau_1; 0, x_0)$ and $c = f(\tau_1, a_1)$. From assumption (1), it follows that $[0,T] \times [r_1, r_2]$ is an invariant set of equation (2.4). Hence $a_1 \in (r_1, r_2)$. From assumption (2), it follows that there exist two $C^1$-smooth functions $\gamma_1, \gamma_2 : [0,T] \longrightarrow (r_1, r_2)$ such that $\gamma_1(t) < \gamma_2(t)$ $(t \in [0,T])$ and $f^{-1}(c) = \{(t,\gamma_1(t)) : t \in [0,T]\} \cup \{(t,\gamma_2(t)) : t \in [0,T]\}$. Moreover, assumption (2) yield the inequality $f(t,x) > 0$ for $(t,x) \in [0,T] \times (r_1, r_2)$. Therefore $a_1 = \gamma_2(\tau_1)$. Writing $b_1 = \gamma_1(\tau_1)$, we choose an impulsive map $\Phi \in C^1(\overline{\tau} \times \mathbb{R}, \mathbb{R})$ such that $\Phi(a_1) = b_1$. From formula (4.4), we conclude that $(\{\tau_1\}, \Phi)$ is the optimal impulsive control for initial problem (2.4), (2.5).

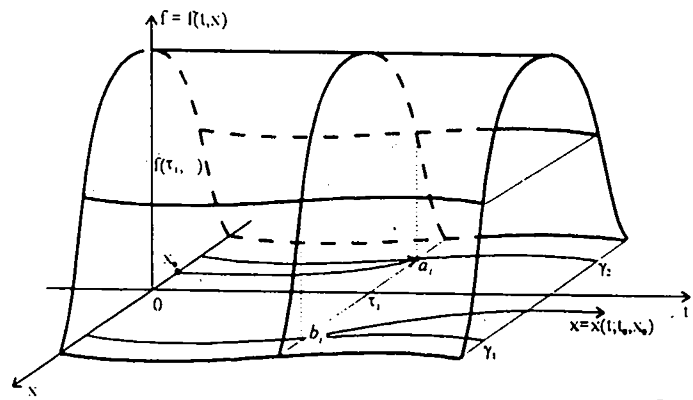

Figure 2

**Example 1.** Let the evolution of an isolated population be described by the *Verhulst equation*

$$\dot{x} = \frac{a}{b} x(b - x) \tag{4.4}$$

where $a = 0.03$ is the reproductive potential of population and $b = 100$ is the capacity of environment. We take from the biomass three times in the time interval $[0, 100]$. We shall determinate the moments $\tau_1, \tau_2$ and $\tau_3$ of subtracting in such a way that the general subtract quantity will be maximum.

Without loss of generality, we shall assume that at the moment $t_0 = 0$ there is $x_0 = 15$ biomass units. Denote $\eta = \eta(t; 0, 15)$ the solution of equation (4.4) with initial condition $x(0; 0, 15) = 15$. So, the impulsive analogue to the considered models is:

$$\begin{aligned} \dot{x} &= \frac{a}{b} x(b - x) \qquad (t \notin \{\tau_1, \tau_2, \tau_3\} \\ x(\tau_i + 0) &= \Phi(\tau_i, x(\tau_i; 0, 15)) \qquad (i \in N_3) \\ x(0; 0, 15) &= 15. \end{aligned}$$

The optimization function is

$$F(x_1, x_2, x_3, y_1, y_2, y_3) = \sum_{i=1}^{3}(x_i - y_i).$$

From Theorem 4 it follows that if $(\tau_1^0, \tau_2^0, \tau_3^0, a_1^0, a_2^0, a_3^0, b_1^0, b_2^0, b_3^0, a_4^0)$ is a solution of the examined optimization impulsive problem, then

$$\begin{aligned} \Psi(\tau_i, a_i) &= \Psi(\tau_{i-1}, b_{i-1}) \qquad (i \in N_4) \\ \Psi_t'(\tau_i, a_i) &= \Psi_t'(\tau_{i-1}, b_{i-1}) \qquad (i \in N_3) \\ f(\tau_i, a_i) &= f(\tau_i, b_i) \qquad (i \in N_3) \end{aligned}$$

where $\Psi(t,x) = \frac{xe^{-0.03t}}{100-x}$ is the first integral of differential equation (4.4). We use Newton's method to solve this system. The initial data is

$$\begin{aligned} \tau_1^0 &= 55 \qquad & a_1^0 &= 60 \qquad & b_1^0 &= 40 \\ \tau_2^0 &= 85 & a_2^0 &= 60 & b_2^0 &= 40 \\ \tau_3^0 &= 100 & a_3^0 &= 60 & b_3^0 &= 0 \end{aligned}$$

The solution is

$$\begin{aligned} \tau_1^0 &= 66.2851 \qquad & a_1^0 &= 56.3004 \qquad & b_1^0 &= 43.6996 \\ \tau_2^0 &= 83.176 & a_2^0 &= 56.3004 & b_2^0 &= 43.6996 \\ \tau_3^0 &= 100 & a_3^0 &= 56.2436 & b_3^0 &= 0 \end{aligned}$$

Thus the subtracting quantity is $F(\overline{a}^0, \overline{b}^0) = 81.4455$.

**Example 2.** In this example we shall study *Gompertz's equation* (1.3) for an isolated population, i.e. $g(t,x) = 0$, where $a = 0.03$, $c = 1$, $x_0 = 0.2$ and $T = 5$. We take from the biomass two times, so $p = 2$. We shall determine the impulsive moments $\tau_1$ and $\tau_2$ such that the general subtract quantity will be maximum.

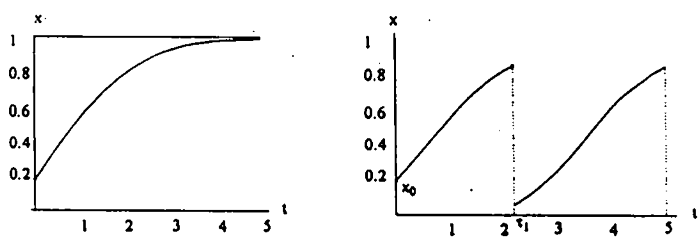


Figure 3

Certainly the optimization function is

$$F(x_1, x_2, y_1, y_2) = \sum_{i=1}^{2}(x_i - y_i)$$

and the first integral of Gompertz's equation (1.3) is

$$\Psi(t,x) = t + \ln(0.03 - \ln(x)).$$

From equalities

$$\Psi''_{tt}(t,x) = \det\begin{pmatrix} \Psi''_{tt}(t,x) & \Psi''_{tx}(t,x) \\ \Psi''_{xt}(t,x) & \Psi''_{xx}(t,x) \end{pmatrix} = 0$$

it follows that the first integral $\Psi(t,x)$ is a convex function. Therefore, we may use Theorem 3. The solution of the corresponding system is

$$\begin{array}{lll} \tau_1^0 = 2.20117 & a_1^0 = 0.859467 & b_1^0 = 0.0523254 \\ \tau_2^0 = 5 & a_2^0 = 0.859467 & b_2^0 = 0 \end{array}$$

The subtracting quantity is $F(\overline{a}^0, \overline{b}^0) = 1.6666086$. From Theorem 3 it follows that there exist only one global solution of the considered problem. On Figure 3, the time portrait of Gompertz's equation and the impulsive analogue are present.

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