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On Some Uniform Convexities and Smoothness in Certain Sequence Spaces

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Abstract. It is proved that any Banach space X with property A_2^ϵ has property A_2 and that a Banach space X is nearly uniformly smooth if and only if it is nearly uniformly *-smooth and weakly sequentially complete. It is shown that if X is a Köthe sequence space the dual of which contains no isomorphic copy of l_1 and has property A_2^ϵ, then X has the uniform Kadec-Klee property. Criteria for nearly uniformly convexity of Musielak-Orlicz spaces equipped with the Orlicz norm are presented. It is also proved that both properties nearly uniformly smoothness and nearly uniformly convexity for Musielak-Orlicz spaces equipped with the Luxemburg norm coincide with reflexivity. Finally, an interpretation of those results for Nakano spaces $l^{(p_i)}$ $(1 < p_i < \infty)$ is given.

Keywords: *Fatou property, order continuity, nearly uniformly convexity, nearly uniformly smoothness, nearly uniformly *-smoothness, Musielak-Orlicz sequence spaces*

AMS subject classification: 46 E 30, 46 E 40, 46 B 20

1. Introduction

Let $(X, \|\cdot\|)$ be a real Banach space and X^* be the dual space of X. By $B(X)$ and $S(X)$ we denote the closed unit ball and the unit sphere of X, respectively. For any subset A of X by $\mathrm{conv}(A)$ we denote the convex hull of A. In the sequel $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_+^e$ stand for the set of natural numbers, the set of reals, the set of non-negative reals and the interval $[0, +\infty]$, respectively.

The following notions used in the paper can be found in [14: Chapter 1].

A sequence (x_n) in a real Banach space X is called a *Schauder basis* of X (or *basis* for short) if for each $x \in X$ there exists a unique sequence (a_n) of reals such that

$$\left\| x - \sum_{n=1}^{k} a_n x_n \right\| \longrightarrow 0 \qquad \text{as } k \to \infty.$$

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A sequence (x_n) which is a Schauder basis of its closed linear span is called a *basic sequence.*

A basis (x_n) of X is said to be an *unconditional basis* if every convergent series $\sum_{n=1}^{\infty} a_n x_n$ with $a_n \in \mathbb{R}$ is unconditionally convergent, i.e. for any permutation $(\pi(n))$ of $\mathbb{N}$ the series $\sum_{n=1}^{\infty} a_{\pi(n)} x_{\pi(n)}$ converges.

For a basis (x_n) of X, its *basic constant* is defined by $K = \sup_n \|P_n\|$, where $P_n : X \to X$ are projections defined by

$$P_n\left(\sum_{i=1}^{\infty} a_i x_i\right) = \sum_{i=1}^{n} a_i x_i.$$

If (x_n) is a basis of X such that the series $\sum_{n=1}^{\infty} a_n x_n$ converges whenever (a_n) is a sequence of reals such that

$$\sup_n \left\|\sum_{i=1}^{n} a_i x_i\right\| < \infty,$$

then (x_n) is said to be a *boundedly complete basis.* It is known that (x_n) is a boundedly complete basis of a Banach space X if and only if (x_n) is an unconditional basis and X is weakly sequentially complete.

Recall that X is said to be *weakly sequentially complete* if for any sequence (y_n) in X such that $\lim_n x^*(y_n)$ exists for every $x^* \in X^*$, there is $y \in X$ such that $y_n \to y$ weakly.

Clarkson [5] introduced the concept of uniform convexity. The norm $\|\cdot\|$ is called *uniformly convex* if for each $\varepsilon > 0$ there is $\delta > 0$ such that for $x, y \in S(X)$ the inequality $\|x - y\| > \varepsilon$ implies $\|\frac{1}{2}(x + y)\| < 1 - \delta$.

A Banach space X is said to have the *Kadec-Klee property* if every sequence from $S(X)$ weakly convergent to an element $x \in S(X)$ is convergent to x in norm. Recall that for a given $\varepsilon > 0$ a sequence (x_n) is said to be *ε-separated* if

$$\mathrm{sep}(x_n) = \inf_{m \neq n} \{\|x_n - x_m\|\} > \varepsilon.$$

A Banach space X is said to have the *uniform Kadec-Klee property* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if x is a weak limit of an ε-separated sequence in $S(X)$, then $\|x\| < 1 - \delta$.

The notion of nearly uniformly convexity for Banach spaces was introduced in [11]. It is an infinite dimensional counterpart of the classical uniform convexity. A Banach space is said to be *nearly uniformly convex* if for every $\varepsilon > 0$ there exists $\delta \in (0, 1)$ such that for every sequence $(x_n) \subset B(X)$ with $\mathrm{sep}(x_n) > \varepsilon$, there holds

$$\mathrm{conv}(\{x_n\}) \cap (1 - \delta)B(X) \neq \phi.$$

It is easy to see that every nearly uniformly convex space has the uniform Kadec-Klee property, and every Banach space with the uniform Kadec-Klee property has the Kadec-Klee property. Huff [11] proved that X is nearly uniformly convex if and only if X is reflexive and has the uniform Kadec-Klee property.

A Banach space X is said to be *nearly uniformly smooth* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for each basic sequence (x_n) in $B(X)$ there is $k > 1$ such that

$$\|x_1 + tx_k\| \leq 1 + t\varepsilon$$

for each $t \in [0, \delta]$ (see [17, 18]). Originally, this property was defined in [20] in a different way. Prus [17] showed that a Banach space X is nearly uniformly convex if and only if X^* is nearly uniformly smooth.

For $x \in S(X)$ and a positive number δ, denote

$$S^*(x, \delta) = \{x^* \in B(X^*) : x^*(x) \geq 1 - \delta\}.$$

Let A be a bounded subset of X . Its *Kuratowski measure of non-compactness* $\alpha(A)$ is defined as the infimum of all numbers $d > 0$ such that A may be covered by finitely many sets of diameter smaller than d (see [1, 2]).

A Banach space X is said to be nearly uniformly *-smooth provided that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in S(X)$, then

$$\alpha(S^*(x, \delta)) \leq \varepsilon.$$

A Banach space X is said to have *property* A_2 if there exists $\Theta \in (0, 2)$ such that for each weakly null sequence (x_n) in $S(X)$, there are $n_1, n_2 \in \mathbb{N}$ satisfying $\|x_{n_1} + x_{n_2}\| < \Theta$. It is well known that if X has property A_2, then it has the weak Banach-Saks property (see [7]).

A Banach space X is said to have *property* A_2^ε if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for each weakly null sequence (x_n) in $B(X)$, there is $k \in \mathbb{N} \setminus \{1\}$ satisfying $\|x_1 + tx_k\| < 1 + t\varepsilon$ whenever $t \in [0, \delta]$. Prus [18] proved that X is nearly uniformly *-smooth if and only if X has property A_2^ε and contains no copy of l_1. Moreover, he also showed that if X is nearly uniformly *-smooth, then it has the weak Banach-Saks property.

The space of all real sequences $x = (x(i))$ is denoted by l^0. A Banach space X is called a *Köthe sequence space* if it is a subspace of l^0 equipped with a norm $\|\cdot\|$ such that for every $x = (x(i)) \in l^0$ and $y = (y(i)) \in X$ satisfying $|x(i)| \leq |y(i)|$ for all $i \in \mathbb{N}$, there hold $x \in X$ and $\|x\| \leq \|y\|$.

X is said to have the *Fatou property*, if $0 \leq x_n \uparrow x$ with $x_n \in X$, $x \in l^0$, $\sup_{n\in\mathbb{N}}\{\|x_n\|\} < \infty$ imply $x \in X$ and $\lim_{n\to\infty} \|x_n\| = \|x\|$.

We say an element $x \in X$ is *order continuous* if for any sequence (x_n) in X such that $|x_n(i)| \to 0$ and $|x_n(i)| \leq |x(i)|$ $(i \in \mathbb{N})$ we have $\lim_{n\to\infty} \|x_n\| = 0$. It is easy to see that x is order continuous if and only if $\lim_{n\to\infty} \|\sum_{i=n}^{\infty} x(i)e_i\| = 0$. The space X is called *order continuous* if every $x \in X$ is order continuous.

A mapping $\Phi : \mathbb{R} \to \mathbb{R}_+^e$ is said to be an *Orlicz function* if Φ is vanishing only at 0, even, convex and left continuous on the whole $\mathbb{R}_+$ (see [13, 16, 19]). An Orlicz function Φ is said to be an *N-function* if $\lim_{u\to 0} \frac{\Phi(u)}{u} = 0$ and $\lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty$. A sequence $\Phi = (\Phi_i)$ of Orlicz functions is called a *Musielak-Orlicz function*. By $\Psi = (\Psi_i)$ we denote the complementary function of Φ in sense of Young, i.e.

$$\Psi_i(v) = \sup\{|v|u - \Phi_i(u) : u \geq 0\} \qquad (i \in \mathbb{N}).$$

For a given Musielak-Orlicz function Φ, we define a convex modular

$$I_\Phi(x) = \sum_{i=1}^{\infty} \Phi_i(x_i)$$

for any $x \in l^0$. A linear space l_Φ defined by

$$l_\Phi = \left\{x \in l^0 : I_\Phi(cx) < \infty \text{ for some } c > 0\right\}$$

is called the *Musielak-Orlicz sequence space generated by* Φ. We consider l_Φ equipped with the *Luxemburg norm*

$$\|x\| = \inf\left\{\varepsilon > 0 : I_\Phi\left(\frac{x}{\varepsilon}\right) \leq 1\right\}$$

or with the *Amemiya-Orlicz norm*

$$\|x\|_0 = \inf\left\{\frac{1}{k}(1 + I_\Phi(kx)) : \, k > 0\right\}.$$

To simplify notations, we assume $l_\Phi = (l_\Phi, \|\cdot\|)$ and $l_\Phi^0 = (l_\Phi, \|\cdot\|_0)$. Both l_Φ and l_Φ^0 are Banach spaces (see [3, 16]).

We say a Musielak-Orlicz function Φ satisfies the δ_2-*condition* ($\Phi \in \delta_2$ for short) if there exist constants $k \geq 2$ and $a > 0$ and a sequence (c_i) in $\mathbb{R}_+$ such that $\sum_{i=1}^{\infty} c_i < \infty$ and the inequality

$$\Phi_i(2u) \leq k\Phi_i(u) + c_i$$

holds for every $i \in \mathbb{N}$ and every $u \in \mathbb{R}$ satisfying $\Phi_i(u) \leq a$.

In the sequel h_Φ stands for the space $\{x \in l^0 : I_\Phi(lx) < \infty \text{ for any } l > 0\}$ equipped with the norm induced from l_Φ. To indicate that it is considered with the Orlicz norm, we write h_Φ^0.

Let us recall three results which will be used in the following.

Lemma 1 (see [9]). *If $\Phi = (\Phi_i)$ is a Musielak-Orlicz function with all Φ_i being finitely valued, Φ satisfies the δ_2-condition and (x_n) is a sequence in l_Φ such that $I_\Phi(x_n) \to 0$ as $n \to \infty$, then $\|x_n\| \to 0$ as $n \to \infty$.*

Lemma 2 (see [6]). *If a Musielak-Orlicz function $\Psi = (\Psi_i)$ satisfies the δ_2-condition, then for each $\lambda, \varepsilon \in (0,1)$ there is $\theta \in (0,1)$ and a sequence (h_i) in $\mathbb{R}_+$ with $\sum_{i=1}^{\infty} \Phi_i(h_i) < \varepsilon$ such that*

$$\Phi_i(\lambda u) \leq \lambda\theta\Phi_i(u)$$

holds for every $i \in \mathbb{N}$ and $u \in \mathbb{R}$ satisfying $\Phi_i(h_i) \leq \Phi_i(u) \leq 1$.

Lemma 3 (see [3, 8, 21]). *If $\Phi = (\Phi_i)$ is a Musielak-Orlicz function with all Φ_i being finitely-valued N-functions, then for each $x \neq 0$ in l_Φ^0 there is $k > 0$ such that*

$$\|x\|_0 = \frac{1}{k}(1 + I_\Phi(kx)).$$

For more details on Musielak-Orlicz spaces we refer to [3] or [16].

2. Results

We start with some general results which improve the result of Prus [18] that nearly uniformly *-smooth Banach spaces have the weak Banach-Saks property.

Theorem 1. *If a Banach space X has property* A_2^ε, then X has property A_2.

Proof. For $\varepsilon = \frac{1}{2}$ there is $\delta \in (0,1)$ such that for each weakly null sequence (x_n) in $S(X)$ there is $k > 1$ such that

$$\|x_1 + tx_k\| < 1 + \frac{t}{2} \qquad (t \in [0,\delta]).$$

Hence

$$\|x_1 + x_k\| = \|x_1 + \delta x_k + (1-\delta)x_k\| < 1 + \frac{\delta}{2} + (1-\delta) = 2 - \frac{\delta}{2} = \Theta < 2$$

and the statement is proved ∎

Now we will present the following useful remark.

Remark 1. A Banach space X is reflexive if and only if X contains no isomorphic copy of l_1 and X is weakly sequentially complete.

Indeed, since l_1 is not reflexive, a reflexive Banach space cannot contain an isomorphic copy of l_1. Moreover, any reflexive Banach space X is weakly sequentially complete. If X contains no isomorphic copy of l_1, by the well known Rosenthal theorem, for every sequence (x_n) in $B(X)$ there exists a subsequence (z_n) of (x_n) which is a weakly Cauchy sequence. So, if X is additionally weakly sequentially complete, we get that (x_n) is relatively weakly sequentially compact. Hence X is reflexive ∎

Corollary 1. *A Banach space X is nearly uniformly smooth if and only if X is nearly uniformly *-smooth and weakly sequentially complete.*

Proof. It is obvious that X is nearly uniformly *-smooth and weakly sequentially complete if it is nearly uniformly smooth. Assume now that X is nearly uniformly *-smooth and weakly sequentially complete. Since nearly uniformly *-smoothness of X implies that X contains no copy of l_1, by Remark 1, X is reflexive, whence nearly uniformly *-smoothness coincides with nearly uniformly smoothness ∎

So, we can now easily understand why c_0 is not nearly uniformly smooth although it is nearly uniformly *-smooth.

Theorem 2. *Let X be a Köthe sequence space. If X^* contains no isomorphic copy of l^1 and has property* A_2^ε, *then X has the uniform Kadec-Klee property.*

Proof. Since X^* contains no isomorphic copy of l^1, for every sequence (x_n^*) in $B(X^*)$ there is a weak Cauchy subsequence $(x_{n_k}^*)$. It is obvious that the sequence $(x_{n_k}^* - x_{n_l}^*)$ is weakly null. By the assumption that X^* has property A_2^ε, there are $n > k > 1$ such that

$$\|x_1^* + t(x_k^* - x_n^*)\| < 1 + \frac{t\varepsilon}{32} \qquad (t \in [0,\delta])$$

(see [18]). Let (x_n) be a sequence in $S(X)$ with $\operatorname{sep}(x_n) > \varepsilon$ and $x_n \to x \in X$ weakly. Then $\operatorname{sep}(x_n - x) > \varepsilon$. We need to show that $\|x\| < 1 - \eta(\varepsilon)$, where $\eta(\varepsilon)$ depends only on ε. Put $K = \frac{32+2\delta\varepsilon}{32+\delta\varepsilon}$. By the Bessaga-Pełczyński selection principle, there exists a subsequence (z_n) of $\{x_n - x, x : n \in \mathbb{N}\}$ with $z_1 = x$ being a basic sequence with basic constant less or equal to K. Put $X_0 = \overline{\operatorname{span}}\{z_n : n \in \mathbb{N}\}$. Let us consider the sequence (z_n^*) of the norm preserving extensions from X_0 to the whole X of the coefficient functionals for the basic sequence (z_n). Then $\langle z, z_n^* \rangle \to 0$ as $n \to \infty$ for any $z \in X_0$. Indeed, $z = \sum_{i=1}^{\infty} z_i^*(z) z_i$ for any $z \in X_0$, whence

$$\begin{aligned}
\|z_n^*(z)z_n\| &= \left\| \sum_{i=n}^{\infty} z_i^*(z)z_i - \sum_{i=n+1}^{\infty} z_i^*(z)z_i \right\| \\
&\le \left\| \sum_{i=n}^{\infty} z_i^*(z)z_i \right\| + \left\| \sum_{i=n+1}^{\infty} z_i^*(z)z_i \right\| \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Since $\|z_n\| > \frac{\varepsilon}{2}$ for all n, this yields $z_n^*(z) \to 0$ as $n \to \infty$.

Let us write $\langle x, z_k^* \rangle$ for $z_k^*(x)$ and take $n > k > 1$ large enough such that $|\langle x, z_k^* \rangle| < \frac{\varepsilon}{32}$ and $|\langle x, z_n^* \rangle| < \frac{\varepsilon}{32}$. Notice that $\|z_1^*\| \le K$ and $\|z_k^*\| \le 2K$ for $k > 1$. Hence, taking into account that $\|x + z_k\| = 1$ for $k > 1$ and applying property $\mathrm{A}_2^{\varepsilon}$ for X^*, we get

$$\left\| z_1^* + \frac{\delta}{2}(z_k^* - z_n^*) \right\| \le K\left(1 + \frac{\delta\varepsilon}{32}\right)$$

and consequently

$$\begin{aligned}
\|x\| &= \langle x, z_1^* \rangle \\
&= \langle x + z_k, z_1^* \rangle + \frac{\delta}{2}\langle x + z_k, z_k^* - z_n^* \rangle - \frac{\delta}{2}\langle x + z_k, z_k^* - z_n^* \rangle \\
&= \left\langle x + z_k, z_1^* + \frac{\delta}{2}(z_k^* - z_n^*) \right\rangle - \frac{\delta}{2}\langle x + z_k, z_k^* - z_n^* \rangle \\
&\le \left\| z_1^* + \frac{\delta}{2}(z_k^* - z_n^*) \right\| - \frac{\delta}{2}\|z_k\| + |\langle x, z_k^* \rangle| + |\langle x, z_n^* \rangle| \\
&< K\left(1 + \frac{\delta\varepsilon}{32}\right) - \frac{\delta\varepsilon}{4} + \frac{\delta\varepsilon}{16} \\
&= K\left(1 + \frac{\delta\varepsilon}{32}\right) - \frac{3\delta\varepsilon}{16} \\
&\le \left(1 + \frac{\delta\varepsilon}{16}\right) - \frac{3\delta\varepsilon}{16} \\
&= 1 - \frac{\delta\varepsilon}{8}
\end{aligned}$$

which finishes the proof ∎

Lemma 4. *Let $\Phi = (\Phi_i)$ be a finitely-valued Musielak-Orlicz function such that Φ^* satisfies the δ_2-condition. Then for every $\varepsilon > 0$, $\lambda \in (0,1)$ and $K \geq 1$ there are $(h_i)_{i=1}^{\infty} \subset \mathbb{R}_+$ and $\theta \in (0,1)$ such that $\sum_{i=1}^{\infty} \Phi_i(h_i) < \varepsilon$ and the inequality*

$$\Phi_i(\gamma u) \leq \gamma\theta\Phi_i(u)$$

holds for all $i \in \mathbb{N}$ and $u \geq 0$ satisfying the inequalities $\Phi_i(h_i) \leq \Phi_i(u) \leq K$ and all $\gamma \in (0, \lambda]$.

Proof. It is known from [6] that our lemma is true for $K = 1$ under the additional assumption that $\Phi_i(1) = 1$ for all $i \in \mathbb{N}$. Let $a_i > 0$ be such that $\Phi_i(a_i) = K$ for all $i \in \mathbb{N}$ and define $\phi_i(u) = \frac{1}{K}\Phi_i(a_i u)$ for all $u \in \mathbb{R}$ and $i \in \mathbb{N}$. Then $\phi = (\phi_i)$ is a Musielak-Orlicz function such that $\phi_i(1) = 1$ for all $i \in \mathbb{N}$. Since, denoting by ϕ_i^* and Φ_i^* the complementary functions of ϕ_i and Φ_i, respectively, there holds

$$\phi_i^*(u) = \frac{1}{K}\Phi_i^*\left(\frac{K}{a_i}u\right)$$

for all $i \in \mathbb{N}$ and $u \in \mathbb{R}$, we know that ϕ^* satisfies the δ_2-condition. By the above mentioned result from [6] there are $(h_i')_{i=1}^{\infty} \subset \mathbb{R}_+$ and $\theta \in (0,1)$ such that

$$\sum_{i=1}^{\infty} \phi_i(h_i') < \frac{\varepsilon}{K} \qquad \text{and} \qquad \phi_i(\gamma u) \leq \gamma\theta\phi_i(u)$$

for all $\gamma \in (0, \lambda]$, $i \in \mathbb{N}$ and $u \in \mathbb{R}$ satisfying $\phi_i(h_i') \leq \phi_i(u) \leq 1$. Setting $h_i = a_i h_i'$ for each $i \in \mathbb{N}$, we easily see that it is just the desired result ∎

Theorem 3. *If $\Phi = (\Phi_i)$ is a Musielak-Orlicz function with all Φ_i being finitely-valued N-functions, then l_Φ^0 is nearly uniformly convex if and only if Φ and Ψ satisfy the δ_2-condition.*

Proof. We need only to prove the sufficiency. Since l_Φ^0 is reflexive, it suffices to prove that l_Φ^0 has the uniform Kadec-Klee property. Let $\varepsilon > 0$ be given and take any sequence $\{x_n\} \subset S(l_\Phi^0)$ with $\operatorname{sep}(x_n) > 2\varepsilon$ and $x_n \xrightarrow{w} x$. It is clear that for any $m \in \mathbb{N}$ there is $n_m \in \mathbb{N}$ such that

$$\operatorname{sep}\left(\left(\sum_{i=m+1}^{\infty} x_n(i)e_i\right)_{n=n_m}^{\infty}\right) > 2\varepsilon.$$

This follows by the fact that $x_n \xrightarrow{w} x$ implies that $x_n \to x$ coordinatewise. Hence for any $m \in \mathbb{N}$ there is $n_m \in \mathbb{N}$ such that

$$\left\|\sum_{i=m+1}^{\infty} x_n(i)e_i\right\|_0 \geq \varepsilon \qquad (n \geq n_m). \tag{1}$$

By Lemma 3 there are $k_n \geq 1$ and $k \geq 1$ such that

$$\|x_n\|_0 = \frac{1}{k_n}(1 + I_\Phi(k_n x_n)) \qquad (n \in \mathbb{N})$$

and

$$\|x\|_0 = \frac{1}{k}(1 + I_\Phi(kx)).$$

Then $K = \sup_n k_n < \infty$. Indeed, since $\|x\|_0 > 1-\delta$, there is $i_0 \in \mathbb{N}$ such that $x_0(i_0) \neq 0$. If $K = \infty$, we can assume without loss of generality that $\lim_{n\to\infty} k_n = \infty$. Hence

$$\begin{aligned} 1 &= \frac{1}{k_n}(1 + I_\Phi(k_n x_n)) \\ &= \lim_{n\to\infty} \frac{1}{k_n} I_\Phi(k_n x_n) \\ &\geq \liminf_{n\to\infty} \frac{1}{k_n} \Phi_{i_0}(k_n x_n(i_0)) \\ &\to \infty \end{aligned}$$

which is a contradiction.

By the δ_2-condition of Φ and inequality (1) there is $\delta > 0$ such that

$$\sum_{i=m+1}^{\infty} \Phi_i(x_n(i)) \geq \delta \qquad (n \geq n_m). \tag{2}$$

Put $\lambda = \frac{K}{K+1}$. Then, by Lemma 4, there is $h = (h_i)_{i=1}^{\infty}$ with $\sum_{i=1}^{\infty} \Phi_i(h_i) \leq \frac{K}{2}$ and a number $\theta \in (0,1)$ such that

$$\Phi_i(\gamma u) \leq \gamma(1-\theta)\Phi_i(u)$$

for all $\gamma \in [0, \lambda]$ and $u \in \mathbb{R}$ satisfying $\Phi_i(h_i) \leq \Phi_i(u) \leq K$. Take m large enough such that

$$\left\| \sum_{i=m+1}^{\infty} x(i) e_i \right\|_0 < \frac{\delta\theta}{8} \tag{3}$$

and

$$\left\| \sum_{i=m+1}^{\infty} h_i e_i \right\|_0 < \frac{\delta\theta}{8}. \tag{4}$$

Since $\frac{k}{k_n+k} \leq \frac{k}{k+1} \leq \frac{K}{K+1}$ for any $n \in \mathbb{N}$, we have

$$\Phi_i\left(\frac{k k_n}{k+k_n} x_n(i)\right) \leq \frac{1-\theta}{k_n+k} k \Phi_i(k_n x_n(i))$$

whenever $|x_n(i)| \geq h_i$. Therefore,

$$\sum_{i=m+1}^{\infty} \Phi_i\left(\frac{k k_n}{k+k_n} x_n(i)\right) \leq \sum_{i=1}^{\infty} \Phi_i(h_i) + \frac{1-\theta}{k_n+k} k \sum_{i=1}^{\infty} \Phi_i(k_n x_n(i)). \tag{5}$$

It is obvious that

$$\begin{aligned}\|x_n + x\|_0 &= \left\|\sum_{i=1}^{m} x(i)e_i + \sum_{i=m+1}^{\infty} x(i)e_i + x_n\right\|_0 \\ &\le \left\|\sum_{i=1}^{m} x(i)e_i + x_n\right\|_0 + \left\|\sum_{i=m+1}^{\infty} x(i)e_i\right\|_0 \\ &\le \left\|\sum_{i=1}^{m} x(i)e_i + x_n\right\|_0 + \frac{\delta\theta}{8}\end{aligned} \tag{6}$$

for m large enough. Moreover, by (3) - (5), we get for $n \ge n_m$

$$\begin{aligned}&\left\|\sum_{i=1}^{m} x(i)e_i + x_n\right\|_0 \\ &\le \frac{k_n + k}{k_n k}\left(1 + \sum_{i=1}^{m} \Phi_i\left(\frac{kk_n}{k_n + k}(x(i) + x_n(i))\right)\right. \\ &\quad \left. + \sum_{i=m+1}^{\infty} \Phi_i\left(\frac{kk_n}{k_n + k}x_n(i)\right)\right) \\ &\le \frac{k_n + k}{k_n k}\left(1 + \frac{k_n}{k_n + k}\sum_{i=1}^{m} \Phi_i(kx(i)) + \frac{k}{k_n + k}\sum_{i=1}^{m} \Phi_i(k_n x_n(i))\right. \\ &\quad \left. + \frac{1-\theta}{k_n + k}k\sum_{i=m+1}^{\infty} \Phi_i\left(\frac{kk_n}{k_n + k}x_n(i)\right) + \sum_{i=m+1}^{\infty} \Phi_i(h_i)\right) \\ &= \frac{1}{k} + \frac{1}{k_n} + \frac{1}{k}\sum_{i=1}^{m} \Phi_i(kx(i)) + \frac{1}{k_n}\sum_{i=1}^{m} \Phi_i(k_n x_n(i)) \\ &\quad + \frac{1}{k_n}\sum_{i=m+1}^{\infty} \Phi_i(k_n x_n(i)) + \sum_{i=m+1}^{\infty} \Phi_i(h_i) - \frac{\theta}{k_n}\sum_{i=m+1}^{\infty} \Phi_i(k_n x_n(i)) \\ &\le \frac{1}{k}\left(1 + \sum_{i=1}^{m} \Phi_i(kx(i))\right) + \frac{1}{k_n}(1 + I_\Phi(k_n x_n)) \\ &\quad + \sum_{i=m+1}^{\infty} \Phi_i(h_i) - \frac{\theta}{k_n}\sum_{i=m+1}^{\infty} \Phi_i(k_n x_n(i)) \\ &\le 2 + \frac{\delta\theta}{8} - \delta\theta.\end{aligned} \tag{7}$$

Therefore, combining (6) and (7), we obtain

$$\|x_n + x\|_0 \le 2 + \frac{\delta\theta}{8} - \delta\theta + \frac{\delta\theta}{8} = 2 - \frac{3}{4}\theta \qquad (n > n_m).$$

Hence, by $x_n \xrightarrow{w} x$ and the lower semicontinuity of the norm with respect to the weak topology, we deduce that

$$\|x\|_0 \leq \lim_{n\to\infty} \left\| \frac{x_n + x}{2} \right\|_0 \leq \frac{1}{2}\left(2 - \frac{3}{4}\theta\right) = 1 - \frac{3}{8}\theta.$$

This contradiction finishes the proof ∎

Theorem 4. *For any Musielak-Orlicz function $\Phi = (\Phi_i)$ with all Φ_i being finitely-valued N-functions the following statements are equivalent:*

(a) *l_Φ is nearly uniformly smooth.*

(b) *l_Φ is nearly uniformly *-smooth.*

(c) *Φ and Ψ satisfy the δ_2-condition.*

Proof. $(c) \Rightarrow (a)$: By Theorem 3, l^0_Ψ is nearly uniformly convex, so its dual l_Φ is nearly uniformly smooth. Therefore, we need only to prove that $(b) \Rightarrow (c)$. We will show that (b) implies the δ_2-condition for Φ. If Φ does not satisfy the δ_2-condition, we can construct $x \in S(l_\Phi)$ such that $I_\Phi(x) \leq 1$ and $I_\Phi((1+\frac{1}{n})x) = \infty$ for every $n \in \mathbb{N}$ (see [12]). Take a sequence (i_k) of natural numbers such that $i_k \uparrow$ and

$$\sum_{i=i_k+1}^{i_{k+1}} \Phi_i\left(\left(1+\frac{1}{k}\right)x(i)\right) \geq 1 \qquad (k \in \mathbb{N}).$$

Put

$$x_k = \Big(0,0,\dots,0,x(i_k+1),x(i_k+2),\dots,x(i_{k+1}),0,0,\dots) \qquad (k \in \mathbb{N}).$$

Then it is obvious that

$$\frac{k}{k+1} \leq \|x_k\| \leq 1 \qquad (k \in \mathbb{N}).$$

Moreover,

$$x_k \longrightarrow 0 \qquad \text{weakly.} \tag{8}$$

Indeed, for every $y \in (l_\Phi)^*$ we have $y^* = y_0^* + y_1^*$ uniquely, where y_0^* is the regular part of y^* and y_1^* is the singular part of y^*, i.e. $y_1^*(x) = 0$ for any $x \in h_\Phi$ (see [10]). The functional y_0^* is generated by some $y_0 \in l_\Psi$ by the formula

$$y_0^*(x) = \langle x, y_0\rangle = \sum_{i=1}^{\infty} x(i)y_0(i) \qquad (x \in l_\Phi).$$

Let $\lambda > 0$ be such that $\sum_{i=1}^{\infty} \Psi_i(\lambda y_0(i)) < \infty$. Since $x_k \in h_\Phi$ for any $k \in \mathbb{N}$, we have

$$\begin{aligned}\langle x_k, y^*\rangle &= \langle x_k, y_0^*\rangle \\ &= \sum_{i=i_k+1}^{i_{k+1}} x(i)y_0(i) \\ &\leq \frac{1}{\lambda}\left(\sum_{i=i_k+1}^{i_{k+1}} \Phi_i(x(i)) + \sum_{i=i_k+1}^{i_{k+1}} \Psi_i(\lambda y_0(i))\right) \\ &\to 0 \quad \text{as} \quad k \to \infty,\end{aligned}$$

i.e. (8) holds.

Since the space l_Φ is nearly uniformly *-smooth, it has property A_2^ε, i.e. for any $\varepsilon > 0$ there exists $\delta \in (0,1)$ such that for each weakly null sequence (z_n) in $B(l_\Phi)$ there is $m > 1$ such that

$$\|z_1 + tz_m\| \leq 1 + t\varepsilon$$

whenever $t \in [0,\delta]$ (see [18]). Take $k_0 \in \mathbb{N}$ such that $\frac{2}{k+1} < (1-\varepsilon)\delta$ if $k \geq k_0$. We have for $k \geq k_0$

$$\begin{aligned} 1 + \delta\varepsilon \geq \|x + \delta x_k\| \geq \|(1+\delta)x_k\| \geq (1+\delta)\frac{k}{k+1} \\ = (1+\delta)\left(1 - \frac{1}{k+1}\right) > 1 + \delta - \frac{2}{k+1} \end{aligned}$$

whence $\frac{2}{k+1} > (1-\varepsilon)\delta$. This is a contradiction which finishes the proof of the fact that (b) implies the δ_2-condition for Φ.

Next, we will show that (b) implies the δ_2-condition for Ψ. By the above part of the proof, we can assume that l_Φ is nearly uniformly *-smooth and Φ satisfies the δ_2-condition. So, l_Φ is order continuous. Moreover, any Musielak-Orlicz space l_Φ has the Fatou property and consequently, it is weakly sequentially complete. So, in view of Corollary 1, l_Φ is nearly uniformly smooth and consequently reflexive. This yields the δ_2-condition for Ψ ∎

Theorem 5. *Let $\Phi = (\Phi_i)$ be a Musielak-Orlicz function with all Φ_i being finitely-valued N-functions. Then Φ and Ψ satisfy the δ_2-condition whenever l_Φ^0 is nearly uniformly *-smooth.*

Proof. Since l_Φ^0 is nearly uniformly *-smooth, it has property A_2^ε, i.e. for any $\varepsilon > 0$ there exists $\delta \in (0,1)$ such that for each weakly null sequence (z_n) in $B(l_\Phi^0)$ there is $m \in \mathbb{N} \setminus \{1\}$ such that

$$\|z_1 + tz_m\|_0 \leq 1 + t\varepsilon$$

for all $t \in [0,\delta]$. Let $\theta \in (0,1)$ be such that $1 + \delta\varepsilon < (1+\delta)\theta$. If Φ does not satisfy the δ_2-condition, then there exists $x \in S(l_\Phi^0)$ and a sequence $\{n_i\}$ of natural numbers $n_i \uparrow$ such that $n_1 = 1$ and

$$\left\|\sum_{i=n_k}^{n_{k+1}} x(i)\right\|_0 \geq \theta \qquad (k \in \mathbb{N})$$

(see [4]). Define

$$x_k = \sum_{i=n_k}^{n_{k+1}} x(i) \qquad (k \in \mathbb{N}).$$

Then we can prove in the same way as for the Luxemburg norm (see the proof of Theorem 4) that (x_k) is a weakly null sequence. Therefore, there is $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$

$$1 + \delta\varepsilon \geq \|x + \delta x_k\|_0 \geq \|(1+\delta)x_k\|_0 \geq (1+\delta)\theta.$$

This is a contradiction which shows the necessity of the δ_2-condition of Φ for the nearly uniformly *-smoothness of l_Φ^0.

The necessity of the δ_2-condition of Ψ can be proved in the same way as for the Luxemburg norm in Theorem 4, since the Amemiya-Orlicz norm has the Fatou property ∎

Recall that the *Nakano space* $l^{(p_i)}$ is the Musielak-Orlicz space l_Φ with $\Phi = (\Phi_i)$, where

$$\Phi_i(u) = |u|^{p_i} \qquad (1 < p_i < +\infty, i \in \mathbb{N}).$$

Corollary 2. *For both the Luxemburg and the Amemiya-Orlicz norms the following statements are equivalent:*

(a) $l^{(p_i)}$ *is nearly uniformly convex.*

(b) $l^{(p_i)}$ *is nearly uniformly smooth.*

(c) $l^{(p_i)}$ *is nearly uniformly *-smooth.*

(d) $1 < \liminf_{i\to\infty} p_i \le \limsup_{i\to\infty} p_i < +\infty$.

Proof. If $\Phi_i(u) = |u|^{p_i}$ for all $u \in \mathbb{R}$ and $i \in \mathbb{N}$, then the complementary functions Ψ_i of Φ_i are defined by the formula

$$\Psi_i(u) = c_i|u|^{q_i}$$

where $\frac{1}{p_i} + \frac{1}{q_i} = 1$ and $c_i = (p_i)^{1/p_i}(q_i)^{1/q_i}$ for all $i \in \mathbb{N}$. It is easy to see that $\Phi = (\Phi_i)$ satisfies the δ_2-condition if and only if $\limsup_{i\to\infty} p_i < +\infty$. Moreover, $\Psi = (\Psi_i)$ satisfies the δ_2-condition if and only if $\liminf_{i\to\infty} p_i > 1$.

Now, we prove the equivalence of the conditions.

$(d) \Rightarrow (a)$: Assume first that $l^{(p_i)}$ is equipped with the Amemiya-Orlicz norm. Then, by Theorem 4, l_Ψ is nearly uniformly smooth. So $l^{(p_i)}$ is nearly uniformly convex as well. It follows in the same way that condition (d) implies that l_Ψ is nearly uniformly convex. Therefore, by the fact that a Banach space X is nearly uniformly convex if and only if X^* is nearly uniformly smooth and that if both Musielak-Orlicz functions Φ and Ψ satisfy the δ_2-condition, then $(l_\Phi)^* \cong l_\Psi^0$ and $(l_\Phi^0)^* \cong l_\Psi$ (see [3, 15, 16, 19]), we deduce that (a) and (b) are equivalent for both norms. By Theorem 4, conditions (b), (c) and (d) are pairwise equivalent. The implication (b) $\Rightarrow$ (c) holds in general and, by Theorem 5, (c) $\Rightarrow$ (d) in the case of the Amemiya-Orlicz norm. This completes the proof ∎

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