Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 17 (1998), No. 4, 907–915

Note on the Fourier-Laplace Transform of $\bar\partial$-Cohomology Classes

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Abstract. We construct the inverse of the Fourier-Laplace transform of $\bar\partial$-cohomology classes (of $(n, n-1)$-forms) in the complement of a convex compact set in $\mathbb{C}^n$, thus giving an analogue of the Borel transform (and its Polya representation) of entire functions of exponential type in several variables. The construction is based on a formula of Berndtsson.

Keywords: *Fourier-Laplace transform, $\bar\partial$-cohomology classes*

AMS subject classification: 32 A

1. Introduction

Let us consider a convex compact set $K \subset \mathbb{C}^n$ and the set $Z^{(n,n-1)}(\mathbb{C}^n \setminus K)$ of $\bar\partial$-closed $(n, n-1)$-forms in $\mathbb{C}^n \setminus K$. Then to each form $\theta \in Z^{(n,n-1)}(\mathbb{C}^n \setminus K)$ we may associate an entire analytic function F_θ (its Fourier-Laplace transform) defined by

$$F_\theta(\zeta) = \int_{z\in S} e^{\langle z,\zeta\rangle}\theta(z) \qquad (\zeta \in \mathbb{C}^n)$$

where $\langle z, \zeta\rangle = \sum z_j\zeta_j$ and S is a smooth $(2n-1)$-dimensional closed surface surrounding K. By the Stokes formula, F_θ does not depend on the choice of the surface S. This function belongs to the space $\mathbb{A}_K(\mathbb{C}^n)$ of entire analytic functions F for which, for every $\delta > 0$, there is a constant $C_\delta > 0$ such that

$$|F(\zeta)| \le C_\delta \exp\left(H_K(\zeta) + \delta|\zeta|\right) \qquad (\zeta \in \mathbb{C}^n)$$

where

$$H_K(\zeta) = \sup\left\{\mathrm{Re}\langle z,\zeta\rangle : z \in K\right\}.$$

Notice also that, in the case $n = 1$, $F_\theta \equiv 0$ precisely when $\theta = f(z)\,dz$ where f extends to an analytic function in $\mathbb{C}$. In the case $n \ge 2$, $F_\theta \equiv 0$ if and only if $\theta \in B^{(n,n-1)}(\mathbb{C}^n \setminus K)$, i.e. when θ is $\bar\partial$-exact in $\mathbb{C}^n \setminus K$ (see Lemma 5). Thus there is defined a one-to-one linear map

$$\mathbb{E} : H^{(n,n-1)}(\mathbb{C}^n \setminus K) \to \mathbb{A}_K(\mathbb{C}^n), \qquad \mathbb{E}([\theta]) = F_\theta$$

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ISSN 0232-2064 / $ 2.50

on the space of $\overline{\partial}$-cohomology classes, i.e.

$$[\theta] \in H^{(n,n-1)}(\mathbb{C}^n \setminus K) = Z^{(n,n-1)}(\mathbb{C}^n \setminus K)/B^{(n,n-1)}(\mathbb{C}^n \setminus K).$$

In the case $n = 1$, if we set

$$\mathbb{A}_0(\mathbb{C} \setminus K) = \mathbb{A}(\mathbb{C} \setminus K)/\mathbb{A}(\mathbb{C})$$

(which is essentially the space of holomorphic functions in $(\mathbb{C} \setminus K) \cup \{\infty\}$ which vanish at ∞), then we have a map

$$\mathbb{E} : \mathbb{A}_0(\mathbb{C} \setminus K) \to \mathbb{A}_K(\mathbb{C})$$

which is one-to-one and onto, with a well-known inversion formula due to Polya (see [4: p. 305]).

In this note we will give an analogous formula in the case $n \geq 2$. In fact this formula will follow from a formula of Berndtsson [1], who constructed explicitly measures whose Fourier-Laplace transform is a given function $F \in \mathbb{A}_K(\mathbb{C}^n)$. So what we do here is to show that these measures coherently define a $\overline{\partial}$-cohomology class in $\mathbb{C}^n \setminus K$ whose Fourier-Laplace transform is F.

Let us examine first what Berndtsson's formula gives in the case $n = 1$. Let us consider the function

$$B_\rho(\xi) = \int_0^\infty e^{-t\xi 2\frac{\partial\rho}{\partial\xi}} F\left(t2\frac{\partial\rho}{\partial\xi}\right) \cdot \frac{\partial\rho}{\partial\xi}(\xi)\, dt \qquad (\xi \in \mathbb{C} \setminus \{\rho < 1\})$$

where $\{\rho < 1\}$ is a strictly convex neighborhood of K. The function ρ is assumed to be smooth convex and homogeneous which guarantees the absolute convergence of the above integral, in view of the assumption on F, i.e. $F \in \mathbb{A}_K(\mathbb{C})$. We claim that B_ρ is analytic and independent of ρ, thus defining an analytic function in $\mathbb{C} \setminus K$. To see that B_ρ is analytic, notice that, by the Lebesgue dominated convergence theorem, B_ρ is of type C^1 and

$$\frac{\partial B_\rho}{\partial\overline{\xi}}(\xi) = \int_0^\infty \frac{\partial}{\partial\overline{\xi}} \left(e^{-t\xi 2\frac{\partial\rho}{\partial\xi}} F\left(t2\frac{\partial\rho}{\partial\xi}\right) \cdot \frac{\partial\rho}{\partial\xi}(\xi)\right) dt. \tag{1}$$

But as a computation shows,

$$\frac{\partial}{\partial\overline{\xi}} \left(e^{-t\xi 2\frac{\partial\rho}{\partial\xi}} F\left(t2\frac{\partial\rho}{\partial\xi}\right) \cdot \frac{\partial\rho}{\partial\xi}(\xi)\right) = \frac{\partial}{\partial t} \left(t \cdot e^{-t\xi 2\frac{\partial\rho}{\partial\xi}} F\left(t2\frac{\partial\rho}{\partial\xi}\right)\right) \cdot \frac{\partial^2\rho}{\partial\overline{\xi}\partial\xi}(\xi).$$

Substituting this into (1), we easily obtain that $\frac{\partial B_\rho}{\partial\overline{\xi}} = 0$ implies $B_\rho \in \mathbb{A}(\mathbb{C} \setminus \{\rho \leq 1\})$. Moreover,

$$\lim_{|\xi|\to\infty} B_\rho(\xi) = 0 \qquad \text{and} \qquad \int_\gamma e^{z\xi} B_\rho(\xi)\, d\xi = F(z)$$

for every $z \in \mathbb{C}$, where γ is a simple closed curve in $\mathbb{C} \setminus \{\rho \le 1\}$ arround K, and the claim follows. The proof in the case $n \ge 2$ is similar, only the computations become more technical.

Closing this introduction we mention that this note is related to the subject of analytic functionals where the central theme is the Ehrenpreis-Martineau theorem in its various forms and levels of generality; for more about it we refer to [3, 4] and the references given there. We also refer to [5, 7] for the theory of hyperfunctions which is also related to this subject.

2. Main result

Now we formulate our result.

Theorem 1. *Let $K \subset \mathbb{C}^n$ be a convex compact set and S a smooth surface around K. Then the transformation $\mathbb{E} : H^{(n,n-1)}(\mathbb{C}^n \setminus K) \to \mathbb{A}_K(\mathbb{C}^n)$ defined by*

$$\mathbb{E}([\theta])(\zeta) = F_\theta(\zeta) = \int_{z \in S} e^{\langle z,\zeta \rangle}\theta(z)\,dz \qquad (\zeta \in \mathbb{C}^n)$$

for $[\theta] \in H^{(n,n-1)}(\mathbb{C}^n \setminus K)$ is one-to-one and onto and defines an isomorphism

$$H^{(n,n-1)}(\mathbb{C}^n \setminus K) \approx \mathbb{A}_K(\mathbb{C}^n)$$

of linear spaces which is independent of S.

Furthermore, the inverse transformation $\mathbb{E}^{-1} : \mathbb{A}_K(\mathbb{C}^n) \to H^{(n,n-1)}(\mathbb{C}^n \setminus K)$ is given by the formula $F \to \mathbb{E}^{-1}(F) = [\theta_F]$, $F \in \mathbb{A}_K(\mathbb{C}^n)$, where the class $[\theta_F]$ restricted to $\mathbb{C}^n \setminus \{\rho \le 1\}$ is equal to $[\theta_F^\rho]$ and

$$\theta_F^\rho(\xi) = c_n \left(\int_0^\infty t^{n-1} e^{-2t\langle \xi, \partial\rho(\xi)\rangle} F\left(t2\frac{\partial\rho}{\partial\xi}\right) dt \right) \partial\rho(\xi) \wedge [\partial\bar{\partial}\rho(\xi)]^{n-1},$$

is defined for $\xi \in \mathbb{C}^n \setminus \{\rho \le 1\}$. (For this formula we assume that $0 \in K$ and that the functions ρ are chosen to be positively homogeneous, i.e. $\rho(\lambda\xi) = \lambda\rho(\xi)$ for $\lambda \ge 0$, and such that $\{\rho < 1\}$ is a strictly convex neighborhood of K. Also, c_n will denote a constant which depends only on n.)

Of course, it is part of the conclusion of the theorem that the classes $[\theta_F^\rho]$ agree in their common domain of definition, as the neighborhood $\{\rho < 1\}$ shrinks to K, thus well-defining the limiting class $[\theta_F]$ in $\mathbb{C}^n \setminus K$; this class is an analogue of the Borel transform of F in several variables.

We will split the proof of the theorem in several steps which we present as lemmas. But let us check first that the integral which defines θ_F^ρ is absolutely convergent and defines a C^∞-form in $\xi \in \mathbb{C}^n \setminus \{\rho \le 1\}$. To do this we will use some facts about convex functions which we recall from [1]. According to this the map $(0,\infty) \times \partial L \to \mathbb{C}^n \setminus \{0\}$

(we have set $L=\{\rho\le 1\}$) defined by $(t,\xi)\to\zeta=t2\partial\rho(\xi)$, is one-to-one and onto with inverse given by $\xi_j=2\frac{\partial\phi}{\partial\zeta_j}(\zeta)$ and $t=\phi(\zeta)$, where $\phi(\zeta)=H_L(\zeta)$.

Now we show that the integral converges absolutely for $\xi\in\partial L$. Fix such a $\xi\in\partial L$. Then, by the convexity of $\phi(\zeta)$,

$$\left|e^{-2t\langle\xi,\partial\rho(\xi)\rangle}\right|=e^{-\mathrm{Re}\langle\xi,2t\partial\rho(\xi)\rangle}=e^{-2\mathrm{Re}\langle\partial\phi(\zeta),\zeta\rangle}\le e^{-\phi(\zeta)}\le\exp(-H_K(\zeta)-\varepsilon|\zeta|)$$

where $\varepsilon=\mathrm{dist}\,(K,\partial L)$. Also, since $F\in\mathbb{A}_K(\mathbb{C}^n)$, we have (with $\delta=\frac{\varepsilon}{2}$)

$$|F(\zeta)|\le C_\delta\exp\left(H_K(\zeta)+\frac{\varepsilon}{2}|\zeta|\right).$$

Therefore

$$\left|e^{-2t\langle\xi,\partial\rho(\xi)\rangle}F\left(t2\frac{\partial\rho}{\partial\xi}(\xi)\right)\right|\le C_\delta\exp\left(-\frac{\varepsilon}{2}t|\zeta|\right)=C_\delta\exp\left(-\varepsilon t\left|\frac{\partial\rho}{\partial\xi}\right|\right)$$

and the absolute convergence of the integral defining θ_F^ρ is immediate. Now if $\xi\in\mathbb{C}^n\setminus L$, then we write $\xi=\lambda\xi'$ where $\xi'\in\partial L$ and $\lambda>1$. Then, by the homogeneity of $\frac{\partial\rho}{\partial\xi_j}$, there follows the quantity

$$\left|e^{-2t\langle\xi,\partial\rho(\xi)\rangle}F\left(t2\frac{\partial\rho}{\partial\xi}\right)\right|=\left|e^{-\lambda2t\langle\xi',\partial\rho(\xi')\rangle}F\left(t2\frac{\partial\rho}{\partial\xi}(\xi')\right)\right|\le C_\delta\exp\left(-\varepsilon t\left|\frac{\partial\rho}{\partial\xi}(\xi')\right|\right)$$

since $\lambda>1$ and $2\mathrm{Re}\langle\xi',\partial\rho(\xi')\rangle\ge\rho(\xi')=1$. It follows that the integral defining θ_F^ρ is absolutely convergent for all $\xi\in\mathbb{C}^n\setminus L$ and it remains so if we differentiate the integrand with respect to the real variables corresponding to ξ. (Notice that if $F\in\mathbb{A}_K(\mathbb{C}^n)$, then any derivative of F also belongs to $\mathbb{A}_K(\mathbb{C}^n)$ which follows from the Cauchy inequalities.) Hence, by the Lebesgue dominated convergence theorem, $\theta_F^\rho(\xi)$ is of type C^∞ in $\xi\in\mathbb{C}^n\setminus\{\rho\le 1\}$.

3. Preparatory lemmas

We begin by proving that θ_F^ρ is $\overline{\partial}$-closed where it is defined. This is done by computing explicitly a $\frac{d}{dt}$-primitive.

Lemma 1. *We have* $\overline{\partial}\theta_F^\rho(\xi)=0$ *for* $\xi\in\mathbb{C}^n\setminus\{\rho\le 1\}$.

Proof. By the previous discussion,

$$\overline{\partial}\theta_F^\rho(\xi)=\int_0^\infty\overline{\partial}_\xi\left[t^{n-1}e^{-2t\langle\xi,\partial\rho(\xi)\rangle}F\left(t2\frac{\partial\rho}{\partial\xi}\right)\cdot\partial\rho(\xi)\wedge[\partial\overline{\partial}\rho(\xi)]^{n-1}\right]dt. \qquad (2)$$

We claim that

$$\begin{aligned}&\overline{\partial}_\xi\left[t^{n-1}e^{-2t\langle\xi,\partial\rho(\xi)\rangle}F\left(t2\frac{\partial\rho}{\partial\xi}\right)\cdot\partial\rho(\xi)\wedge[\partial\overline{\partial}\rho(\xi)]^{n-1}\right]\\&\quad=\frac{d}{dt}\left[a_n\left(t^ne^{-2t\langle\xi,\partial\rho(\xi)\rangle}F\left(t2\frac{\partial\rho}{\partial\xi}\right)\right)\overline{\partial}\gamma_1\wedge\ldots\wedge\overline{\partial}\gamma_n\wedge\omega\right]\end{aligned} \qquad (3)$$

where $\gamma_j = \frac{\partial\rho}{\partial\xi_j}$, $\omega = d\xi_1 \wedge \ldots \wedge d\xi_n$ and $a_n = (-1)^{\frac{n(n-1)}{2}} \frac{1}{(n-1)!}$. To prove this notice first that

$$\partial\rho(\xi) \wedge [\partial\overline{\partial}\rho(\xi)]^{n-1} = a_n \sum_{j=1}^{n} (-1)^{j-1} \gamma_j \overline{\partial}\gamma_1 \wedge \ldots (j) \ldots \wedge \overline{\partial}\gamma_n \wedge \omega.$$

Therefore (3) is equivalent to

$$\begin{aligned} &\overline{\partial}_\xi \left[t^{n-1} e^{-2t\langle\xi,\partial\rho(\xi)\rangle} F\left(t2\frac{\partial\rho}{\partial\xi}\right) \cdot \sum_{j=1}^{n} (-1)^{j-1} \gamma_j \overline{\partial}\gamma_1 \wedge \ldots (j) \ldots \wedge \overline{\partial}\gamma_n \wedge \omega \right] \\ &\qquad = \frac{d}{dt}\left[\left(t^n e^{-2t\langle\xi,\partial\rho(\xi)\rangle} F\left(t2\frac{\partial\rho}{\partial\xi}\right)\right) \overline{\partial}\gamma_1 \wedge \ldots \wedge \overline{\partial}\gamma_n \wedge \omega \right]. \end{aligned} \tag{4}$$

Now (4) follows from the following three observations:

Observation 1:

$$\overline{\partial}_\xi \left(\sum (-1)^{j-1} \gamma_j \overline{\partial}\gamma_1 \wedge \ldots (j) \ldots \wedge \overline{\partial}\gamma_n \wedge \omega \right) = n \overline{\partial}\gamma_1 \wedge \ldots \wedge \overline{\partial}\gamma_n \wedge \omega$$

and therefore the term which we obtain when $\overline{\partial}_\xi$ (in (4)) hits the sum $\sum_{j=1}^{n}$ is equal to the term obtained when $\frac{d}{dt}$ hits the term t^n.

Observation 2:

$$\begin{aligned} &\left[\overline{\partial}_\xi \left(e^{-2t\langle\xi,\partial\rho(\xi)\rangle}\right)\right] \wedge \left(\sum_{j=1}^{n} (-1)^{j-1} \gamma_j \overline{\partial}\gamma_1 \wedge \ldots (j) \ldots \wedge \overline{\partial}\gamma_n \wedge \omega \right) \\ &\quad = (-2t)\left(e^{-2t\langle\xi,\partial\rho(\xi)\rangle}\right)\left(\sum \xi_j \overline{\partial}\gamma_j\right) \wedge \left(\sum_{j=1}^{n} (-1)^{j-1} \gamma_j \overline{\partial}\gamma_1 \wedge \ldots (j) \ldots \wedge \overline{\partial}\gamma_n \wedge \omega \right) \\ &\quad = (-2t)\left(e^{-2t\langle\xi,\partial\rho(\xi)\rangle}\right)\left(\sum_{j=1}^{n} \xi_j \gamma_j\right) \left(\overline{\partial}\gamma_1 \wedge \ldots \wedge \overline{\partial}\gamma_n \wedge \omega\right) \end{aligned}$$

and therefore the terms obtained when $\overline{\partial}_\xi$ and $\frac{d}{dt}$ hit the exponentials are equal.

Observation 3:

$$\begin{aligned} &\left[\overline{\partial}_\xi \left(F\left(t2\frac{\partial\rho}{\partial\xi}\right)\right)\right] \wedge \left(\sum_{j=1}^{n} (-1)^{j-1} \gamma_j \overline{\partial}\gamma_1 \wedge \ldots (j) \ldots \wedge \overline{\partial}\gamma_n \wedge \omega \right) \\ &\quad = (2t)\left(\sum_{j=1}^{n} \frac{\partial F}{\partial\zeta_j}(2t\gamma_1, \ldots, 2t\gamma_n)\overline{\partial}\gamma_j \right) \left(\sum_{j=1}^{n} (-1)^{j-1} \gamma_j \overline{\partial}\gamma_1 \wedge \ldots (j) \ldots \wedge \overline{\partial}\gamma_n \wedge \omega \right) \\ &\quad = (2t)\left(\sum_{j=1}^{n} \gamma_j \frac{\partial F}{\partial\zeta_j}(2t\gamma_1, \ldots, 2t\gamma_n) \right) \left(\overline{\partial}\gamma_1 \wedge \ldots \wedge \overline{\partial}\gamma_n \wedge \omega\right) \end{aligned}$$

and therefore the terms obtained when $\overline{\partial}_\xi$ and $\frac{d}{dt}$ hit the quantity $F\left(t2\frac{\partial\rho}{\partial\xi}\right)$ are equal.

This proves (4) and, therefore, (3) holds. Now substituting (3) into (2) and integrating from $t = 0$ to $t = \infty$ we easily obtain the assertion of the lemma ∎

The following two lemmas are quite standard; the proof of the first one may be found in [6: p. 217] while we outline a proof of the second lemma for completeness.

Lemma 2. *Let $D \subset \mathbb{C}^n$ be an open set and let there be compact sets K_j $(j \in \mathbb{N})$ with $K_j \subset \operatorname{int}(K_{j+1})$ and $D = \cup_{j=1}^{\infty} K_j$. Let u be a $(0,q)$-form in D which is $\overline{\partial}$-exact in a neighborhood of K_j for all j. Let us also assume the following:*

(i) *In the case $q \geq 2$, $H^{(0,q-1)}(K_j) = 0$ for all j.*

(ii) *In the case $q = 1$, every function in $\mathbf{A}(K_j)$ (i.e. analytic in a neighborhood of K_j) can be approximated, uniformly on K_j, by functions in $\mathbf{A}(K_{j+1})$.*

Then u is $\overline{\partial}$-exact in all of D.

Lemma 3. *Let D_1 and D_2 be two convex open sets in $\mathbb{C}^n$, $D_2 \subset\subset D_1$, and set $D = D_1 \setminus \overline{D}_2$. Then every analytic function in D can be extended to an analytic function in D_1 $(n \geq 2)$ and approximated, uniformly on compact sets of D, by entire functions. Also, $H^{(0,q)}(D) = 0$ for $1 \leq q \leq n-2$ $(n \geq 3)$.*

Proof. We will prove only the last assertion of the lemma. The proof will be based on the Cauchy-Leray formula which we recall first (the first assertion can also be proved using the same formula, but we omit its proof since it is well-known).

Let $\Omega \subset \mathbb{C}^n$ be a bounded open set with smooth boundary and $\gamma : (\partial\Omega) \times \Omega \to \mathbb{C}^n$ a C^2-map such that

$$\sum_{j=1}^{n} (\zeta_i - z_i)\gamma_i(\zeta, z) \neq 0 \qquad \text{for } (\zeta, z) \in (\partial\Omega) \times \Omega.$$

For $u \in C_{(0,q)}(\overline{\Omega})$ $(0 \leq q \leq n)$ let us set

$$L_q^{\gamma}(u) = \int_{\partial\Omega} u \wedge \omega_q^1(\gamma)$$
$$T_{q-1}u = (-1)^{q-1} \int_{\partial\Omega} u \wedge \omega_{q-1}^2(\gamma, \beta) - \int_{\Omega} u \wedge \omega_{q-1}^1(\beta)$$

where $\beta_i = \overline{\zeta}_i - \overline{z}_i$ and

$$\omega_q^1(\gamma) = c_n(-1)^q \binom{n-1}{q} \left(\sum_{i=1}^{n} \gamma_i(\zeta_i - z_i)\right)^{-n} \det\left[\gamma_i, \overbrace{\overline{\partial}_z\gamma_i}^{q}, \overbrace{\overline{\partial}_\zeta\gamma_i}^{n-q-1}\right] \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n.$$

The formula for $\omega_{q-1}^2(\gamma, \beta)$ is similar (see, for example, [2: p. 85] where the notation is similar), but we will not write it down since its explicit form will not be important here. In this setting every $u \in C_{(0,q)}^1(\overline{\Omega})$ can be decomposed (in $C_{(0,q)}(\Omega)$) as

$$u = \overline{\partial}_z(T_{q-1}u) + T_q(\overline{\partial}u) + L_q^{\gamma}(u).$$

Now D can be exhausted with compact sets of the form $K_j = \{\lambda \leq 0\} \setminus \{\rho < 0\}$ where the sets $\{\lambda < 0\}$ and $\{\rho < 0\}$ are convex with smooth boundary. Appling the Cauchy-Leray formula in sets of the form $\Omega = \{\lambda < 0\} \setminus \{\rho \leq 0\}$ with

$$\gamma_i(\zeta, z) = \begin{cases} \dfrac{\partial\lambda}{\partial\zeta_i}(\zeta) & \text{if } \zeta \in \{\lambda = 0\} \\ \dfrac{\partial\rho}{\partial\zeta_i}(z) & \text{if } \zeta \in \{\rho = 0\} \ (z \in \Omega) \end{cases}$$

we obtain $H^{(0,q)}(K_j) = 0 \ (1 \leq q \leq n-2)$. The point here is that with this choice of γ, $\omega_q^1(\gamma) = 0$ when $\zeta \in \partial\Omega$. Now $H^{(0,q)}(D) = 0$ follows from Lemma 2 ∎

The following lemma can be proved exactly as Lemma 2. It suffices to consider some compact sets K_j between the D_j's.

Lemma 4. *Let $D_1 \subset\subset D_2 \subset\subset \ldots D_j \subset\subset D_{j+1} \subset\subset \ldots$ be a sequence of open subsets of $\mathbb{C}^n$ and $q \geq 1$, and let us assume the following:*

(i) *If $q \geq 2$, then $H^{(0,q-1)}(D_j) = 0$ for all j.*

(ii) *If $q = 1$, then $\mathbb{A}(D_{j+1})$ should be dense in $\mathbb{A}(D_j)$.*

Under these assumptions if θ is a $(0,q)$-form in D which is $\overline{\partial}$-exact in every D_j, then θ is $\overline{\partial}$-exact in the whole D, i.e. $\operatorname{inv\,lim} B^{(0,q)}(D_j) = B^{(0,q)}(D)$. In particular, if moreover $H^{0,q-1)}(D_j) = 0$ for all j, then $H^{(0,q)}(D) = 0$.

Lemma 5. *The transformation $\mathbb{E}$ is one-to-one, i.e. if*

$$\int_{z\in S} e^{\langle z,\zeta\rangle}\theta(z) = 0 \qquad \textit{for every } \zeta \in \mathbb{C}^n, \tag{5}$$

then θ is $\overline{\partial}$-exact in $\mathbb{C}^n \setminus K$.

Proof. Since the linear combinations of the functions $e^{\langle z,\zeta\rangle}$ $(\zeta \in \mathbb{C}^n)$ is dense in $\mathbb{A}(\mathbb{C}^n)$, it follows from (5) that

$$\int_{z\in S} \phi(z)\theta(z) = 0 \qquad \text{for every } \phi \in \mathbb{A}(\mathbb{C}^n). \tag{6}$$

Let us exhaust now the set $\mathbb{C}^n \setminus K$ by compact sets of the form $K_j = \{\lambda \leq 0\} \setminus \{\rho < 0\}$ (as in Lemma 3). By the Cauchy-Leray formula in $\Omega = \{\lambda < 0\} \setminus \{\rho \leq 0\}$ we have

$$\theta = \overline{\partial}_z(T_{n-2}\theta) + T_{n-1}(\overline{\partial}\theta) + L^\gamma_{n-1}(\theta) \tag{7}$$

where γ is as in Lemma 3 and where we identify $(n, n-1)$-forms with $(0, n-1)$-forms in the obvious way. By the definition of the kernels,

$$L^\gamma_{n-1}(\theta) = c_n \int_{\zeta\in\{\rho=0\}} \left(\sum_{i=1}^n \frac{\partial\rho}{\partial\zeta_i}(z)(\zeta_i - z_i)\right)^{-n} \theta(\zeta) \wedge \det\left[\frac{\partial\rho}{\partial z_i}, \overbrace{\overline{\partial}_z\left[\frac{\partial\rho}{\partial z_i}\right]}^{n-1}\right],$$

since the integral over $\{\lambda = 0\}$ vanishes. But for each fixed $z \in \Omega$ the function $[\sum \frac{\partial\rho}{\partial z_i}(\zeta_i - z_i)]^{-n}$, as a function of ζ, is analytic in $\{\rho < \rho(z)\}$, and therefore it can be approximated by entire functions. It follows from (6) that $L^\gamma_{n-1}(\theta) = 0$, and since $\overline{\partial}\theta = 0$, (7) becomes $\theta = \overline{\partial}_z(T_{n-2}\theta)$, i.e. θ is $\overline{\partial}$-exact in Ω. Finally, since $\mathbb{C}^n \setminus K$ can be exhausted by sets like Ω, it follows from Lemma 4 that θ is $\overline{\partial}$-exact in $\mathbb{C}^n \setminus K$ ∎

The next lemma is quite elementary and we state it in $\mathbb{R}^n$ for C^∞-functions (and we will use it in $\mathbb{C}^n$ for differential forms).

Lemma 6. *Let $D_1 \subset\subset D_2 \subset\subset \dots D_j \subset\subset D_{j+1} \subset\subset \dots$ be a sequence of open subsets of $\mathbb{R}^n$ and $f_j \in C^\infty(D_j)$. Then there exist functions $g_j \in C^\infty(D_j)$ such that $g_j - g_{j+1} = f_j$ in D_j for every j.*

Proof. Let us choose functions $\chi_j \in C_0^\infty(\mathbb{R}^n)$ so that $\operatorname{supp}(\chi_j) \subset D_{j+1}$ and $\chi_j \equiv 1$ in a neighborhood of $\overline{D}_j$, and let us define $h_j = -\chi_j f_{j+1}$ for all j. Then every h_j has a C^∞ extension in $\mathbb{R}^n$ (by setting it equal to 0 in $\mathbb{R}^n \setminus D_{j+1}$) which we denote also by h_j. Then the functions $g_1 = f_1$, $g_2 = f_2 + h_1$, $g_3 = f_3 + h_1 + h_2, \dots$ satisfy the required relations ∎

Lemma 7. *Let D_j (and q) be as in Lemma 4. Then* $\operatorname{inv\,lim} H^{(0,q)}(D_j) \approx H^{(0,q)}(D)$. *Indeed, the map*

$$\sigma : H^{(0,q)}(D) \to \operatorname{inv\,lim} H^{(0,q)}(D_j)$$

defined by the restriction of cohomology classes, i.e.

$$\sigma([\theta]) = ([\theta|_{D_1}], [\theta|_{D_2}], [\theta|_{D_3}], \dots) \qquad ([\theta] \in H^{(0,q)}(D))$$

is an isomorphism.

Proof. Let us consider the map

$$\sigma' : \operatorname{inv\,lim} Z^{(0,q)}(D_j) = Z^{(0,q)}(D) \to \operatorname{inv\,lim} H^{(0,q)}(D_j)$$

with

$$\sigma'(\eta_1, \eta_2, \eta_3, \dots) = ([\eta_1], [\eta_2], [\eta_3], \dots), \qquad (\eta_1, \eta_2, \eta_3, \dots) \in \operatorname{inv\,lim} Z^{(0,q)}(D_j).$$

Then

$$\ker \sigma' = \operatorname{inv\,lim} B^{(0,q)}(D_j) = B^{(0,q)}(D),$$

by Lemma 4. Also, σ' is onto. Indeed, let

$$([\theta_1], [\theta_2], [\theta_3], \dots) \in \operatorname{inv\,lim} H^{(0,q)}(D_j).$$

Then there exist $(0, q-1)$-forms u_j in D_j such that

$$\begin{aligned} \theta_2 &= \theta_1 + \overline{\partial} u_1 && \text{in } D_1 \\ \theta_3 &= \theta_2 + \overline{\partial} u_2 && \text{in } D_2 \\ &\ \vdots \end{aligned}$$

By Lemma 6, there exist $(0, q-1)$-forms v_j in D_j such that $v_j - v_{j+1} = u_j$ in D_j for every j. Then $\overline{\partial} v_j - \overline{\partial} v_{j+1} = \overline{\partial} u_j$ in D_j and therefore

$$\theta_{j+1} = \theta_j + \overline{\partial} u_j = \theta_j + [\overline{\partial} v_j - \overline{\partial} v_{j+1}]$$

hence

$$\theta_j + \overline{\partial} v_j = \theta_{j+1} + \overline{\partial} v_{j+1} \qquad \text{in } D_j.$$

Thus

$$(\theta_1 + \overline{\partial} v_1, \theta_2 + \overline{\partial} v_2, \dots) \in \operatorname{inv\,lim} Z^{(0,q)}(D_j)$$

and

$$\sigma'(\theta_1 + \overline{\partial} v_1, \theta_2 + \overline{\partial} v_2, \dots) = ([\theta_1], [\theta_2], \dots)$$

which shows that σ' is onto. It follows now that σ is an isomorphism ∎

4. Proof of Theorem 1

In view of the previous lemmas what remains to show is that the transform $\mathbb{E}$ is onto. We may also describe, somehow more precisely now, the inverse of the transform $\mathbb{E}$.

Proof of Theorem 1. Let us exhaust the set $\mathbb{C}^n \setminus K$ with open sets of the form

$$\{\lambda_1 < 1\} \setminus \{\rho_1 \leq 1\} \subset\subset \{\lambda_2 < 1\} \setminus \{\rho_2 \leq 1\} \subset\subset \ldots \subset\subset \mathbb{C}^n \setminus K$$

with the sets $\{\lambda_j < 1\}$ and $\{\rho < 1\}$ being strictly convex. Then, using the functions ρ_j, we define the differential forms $\theta^j_F(\xi)$ for ξ in the set $D_j = \{\lambda_j < 1\} \setminus \{\rho_j \leq 1\}$. By Lemma 1, these forms define classes $[\theta^j_F] \in H^{(n,n-1)}(D_j)$. Since, by the formula of the theorem,

$$\int_{z\in S} e^{\langle z,\zeta\rangle}(\theta^{j+1}_F(z) - \theta^j_F(z)) = 0 \qquad (\zeta \in \mathbb{C}^n)$$

(S is a closed surface in D_j), it follows from the proof of Lemma 5 that the restriction of the class $[\theta^{j+1}_F]$ to D_j is equal to $[\theta^j_F]$. Therefore there is defined an element

$$([\theta^1_F], [\theta^2_F], [\theta^3_F], \ldots) \in \text{inv lim}\, H^{(n,n-1)}(D_j).$$

Thus the inverse transformation is defined by the formula

$$\mathbb{E}^{-1}(F) = \sigma^{-1}([\theta^1_F], [\theta^2_F], [\theta^3_F], \ldots)$$

where σ is the isomorphism $H^{(n,n-1)}(D) \xrightarrow{\sigma} \text{inv lim}\, H^{(n,n-1)}(D_j)$ of Lemma 7 ∎

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Received 16.02.1998; in revised form 13.07.1998