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Paul Günther's Work on Hadamard's Conjecture

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While Professor Günthers mathematical work embraces many aspects of differential geometry and partial differential equations his work on 2nd order hyperbolic equations in relation to Huygens principle stands out as particularly significant. The wave equation in flat space $\mathbb{R}^n$ has the remarkable property (Huygens' principle) that the value $u(x,t)$ of the solution at time t only depends on the initial data on the surface of a sphere with center x and radius t. This is verified by everyday experience: a flash of light is seen and the sound of a shot is heard at a distance t after time t (the velocity of light and sound taken as 1) but then no more. This three-dimensional phenomenon contrasts markedly with wave propagation in two dimensions as illustrated by the *sequence* of spherical wave from a pebble falling on a flat water surface.

In his theory of the Cauchy problem for second order hyperbolic equations Hadamard found that Huygen's principle is a very rare occurrence. He proved that it could never happen in an even-dimensional space and for odd-dimensional space only if the logarithmic term of his "elementary solution" vanishes. It came to be believed (in Courant and Hilbert: *Methoden der Mathematischen Physik*, Vol. II. Berlin: Springer-Verlag 1937, p. 438, this was stated as a conjecture, attributed to Hadamard) that the wave equation in $\mathbb{R}^{2k+1}$ gives the only example of Huygens principle up to certain simple transformations (non-singular transformations of the independent variables, gauge transformations and conformal transformations) which clearly preserve the Huygensian property. In a later edition of Courant and Hilbert (1962) it is however stated (Vol. II, footnote p. 765) that Hadamard had never categorically made this conjecture. In 1953 Stellmacher found examples of equations for space dimensions 5, 7, etc. which were not equivalent to the flat wave equations, yet satisfied Huygen's principle. However, his examples did not settle the important case $n = 3$. This had to wait until Günther's remarkable discovery in 1965 that the Laplacian on an even-dimensional Lorenzian plane wave manifold is always Huygensian, yet not in general equivalent to the flat wave operator. A complete proof of this result can be found in Günther's *magnum opus* "*Huygens' principle and Hyperbolic Equations*" (Boston et al.: Academic Press 1988). This book is a remarkable achievement. Not only does it give a clear and careful account of Günther's own work and of recent work of others in this area but in addition it ties

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together Hadamard's pioneering work during the pre-distribution period with the modern hyperbolic equation theory within the framework of distributions. This detailed authoritative account is therefore of utmost value to present and future generations. Over ten years ago the present writer suggested to Günther that a full volume on this topic would be welcome by everyone working in the field of hyperbolic equations. Professor Günther said that he would consider this seriously and in a few years produced the 850 page volume mentioned above on the subject. The introduction alone is about 35 pages. Klaus Peters, then in charge of Academic Press' mathematical book program, liked the introduction so much that he thought that an article in the Mathematical Intelligencer along those lines would constitute an attractive survey of the subject for the broad mathematical public. Again Professor Günther was responsive to this suggestion and published in "The Mathematical Intelligencer" (1991) a masterly, well-documented report on the status of Hadamard's problem concerning Huygens' principle. This I believe is by far the best introduction to Hadamard's fascinating problem.