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Remark on the Normal Forms of Diversors of Second Order Differential Equations of Normal Hyperbolic Type

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Dedicated to the memory of Professor Paul Günther (1926 - 1996)

Abstract. With respect to the monograph of P. Günther "Huygens' Principle and Hyperbolic Equations" this paper contains a supplement to diversors of second order differential equations of normal hyperbolic type [3: Chapter IV]. We construct a "normal form" of a diversor and consider the coefficients of this form in a certain neighbourhood of the characteristic backward conoid $C_-(\xi)$ of a point ξ.

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Let (M, g) be a pseudo-Riemannian manifold with finite dimension $m = \dim M > 2$ whose metric g has Lorentz signature $\{+, -, \dots, -\}$. It is always assumed that M is of class C^∞, connected and satisfying the second axiom of countability; g is of class C^∞ on M. ∇ denotes the Levi-Civita connection of (M, g).

Let $\Omega \subseteq M$ be a geodesically normal domain and $\Omega_0 \subseteq \Omega$ any causal domain in Ω (see also [3: p. 15]). We consider any domain Ω and choose in Ω any coordinate system $\rho : \Omega \to \mathbb{R}^m$, where $\Omega \subseteq M$ is open. We denote the second order differential operator of normal hyperbolic type of (M, g), acting on scalar functions u, by P:

$$\begin{aligned} P[u] &= g^{ij}\nabla_i\nabla_j u + A^i\nabla_i u + Cu \\ &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij}\frac{\partial u}{\partial x^j}\right) + A^i\frac{\partial u}{\partial x^i} + Cu \qquad (i,j = 1,2,\dots,m) \end{aligned} \tag{1}$$

and the invariant measure associated to the metric g by μ which is given in these coordinates $\{x^1, \dots x^m\}$ by

$$\mu = \sqrt{g}\, dx^1 \wedge \dots \wedge dx^m.$$

Let the point $\xi \in \Omega$ be fixed. We denote the characteristic conoid by $C(\xi)$ given by the equation $\Gamma(\xi, x) = 0$ where $\Gamma(\xi, x)$ is the quadratic geodesic distance function.

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The notion "diversor" is due to L. Asgeirsson [1]. He defines a diversor D of P as a differential operator D, such that $D \circ P$ can be written as divergence expression on the characteristic conoid $C(\xi)$, the vertex ξ excluded.

Now we consider the characteristic backward conoid $C_-(\xi)$. Let $\Omega_0' \subseteq \Omega_0$ be a domain such that $C(\xi) \cap \Omega_0' = (C_-(\xi) \setminus \{\xi\}) \cap \Omega_0$, i. e. the vertex $\xi \notin \Omega_0'$.

Definition 1. Let $\phi \in C_0^\infty(\Omega_0')$ be any test function. A differential operator D is said to be a *diversor* of P with respect to $C_-(\xi)$ if

$$\int_{C_-(\xi)} (D \circ P)[\phi](x)\nu(x) = 0, \tag{2}$$

i.e. the distribution $v \in \mathcal{D}'(\Omega_0')$ with

$$(v, \phi) = \int_{C_-(\xi)} D[\phi](x)\nu(x) \tag{3}$$

is a solution of $P^*[v] = 0$ in Ω_0' with $\operatorname{supp} v \subseteq C_-(\xi) \setminus \{\xi\}$ where $\nu(x)$ denotes the Leray form of the submanifold $C_-(\xi)$ (see also [3: Chapter II, §2]), P^* denotes the (invariantly) formally adjoint operator of P.

Such an identity (2) is only possible if $(D \circ P)[\phi]$ can be written in divergence form with respect to the submanifold $C_-(\xi)$ in Ω_0'.

Definition 2. Two diversors D_1 and D_2 in Ω_0' are called *equivalent* if

$$\int_{C_-(\xi)} D_1[\phi](x)\nu(x) = \int_{C_-(\xi)} D_2[\phi](x)\nu(x) \qquad (\phi \in C_0^\infty(\Omega_0')),$$

i.e. $D_1[\phi] - D_2[\phi]$ is a divergence expression on the characteristic semiconoid $C_-(\xi)$.

Proposition 1. *Let $\{\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^m\}$ be a local coordinate system in Ω_0', such that $C_-(\xi)$ is given by $\bar{x}^1 = 0$, i.e.*

$$\left.\begin{aligned} \bar{x}^1 &= \Gamma(\xi, x) \\ \bar{x}^\alpha &= x^\alpha \quad (\alpha = 2, 3, \ldots, m) \end{aligned}\right\}.$$

For each diversor D there exists an equivalent linear differential operator which is called D_N of the form

$$D_N[\phi] = \frac{1}{\sqrt{g}} \sum_{\nu=0}^{\kappa} \frac{\partial^\nu}{\partial \Gamma^\nu}(w_{\kappa-\nu} \cdot \phi). \tag{4}$$

The coefficients $w_{\kappa-\nu}$ are of class C^∞ in Ω_0' and are uniquely determined on $C_-(\xi)$. The form (4) of a diversor is said to be normal form *D_N of D of order κ.*

Proof. The proof is obvious. The derivates of highest order in $D[\phi]$ are not all interior derivates $\partial^\alpha / \partial \bar{x}^\alpha$ with respect to the manifold $\bar{x}^1 = 0$, consequently, the order of D cannot be reduced with the help of integration by parts [3: pp. 270, 271] ∎

Proposition 2. *To each diversor D of order κ of P with respect to $C_-(\xi)$ there exists an equivalent diversor in normal form* (4) *whose "modified coefficients" W_ν with*

$$W_\nu := \frac{\partial_1 \Gamma(\xi, x)}{\sqrt{g}} w_\nu \qquad (\nu = 0, 1, 2, \ldots \kappa) \tag{5}$$

in Ω_0' are given by

$$\begin{aligned} g^{ij}\nabla_i\Gamma\nabla_j W_0 + (M^* + n - 4 - 2\kappa)W_0 &= 0 \\ g^{ij}\nabla_i\Gamma\nabla_j W_\nu + (M^* + n - 4 - 2\kappa + 2\nu)W_\nu &= \tfrac{1}{2} P^*[W_{\nu-1}] \quad (\nu = 1, 2, \ldots, \kappa) \\ L^*[W_\kappa] &= 0 \quad \text{on } C_-(\xi) \end{aligned} \tag{6}$$

with

$$M^*(\xi, x) = \tfrac{1}{2} g^{ij}\nabla_i\nabla_j\Gamma - \tfrac{1}{2} A^i\nabla_i\Gamma - n.$$

Proof. Let $\Omega_0'' \subseteq \Omega_0'$ be a neighbourhood of $C_-(\xi)$ with the condition $\partial_1\Gamma \neq 0$. ($\Delta_2 = g^{ij}\nabla_i\nabla_j$ denotes the 2. Beltrami operator.) In Ω_0'' we obtain by the (regular) transformation to the coordinates $\bar{x}^i$

$$\bar{g}^{11} = 4\Gamma, \qquad \bar{g}^{1\beta} = g^{i\beta}\partial_i\Gamma, \qquad \bar{g}^{\alpha 1} = g^{\alpha j}\partial_j\Gamma, \qquad \bar{g}^{\alpha\beta} = g^{\alpha\beta}$$

$$\sqrt{g} = |\partial_1\Gamma|\sqrt{\bar{g}}, \qquad \sqrt{\bar{g}} = \frac{\sqrt{g}}{|\partial_1\Gamma|}$$

$$\bar{\Gamma}^1 = -\Delta_2\Gamma, \qquad \bar{\Gamma}^\alpha = \Gamma^\alpha \qquad (\Gamma^i = g^{kj}\Gamma^i_{kj})$$

$$\bar{A}^1 = A^i\nabla_i\Gamma, \qquad \bar{A}^\alpha = A^\alpha$$

$$\frac{\partial}{\partial\bar{x}^1} = \frac{1}{|\partial_1\Gamma|}\frac{\partial}{\partial x^1}, \qquad \frac{\partial}{\partial\bar{x}^\alpha} = -\frac{\partial_\alpha}{|\partial_1\Gamma|}\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^\alpha}$$

$$\bar{g}^{1j}\frac{\partial}{\partial\bar{x}^j} = g^{ij}\nabla_i\Gamma\nabla_j$$

and by explicit calculations the expression

$$\begin{aligned} D \circ P[\phi] &= \frac{1}{\sqrt{\bar{g}}}\sum_{\nu=0}^{\kappa}\frac{\partial^\nu}{\partial\Gamma^\nu}(w_{\kappa-\nu}\cdot P[\phi]) \\ &= \frac{1}{\sqrt{\bar{g}}}\cdot \mathrm{Div}\,[\phi] \\ &\quad + \frac{1}{\sqrt{\bar{g}}}\phi\left[P^0[\phi] + \sum_{\nu=1}^{\kappa+1}\frac{\partial^\nu}{\partial\Gamma^\nu}\left(P^0[w_{\kappa-\nu}] + (N + 4\nu + 4)[w_{\kappa-\nu+1}]\right)\right] \\ &\quad + \frac{1}{\sqrt{\bar{g}}}\sum_{r=1}^{\kappa}\frac{\partial^r\phi}{\partial\Gamma^r}\left[\sum_{\nu=r}^{\kappa+1}\binom{\nu}{r}\frac{\partial^{\nu-r}}{\partial\Gamma^{\nu-r}}\left(P^0[w_{\kappa-\nu}] + (N + 4\nu + 4)[w_{\kappa-\nu+1}]\right)\right] \\ &\quad + \frac{1}{\sqrt{\bar{g}}}\frac{\partial^{\kappa+1}\phi}{\partial\Gamma^{\kappa+1}}\cdot(N + 4\kappa + 8)[w_0] \end{aligned}$$

with

$$\begin{aligned} N[\phi] :&= -2\frac{\partial(\bar{g}^{1j}\phi)}{\partial \bar{x}^j} + (\bar{A}^1 - \bar{\Gamma}^1)\phi \\ &= -2\frac{\sqrt{g}}{|\partial_1\Gamma|} g^{ij}\nabla_i\Gamma\nabla_j\left(\frac{|\partial_1\Gamma|}{\sqrt{g}}\phi\right) + (-\Delta_2\Gamma + (\nabla_i\Gamma)A^i)\phi \qquad (7) \\ (N+k)[\phi] :&= N[\phi] + k\phi \qquad (k \in \mathbb{N}) \qquad (8) \\ P^0[\phi] :&= \sqrt{\bar{g}}\, P^*\left[\frac{\phi}{\sqrt{\bar{g}}}\right] = \frac{\sqrt{g}}{|\partial_1\Gamma|} P^*\left[\frac{|\partial_1\Gamma|}{\sqrt{g}}\phi\right]. \qquad (9) \end{aligned}$$

Because (2) we obtain that the coefficient w_0 satisfies the equation

$$(N + 4\kappa + 8)[w_0] = 0 \qquad (10)$$

at first on $C_-(\xi)$. Now (10) (and w_0) can be extended to Ω_0'. (It is a transition to an equivalent diversor.) Successively, in Ω_0' we obtain that the coefficients $w_1, w_2, \ldots, w_\kappa$ are solutions of

$$(N + 4\kappa - 4\nu + 8) = -P^0[w_{\nu-1}], \qquad (11)$$

and, finally,

$$P^0[w_\kappa) = 0 \qquad \text{on } C_-(\xi). \qquad (12)$$

Consequently, from (10), (11), (12) and with respect to (7), (8), (9) the assertion follows ∎

In the case of order $\kappa = \frac{n-4}{2}$ a comparision of (6) with the equations for the Hadamard coefficients V_ν of the Riesz distributions (see also [5, 7, 8])

$$\begin{aligned} g^{ij}\nabla_i\Gamma\nabla_j W_0 + M^* V_0 &= 0 \\ g^{ij}\nabla_i\Gamma\nabla_j W_\nu + (M^* + 2\nu)V_\nu &= -P^*[W_{\nu-1}] \qquad (\nu = 1, 2, \ldots) \end{aligned} \qquad (13)$$

shows the relations

$$W_\nu(\xi, x) = (-1)^\nu \frac{1}{2^\nu} V_\nu(\xi, x). \qquad (14)$$

Consequently, in $\Omega_0'' \subseteq \Omega_0'$ the coefficients w_ν are smooth.

Now we consider (2), respectively (3), but $\phi \in C_0^\infty(\Omega_0)$ (vertex $\xi \in \Omega_0$!):

$$\int_{C_-(\xi)} (D \circ P)[\phi](x)\nu(x) = 0 \qquad (\phi \in C_0^\infty(\Omega_0)) \qquad (15)$$

with D in normal form (4) with (5) and (6). However, because these singularities of w_ν (for $x \to \xi$ on $C_-(\xi)$) are algebraic, it is possible to show (see [2: pp. 21, 22, 53]) that the integral (15) exists or can be regularized (in the sense of distributions). Consequently, the distribution $v \in \mathcal{D}'(\Omega_0')$ in (3) can be extended to a distribution $v \in \mathcal{D}'(\Omega_0)$ over Ω_0. Then the results about diversors in [3: Chapter IV, §3] of P. Günther are applicable.

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