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Conformal Completion of $\mathbb{U}(n)$-invariant Ricci-Flat Kähler Metrics at Infinity

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Dedicated to the memory of Professor Paul Günther

Abstract. For every $n \geq 2$ we give an example of a complete $\mathbb{U}(n)$-invariant cohomogeneity one metric on $\mathbb{R}^{2n}$ which is not conformally flat and which carries twistor spinors with zeros. The construction uses a conformal completion at infinity of a $\mathbb{U}(n)$-invariant Ricci-flat Kähler metric on $\mathbb{R}^{2n} \setminus \{0\}$ given by Calabi [2] and by Freedman and Gibbons [4]. This extends our results in [6] for $n = 2$ to all even dimensions.

Keywords: *Ricci-flat Kähler metrics, conformal completion, twistor spinor, cohomogeneity one metric, asymptotic locally Euclidean metric*

AMS subject classification: 53 C 25, 83 C 60

1. Introduction

Twistor spinors are solutions of a conformally invariant field equation on a Riemannian spin manifold (cf. [1, 5, 7]). A simply-connected four-dimensional manifold carries a twistor spinor if and only if it is *half conformally flat*, i.e. if the canonical almost complex structure on the *twistor space* is integrable. In our recent paper [6] we constructed a conformal completion at infinity of the *Eguchi-Hanson metric* given on the complement of the unit ball in $\mathbb{R}^4$ (cf. [3]). This was the first example of a Riemannian spin manifold which is not conformally flat and which carries twistor spinors with zeros. After the conformal change the two linearly independent *parallel spinors* of the Eguchi-Hanson metric become twistor spinors with zero at infinity. Note that by a result of Lichnerowicz [7: Theorem 7] a *compact* Riemannian spin manifold carrying a twistor spinor with zero is conformally equivalent to the standard sphere.

In this note we extend our results in [6] to all even dimensions. We use $\mathbb{U}(n)$-invariant Ricci flat Kähler metrics on $\mathbb{C}^n \setminus \{0\}$ which in this form were given first by Calabi [2] and Freedman and Gibbons [4].

Theorem. *For every $n \geq 2$ there is a complete $\mathbb{U}(n)$-invariant cohomogeneity one metric on $\mathbb{R}^{2n}$ which is not conformally flat and which carries a two-dimensional space of twistor spinors with common zero point.*

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2. $\mathbb{U}(n)$-invariant metrics on $\mathbb{R}^{2n}$

On $\mathbb{C}^n \setminus \{0\}$ with complex coordinates z^α ($\alpha = 1, \ldots, n$) and their conjugates $\overline{z}^\alpha$ we consider the *Kähler metric*

$$g = 2 \sum_{\alpha,\beta=1}^{n} g_{\alpha\overline{\beta}} dz^\alpha dz^{\overline{\beta}}$$

where $g_{\alpha\overline{\beta}} = \partial_\alpha \partial_{\overline{\beta}} F$ and F is a potential function of the Kähler metric. Let $r^2 = \sum_{\alpha=1}^{n} z^\alpha z^{\overline{\alpha}}$. Now we consider the case of a radially symmetric potential function F, i.e. $F(z) = \tilde{F}(r^2)$, and we choose for a real parameter $a > 0$

$$\tilde{F}(s) = \int_1^s \frac{(a^n + \xi^n)^{\frac{1}{n}}}{\xi} \, d\xi.$$

Then

$$g = 2 \, \frac{(a^n + r^{2n})^{\frac{1}{n}}}{r^2} \left\{ \sum_{\alpha=1}^{n} dz^\alpha dz^{\overline{\alpha}} - \frac{1}{r^{2n}(a^n + r^{2n})} \sum_{\alpha=1}^{n} \overline{z}^\alpha dz^\alpha \sum_{\beta=1}^{n} z^\beta d\overline{z}^\beta \right\} \tag{1}$$

is a *Ricci-flat Kähler metric*, since $\det \partial_\alpha \partial_{\overline{\beta}} F = 0$ (cf. [4]). This metric is invariant under the canonical $\mathbb{U}(n)$-action on $\mathbb{C}^n$, hence the induced metrics on the distance spheres $S_c^{2n-1} = \{z \in \mathbb{C} | r = c\}$ for positive c are homogeneous with respect to the $\mathbb{U}(n)$-action, i.e. as homogeneous spaces they are of the form $S_c^{2n-1} = \mathbb{U}(n)/\mathbb{U}(n-1)$. There is a one-parameter family $\{h_t\}$ of these metrics which are also called *Berger metrics* or *canonical variation* of the standard metric on S^{2n-1} with respect to the *Hopf fibration* $S^{2n-1} \to \mathbb{C}P^{n-1}$.

More precisely, if $S^{2n-1} = \{z \in \mathbb{C}^n | r = 1\} \subset \mathbb{C}^n$, then

$$y \in S^{2n-1} \longmapsto V(y) = i\partial_r = i \frac{y}{\|y\|}$$

is the *Hopf vector field*. Then $h_t(V, V) = t$, and on the orthogonal complement of V the metric h_t coincides with the standard one. For $t \to 0$ the metric h_t on S^{2n-1} *collapses* (with bounded curvature) to the Fubini-Study metric on the $(n-1)$-dimensional complex projective space $\mathbb{C}P^{n-1}$.

Fix $z^* = r(1, 0, \ldots, 0) \in \mathbb{C}^n \setminus \{0\}$ with $r \in \mathbb{R}^+$. Then

$$\partial_r(z^*) = \frac{\partial}{\partial r}(z^*) = \frac{1}{2}\left(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial \overline{z}_1} \right)(z^*).$$

We conclude from equation (1) that

$$g_{z^*}(\partial_r, \partial_r) = \frac{a^n - 1 + r^{2n}}{r^2 (a^n + r^{2n})^{1-\frac{1}{n}}}.$$

The Hopf vector field V on $\mathbb{C}^n \setminus \{0\}$ is generated by the $\mathbb{U}(1)$-action $(\exp(i\phi), z) \mapsto \exp(i\phi) \cdot z$, i.e. at z^*

$$V(z^*) = V((r,0,\ldots,0)) = \frac{d}{dt}\Big|_{\phi=0} \big(\exp(i\phi)r, 0, \ldots, 0\big) = ir\frac{\partial}{\partial r}.$$

At z^* the vectors

$$X_\alpha = \frac{1}{2}\left(\frac{\partial}{\partial z_\alpha} + \frac{\partial}{\partial \bar{z}_\alpha}\right) \quad (\alpha \geq 2) \qquad \text{and} \qquad X_{\alpha+n} = \frac{1}{2}i\left(\frac{\partial}{\partial z_\alpha} - \frac{\partial}{\partial \bar{z}_\alpha}\right) \quad (\alpha \geq 2)$$

form a basis of pairwise orthogonal vectors spanning the orthogonal complement of the complex plane spanned by ∂_r and $i\partial_r$. With respect to the Euclidean metric they form an orthonormal basis. Since $z_2 = z_3 = \ldots = z_n$ in $z^* = r$ we obtain

$$g(X_\alpha, X_\alpha) = \frac{(a^n + r^{2n})^{\frac{1}{n}}}{r^2}.$$

Therefore we can write down the metric in the form

$$g = \frac{a^n - 1 + r^{2n}}{r^2(a^n + r^{2n})^{1-\frac{1}{n}}}\, dr^2 + (a^n + r^{2n})^{\frac{1}{n}}\, h_{\frac{a^n - 1 + r^{2n}}{a^n + r^{2n}}}. \tag{2}$$

It is defined for $a \in (0,1)$ only for $r^{2n} > 1 - a^n$. One can show that after dividing out a free $\mathbb{Z}_n$-action and by adding a $\mathbb{C}P^{n-1}$ at $r = 0$ one obtains a complete Ricci flat Kähler metric on a complex line bundle over $\mathbb{C}P^{n-1}$, which is for $r \to \infty$ asymptotic to $\mathbb{C}^n/\mathbb{Z}_n$, i.e. it is *asymptotic locally Euclidean* (cf. [4]).

Remark. If $n = 2$ and $a = 1$, we obtain

$$g = \frac{1}{\sqrt{1 + \frac{1}{r^4}}}\, dr^2 + \sqrt{1 + r^4}\, h_{\left(1 - \frac{1}{1+r^4}\right)}.$$

If we let $\rho(r) := (1 + r^4)^{\frac{1}{4}}$, we get

$$g = \frac{d\rho^2}{1 - \frac{1}{\rho^4}} + \rho^2\, h_{\left(1 - \frac{1}{\rho^4}\right)} \tag{3}$$

which is the form of the Eguchi-Hanson metric outside the unit ball in $\mathbb{R}^4$ for the parameter $a = 1$ as given in [3] (cf. also [6: Chapter 2]).

3. Conformal completion at infinity

Now we choose in equation (2) the parameter $a = 1$:

$$g = \frac{1}{\left(1 + \frac{1}{r^{2n}}\right)^{1-\frac{1}{n}}} dr^2 + (1 + r^{2n})^{\frac{1}{n}} h_{\left(1 - \frac{1}{1+r^{2n}}\right)}. \tag{4}$$

We change the radial coordinate $R = \frac{1}{r}$ and obtain on $\mathbb{R}^{2n} \setminus \{0\}$ the following metric for $(R, y) \in \mathbb{R}^+ \times S^{2n-1}$:

$$g_1(R, y) = g\left(\frac{1}{R}, y\right) = \frac{dR^2}{R^4(1 + R^{2n})^{1-\frac{1}{n}}} + \frac{(1 + R^{2n})^{\frac{1}{n}}}{R^2} h_{\frac{1}{1+R^{2n}}}.$$

Then we consider the following conformally equivalent metric on $\mathbb{R}^{2n} \setminus \{0\}$:

$$g_2(R, y) = R^4(1 + R^{2n})^{1-\frac{1}{n}} g_1(R, y),$$

hence

$$g_2(R; y) = dR^2 + R^2(1 + R^{2n}) h_{\frac{1}{1+R^{2n}}}. \tag{5}$$

Now we use the following

Lemma (cf. [6: Lemma 3.1]). *Let $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ be C^∞-functions. The metric*

$$g = dr^2 + r^2 \alpha(r^2) h_{\beta(r^2)}$$

on $\mathbb{R}^{2n} \setminus \{0\}$ given in polar coordinates $(r, y) \in \mathbb{R}^+ \times S^{2n-1}$ extends to a C^∞-metric on $\mathbb{R}^{2n}$ if and only if $\alpha(0) = \beta(0) = 1$.

Proof. h_1 is the standard metric on S^{2n-1}, we denote by σ the dual 1-form on S^{2n-1} with respect to h_1 of the Hopf vector field V. Then we can write the difference $g - g_0$ of the metric g and the Euclidean metric $g_0 = dr^2 + r^2 h_1$ as

$$g - g_0 = r^2 \left(\alpha(r^2) - 1\right) h_1 + r^2 \alpha(r^2) \left(\beta(r^2) - 1\right) \sigma^2.$$

If $(x_1, x_2, \ldots, x_n)$ are Cartesian coordinates on $\mathbb{R}^{2n}$, then

$$r\,dr = \sum_{j=1}^{2n} x_j dx_j.$$

We conclude that dr^2 is not continuous in 0, but $r^2 dr^2$ is a smooth 2-form on $\mathbb{R}^{2n}$, where *smoothness* means C^∞-differentiability. Since $r^2 h_1 = g_0 - dr^2$ the 2-form $r^2 h_1$ is not continuous in 0 but the 2-form $r^4 h_1$ is a smooth one on $\mathbb{R}^{2n}$. Then it follows from equation (6) for directions orthogonal to ∂_r and $i\partial_r$ that $\alpha(0) = 1$, provided g is smooth on $\mathbb{R}^{2n}$. Since

$$\sigma = \frac{1}{r^2} \sum_{j=1}^{2n} \left(- x_{2j} dx_{2j-1} + x_{2j-1} dx_{2j} \right)$$

we conclude that $r^2 \sigma^2$ is not continuous in 0 but $r^4 \sigma^2$ is a smooth 2-form on $\mathbb{R}^{2n}$. Then it follows from equation (6) that the smoothness of g in 0 implies $\beta(0) = 1$. On the other hand it follows that g is smooth on $\mathbb{R}^{2n}$ if $\alpha(0) = \beta(0) = 1$ ∎

Proof of the Theorem. We conclude from the Lemma that

$$g_2 = dR^2 + R^2(1 + R^{2n})h_{\frac{1}{1+R^{2n}}}$$

given in equation (5) is a complete metric on $\mathbb{R}^{2n}$ which outside 0 is conformally equivalent to a Ricci flat, non-flat Kähler metric. The function

$$u(R) = R^2(1 + R^{2n})^{\frac{n-1}{n}}$$

is smooth on $\mathbb{R}^{2n}$ and $u(R)^2 g(R, y)$ is the Ricci flat Kähler metric g_1 for $R > 0$. Since a Ricci flat Kähler metric carries two linearly independent *parallel spinors* ψ_1 and ψ_2 the metric g carries two linearly independent *twistor spinors* $u(R)^{\frac{1}{2}}\,\overline{\psi_1}$ and $u(R)^{\frac{1}{2}}\,\overline{\psi_2}$ with 0 as common zero point where $\psi \mapsto \overline{\psi}$ is the canonical bundle isometry between the spinor bundles of the conformally equivalent metrics g_1 and g_2. This follows from the conformal invariance of twistor spinors (cf. [1]) ∎

We also conclude that u is a solution of the partial differential equation

$$-u\,\mathrm{Ric}^0 = (d-2)(\mathrm{Hess}\,u)^0$$

where Ric^0 is the tracefree part of the Ricci tensor of the metric g_2, $\mathrm{Hess}\,u^0$ is the tracefree part of the Hessian of the function u with respect to the metric g_2, and $d = \dim M = 2n$ (cf. [5: Proposition 2.1]).

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