

# On some Applications of Theorems on the Spectral Radius to Differential Equations

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**Abstract.** In this paper we obtain uniqueness theorems for the Darboux problem of neutral type in an implicit form. Our proofs are based on a fixed point theorem and theorems on the spectral radius of the sum of operators.

**Keywords:** *Darboux problem of neutral type, spectral radius, uniqueness of solutions*

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## 1. Introduction

Let  $E$  be a Banach space. It is well known that if  $A_1, A_2 : E \rightarrow E$  are linear bounded and commutative operators, then

$$r(A_1 + A_2) \leq r(A_1) + r(A_2) \quad (1)$$

where  $r(A_1)$ ,  $r(A_2)$  and  $r(A_1 + A_2)$  denote the spectral radii of  $A_1$ ,  $A_2$  and  $A_1 + A_2$ , respectively. Classical examples show that in the above inequality the assumption of the commutativity is essential (cf. [3: Chapter II]). But there are many theorems which give sufficient conditions, different from the global commutativity, under which the above inequality is satisfied. In this paper we shall apply the following two theorems of this type. For the concepts of the first one we refer, e.g., to [3: Chapter II].

**Theorem 1** (see [2]). *Let  $K$  be a generating and normal cone in a Banach space  $E$  and let linear operators  $A_1, A_2 : E \rightarrow E$  be positive on  $K$ , semicommutative and  $u_0$ -upper bounded. Then the inequality (1) is satisfied.*

In fact, Esayan [2] has obtained a more general result but the above quoted one is enough for our considerations.

Now, let  $(X, \|\cdot\|, \prec)$  be a Banach space with a binary relation  $\prec$ . Assume the following:

- 1<sup>o</sup> The relation  $\prec$  is reflexive and transitive.
- 2<sup>o</sup> The norm  $\|\cdot\|$  is monotone, that is, if  $\theta \prec x \prec y$ , then  $\|\theta\| \leq \|y\|$ .
- 3<sup>o</sup> If  $x \prec y$ , then  $x + z \prec y + z$  for  $x, y, z \in X$ .

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**Theorem 2** (see [5]). *Let  $(X, \|\cdot\|, \prec)$  be a Banach space with a binary relation  $\prec$  satisfying the assumptions  $1^0 - 3^0$ , and let be given linear and bounded operators  $A_1, A_2 : E \rightarrow E$  satisfying the following two conditions:*

$4^0$  *If  $\theta \prec w$ , then  $\theta \prec A_1 w$  and  $\theta \prec A_2 w$ .*

$5^0$  *There exists an element  $\theta \prec w_0 \in X$  such that*

$$r(A_1 + A_2) = \lim_{n \rightarrow \infty} \|(A_1 + A_2)^n w_0\|^{\frac{1}{n}}$$

and

$$A_2 A_1^j A_2^k w_0 \prec A_1^j A_2^{k+1} w_0 \quad (j \geq 1, k \geq 0).$$

Then the inequality (1) holds.

**Remark 1.** In an implicit manner, Theorem 1 contains the additional assumption  $r(A_2) = \lim_{n \rightarrow \infty} \|A_2^n x_0\|^{\frac{1}{n}}$ . But a slight technical change in the proof of Theorem 2 shows that this assumption can be omitted.

In this paper we shall consider the following Darboux problem of neutral type:

$$\left. \begin{aligned} z_{xy}(x, y) &= f(x, y, z(h(x, y)), z_{xy}(H(x, y))) && \text{for } (x, y) \in I^2 \\ z(x, 0) &= 0 && \text{for } x \in I \\ z(0, y) &= 0 && \text{for } y \in I \end{aligned} \right\} \quad (2)$$

where  $z_{xy}$  denotes the second mixed derivative and  $I = [0, a]$  with  $a > 0$ . Similar problems (with or without translation of arguments) have been considered, e.g., in [1, 4, 5].

Our proofs are based on the following extension of the Banach fixed point theorem.

**Theorem 3** (see [4]). *Let  $(X, \|\cdot\|, \prec, m)$  be a Banach space with a binary relation  $\prec$  and a mapping  $m : X \rightarrow X$ . Suppose that assumptions  $1^0, 2^0$  and*

$6^0$   *$\theta \prec m(w)$  and  $\|m(w)\| = \|w\|$  for every  $w \in X$*

*hold. Moreover, let  $F : X \rightarrow X$  and  $A : X \rightarrow X$  be operators with the following properties:*

$7^0$  *A linear and bounded with  $r(A) < 1$ .*

$8^0$  *If  $\theta \prec w \prec v$ , then  $Aw \prec Av$ .*

$9^0$   *$m(Fw - Fv) \prec Am(w - v)$  for every  $w, v \in X$ .*

Then the mapping  $F$  has a unique fixed point.

Recall that to prove Theorem 3 one show that, for some  $k \in \mathbb{N}$ , the  $k$ -th iteration of  $F$  is a contraction.

## 2. The Darboux problem

Consider now the problem (2). Assume the following:

(i)  $h = (h_1, h_2) : I^2 \rightarrow I^2$  is a continuous function and  $h(x, y) \leq (x, y)$  for every pair  $(x, y) \in I^2$ , where  $(x_1, y_1) \leq (x_2, y_2)$  stands for  $x_1 \leq x_2$  and  $y_1 \leq y_2$ .

(ii)  $H : I^2 \rightarrow I^2$  is a continuous mapping such that  $H(\text{int}I^2) = \text{int}I^2$  and  $H|_{\text{int}I^2}$  is a diffeomorphism.

(iii)  $(x, y, u, v) \rightarrow f(x, y, u, v)$  is a continuous real function defined on the product  $I^2 \times \mathbb{R}^2$ , satisfying the Lipschitz condition

$$|f(x, y, u_1, v_1) - f(x, y, u_2, v_2)| \leq L_1|u_1 - u_2| + L_2|v_1 - v_2|$$

for  $(x, y, u_1, v_1), (x, y, u_2, v_2) \in I^2 \times \mathbb{R}^2$ , where  $L_1 > 0$  and  $0 < L_2 < 1$  are some constants.

Our first result is given by the following

**Theorem 4.** *Assume additionally to (i) - (iii) that the functions  $h, H$  satisfy one of the following conditions:*

(iv)  $|H'(x, y)| \leq 1$  and  $D(h(H(x, y))) \subset H(D(h(x, y)))$  for every  $(x, y) \in \text{int}I^2$  where  $|H'(x, y)|$  denotes the absolute value of the Jacobian of the mapping  $H$  in  $(x, y)$  and  $D(x, y) = \{(t, s) : 0 \leq t \leq x \text{ and } 0 \leq s \leq y\}$ .

(v)  $|H'(x, y)| \geq 1$  and  $D(h(H(x, y))) \supset H(D(h(x, y)))$  for every  $(x, y) \in \text{int}I^2$ .

Then the problem (2) has exactly one solution defined on  $I^2$ .

**Proof.** It can be easily verified that problem (2) is equivalent to the functional integral equation

$$w(x, y) = f \left( x, y, \int_{D(h(x, y))} w(t, s) dt ds, w(H(x, y)) \right) \quad ((x, y) \in I^2). \quad (3)$$

Denote by  $C = C(I^2, \mathbb{R})$  the space of all continuous functions  $z : I^2 \rightarrow \mathbb{R}$  with the supremum norm  $\|\cdot\|_c$ . Consider the operator  $F$  given by

$$F(w)(x, y) = f \left( x, y, \int_{D(h(x, y))} w(t, s) dt ds, w(H(x, y)) \right) \quad (4)$$

where  $w \in C$  and  $(x, y) \in I^2$ . It is clear that  $F(C) \subset C$ . Further, in view of assumption (iii) we have

$$|F(w)(x, y) - F(v)(x, y)| \leq ((A_1 + A_2)|w - v|)(x, y) \quad (5)$$

where

$$(A_1 u)(x, y) = L_1 \int_{D(h(x, y))} u(t, s) dt ds \quad \text{and} \quad (A_2 u)(x, y) = L_2 u(H(x, y))$$

for  $u \in C$  and  $(x, y) \in I^2$ .

Now we prove that the operators  $A_1 + A_2$  and  $F$  satisfy the assumptions of Theorem 3. Obviously,  $(A_1 + A_2)(C) \subset C$  and the operator  $A_1 + A_2$  is linear and bounded. Moreover,  $A_1 + A_2$  is an increasing operator with respect to the binary relation  $w \prec u$  if and only if  $w(x, y) \leq u(x, y)$  for every  $(x, y) \in I^2$ . Let us remark that in view of (5) the assumption  $9^0$  of Theorem 3 is satisfied taking  $(m(u))(x, y) := |u(x, y)|$ .

To end our proof it is enough to show that  $r(A_1 + A_2) < 1$ . We have

$$\begin{aligned} (A_1 A_2 u)(x, y) &= L_1 L_2 \int_{D(h(x, y))} u(H(t, s)) dt ds \\ (A_2 A_1 u)(x, y) &= L_1 L_2 \int_{D(h(H(x, y)))} u(t, s) dt ds \end{aligned}$$

for  $u \in C$  and  $(x, y) \in I^2$ . Let  $K$  denote the cone of non-negative functions in  $C$ , i.e.

$$K = \left\{ w \in C : w(x, y) \geq 0 \text{ for every } (x, y) \in I^2 \right\}.$$

This cone is normal and generating. Obviously,  $A_1$  and  $A_2$  are positive on  $K$  and  $u_0$ -upper bounded. Further, in view of assumptions (ii) and (iv) we obtain

$$\begin{aligned} (A_2 A_1 u)(x, y) &= L_1 L_2 \int_{D(h(H(x, y)))} u(t, s) dt ds \\ &= L_1 L_2 \int_{H^{-1}(D(h(H(x, y))))} u(H(t, s)) |H'(t, s)| dt ds \\ &\leq L_1 L_2 \int_{H^{-1}(D(h(H(x, y))))} u(H(t, s)) dt ds \\ &\leq L_1 L_2 \int_{D(h(x, y))} u(H(t, s)) dt ds \\ &= (A_1 A_2 u)(x, y) \end{aligned}$$

for  $u \in K$  and  $(x, y) \in I^2$ , so the operators  $A_1$  and  $A_2$  are semicommutative on  $K$ . Analogously, if assumption (v) holds, one obtain  $A_1 A_2 \prec A_2 A_1$ , so  $A_1$  and  $A_2$  are again semicommutative on  $K$ . In view of Theorem 1 we obtain

$$r(A_1 + A_2) \leq r(A_1) + r(A_2).$$

It can be easily verified that

$$\|A_1^n w_0\|_c \leq \frac{L_1^n a^{2n}}{(n!)^2} \|w_0\| \quad \text{and} \quad \|A_2^n w_0\|_c = L_2^n \|w_0\|$$

for every  $n \in \mathbb{N}$  where  $w_0(x, y) \equiv 1$  on  $I^2$ . Thus  $r(A_1) = 0$  and  $r(A_2) = L_2$ . In view of the assumption  $L_2 < 1$  we have  $r(A_1 + A_2) < 1$ . Hence, applying Theorem 3 we deduce that the operator (4) has a unique fixed point in  $C$ , which obviously is a solution of the problem (2). Hence the proof of Theorem 4 is completed. ■

Now we consider the problem (2) and assume that the assumptions (i) and (iii) are satisfied. Moreover, assume that

(vi)  $H : I^2 \rightarrow I^2$  is a continuous function and  $h(H(x, y)) \leq h(x, y)$  for  $(x, y) \in I^2$ .

Our next result is the following

**Theorem 5.** *Under the assumptions (i), (iii), (vi) the problem (2) has exactly one solution on  $I^2$ .*

**Proof.** The idea of this proof is exactly the same as in the previous case. Hence it is enough to show that  $r(A_1 + A_2) \leq r(A_1) + r(A_2)$ . For this we will use Theorem 2. Obviously, for  $\theta \prec w$  we have  $\theta \prec A_1 w$  and  $\theta \prec A_2 w$ . Let  $w_0(x, y) \equiv 1$  on  $I^2$ . Since  $K$  is normal and  $w_0 \in \text{int}K$ ,

$$r(A_1 + A_2) = \lim_{n \rightarrow \infty} \|(A_1 + A_2)^n w_0\|^{\frac{1}{n}}$$

(cf. [3: Chapter II]). Moreover, for  $j \geq 1$  and  $k \geq 0$  we obtain

$$\begin{aligned} &(A_2 A_1^j A_2^k w_0)(x, y) \\ &= L_1^j L_2^{k+1} \int_{D(h(H(x, y)))} \int_{D(h(x_1, y_1))} \dots \int_{D(h(x_{j-1}, y_{j-1}))} 1 \, dx_j dy_j \dots dx_1 dy_1 \end{aligned}$$

and

$$\begin{aligned} &(A_1^j A_2^{k+1} w_0)(x, y) \\ &= L_1^j L_2^{k+1} \int_{D(h(x, y))} \int_{D(h(x_1, y_1))} \dots \int_{D(h(x_{j-1}, y_{j-1}))} 1 \, dx_j dy_j \dots dx_1 dy_1. \end{aligned}$$

Thus, in view of assumption (vi),  $A_2 A_1^j A_2^k w_0 \prec A_1^j A_2^{k+1} w_0$ . By Theorem 2 we infer that  $r(A_1 + A_2) \leq r(A_1) + r(A_2)$ . The proof of Theorem 5 is completed ■

### 3. Remarks

In what follows we indicate differences between Theorem 4 and Theorem 5, and compare our methods of a proof with a method based on direct application of the Banach contraction principle.

**Remark 2.** In Theorem 5 we do not assume that  $H|_{\text{int}I^2}$  is a diffeomorphism and  $H(\text{int}I^2) = \text{int}I^2$ . Hence Theorem 5 does not follow from Theorem 4, obviously. For example, the functions  $h$  and  $H$  given by  $h(x, y) = H(x, y) = (\frac{x}{2}, \frac{y}{2})$  ( $(x, y) \in I^2$ ) satisfy assumptions (i) and (vi), so they can illustrate Theorem 5. Now let

$$H(x, y) = (y, x) \quad \text{and} \quad h(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right) \quad ((x, y) \in I^2).$$

Obviously, the function  $H$  satisfies the assumption (ii). Further, we have  $|H'(x, y)| = 1$  for every  $(x, y) \in \text{int}I^2$  and

$$\begin{aligned} D(h(H(x, y))) &= D(h(y, x)) = D\left(\frac{y}{2}, \frac{x}{2}\right) \\ H(D(h(x, y))) &= H\left(D\left(\frac{x}{2}, \frac{y}{2}\right)\right) = D\left(\frac{y}{2}, \frac{x}{2}\right) \end{aligned}$$

so the function  $H$  satisfies the assumptions (iv) and (v). Let us remark that the inequality  $h(H(x, y)) \leq h(x, y)$  is satisfied only if  $x = y$ . Hence the functions  $h$  and  $H$  do not satisfy the assumption (vi) in Theorem 5.

**Remark 3.** To prove an existence theorem for the problem (2) one can apply the Banach contraction principle. In this case it is enough to assume that the functions  $h$  and  $H$  are continuous. But this method requires the assumption  $L_1 a^2 + L_2 < 1$ , where  $a$ ,  $L_1$  and  $L_2$  denote the same constants as in Section 2. Recall that in Theorems 4 and 5 we assume that  $L_1 > 0$  and  $0 < L_2 < 1$ .

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