Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 16 (1997), No. 3, 565–573

# A Multiplier Approach to the Lance-Blecher Theorem

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**Abstract.** A new approach to a theorem of E. C. Lance and D. P. Blecher in Hilbert C*-module theory and to two extensions of it is presented resting on a reinterpretation of key structural elements in terms of multiplier theory of operator C*-algebras. In the course of proving further facts are obtained.

**Keywords:** *Hilbert C*-modules, isometric isomorphisms, multiplier theory of C*-algebras, norms and C*-valued inner products*

**AMS subject classification:** Primary 46 L 99, secondary 46 H 25

## 1. Introduction

We give an alternative purely C*-algebraic proof of the following fact which was first discovered by E. C. Lance [11: Theorem] and D. P. Blecher ([2: Theorems 3.1 and 3.2] and [1]): the Hilbert norm on a Hilbert C*-module allows to recover the values of the inducing C*-valued inner product in a unique way, and two Hilbert C*-modules $\{\mathcal{M}_1, \langle\cdot,\cdot\rangle_1\}$, $\{\mathcal{M}_2, \langle\cdot,\cdot\rangle_2\}$ are isometrically isomorphic as Banach C*-modules if and only if there exists a bijective C*-linear map $S : \mathcal{M}_1 \to \mathcal{M}_2$ such that the identity $\langle\cdot,\cdot\rangle_1 \equiv \langle S(\cdot), S(\cdot)\rangle_2$ is valid. Extending this result, we obtain that two C*-valued inner products on a Banach C*-module inducing norms equivalent to the given one give rise to isometrically isomorphic Hilbert C*-modules if and only if the derived C*-algebras of "compact" module operators are $*$-isomorphic. Moreover, the dual norm on the $A$-dual Banach $A$-module $\mathcal{M}'$ which is induced by the Hilbert norm on $\mathcal{M}$ allows to recover the $A$-valued inner product on $\mathcal{M}$ up to unitary equivalence. The involution and the C*-norm of the C*-algebra of "compact" module operators on a Hilbert C*-module determine the original C*-valued inner product on the module up to the following equivalence relation: $\langle\cdot,\cdot\rangle_1 \sim \langle\cdot,\cdot\rangle_2$ if and only if there exists an invertible, positive element $a$ of the center of $\mathrm{M}(A)$ such that the identity $\langle\cdot,\cdot\rangle_1 \equiv a \cdot \langle\cdot,\cdot\rangle_2$ holds. If the center of $\mathrm{M}(A)$ is trivial, then one has only to fix the Hilbert norm for one singular non-zero element of the Hilbert C*-module to make the choice unique.

The importance of these assertions is caused by examples of unitarily non-isomorphic C*-valued inner products on some Banach C*-modules which nevertheless induce equivalent norms to the given one (L. G. Brown [3: Examples 6.2 and 6.3]; cf. [14: Example 2.3]).

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ISSN 0232-2064 / $ 2.50 

The basic observation of our approach is the reinterpretation of C*-valued inner products on a given Hilbert C*-module which induce norms equivalent to the given one in terms of positive invertible quasi-multipliers of the C*-algebra of "compact" module operators arising from the original C*-valued inner product. Involving results on quasi-multipliers of C*-algebras due to L. G. Brown [3] we can formulate our multiplier C*-theory based proof of the statements above.

## 2. Preliminaries

We start our investigations recalling some definitions and basic facts from the literature (cf. [8 - 10, 12, 15]). We consider Hilbert C*-modules $\{\mathcal{M}, \langle\cdot,\cdot\rangle\}$ over general C*-algebras $A$, i.e. (left) $A$-modules $\mathcal{M}$ together with an $A$-valued inner product $\langle\cdot,\cdot\rangle : \mathcal{M}\times\mathcal{M} \to A$ satisfying the following conditions:

**(i)** $\langle x,x\rangle \geq 0$ for every $x \in \mathcal{M}$.

**(ii)** $\langle x,x\rangle = 0$ if and only if $x = 0$.

**(iii)** $\langle x,y\rangle = \langle y,x\rangle^*$ for every $x,y \in \mathcal{M}$.

**(iv)** $\langle ax+by,z\rangle = a\langle x,z\rangle + b\langle y,z\rangle$ for every $a,b \in A$ and $x,y,z \in \mathcal{M}$.

**(v)** $\mathcal{M}$ is complete with respect to the norm $\|x\| = \|\langle x,x\rangle\|_A^{\frac{1}{2}}$.

We always suppose that the linear structures of the C*-algebra $A$ and of the (left) $A$-module $\mathcal{M}$ are compatible, i. e. $\lambda(ax) = (\lambda a)x = a(\lambda x)$ for every $\lambda \in \mathbb{C}$, $a \in A$ and $x \in \mathcal{M}$. A Hilbert C*-module is said to be *full* if the norm-closed linear span of the values of the C*-valued inner product coincides with its C*-algebra of coefficients.

Let us denote the $A$-dual Banach $A$-module of a Hilbert $A$-module $\{\mathcal{M}, \langle\cdot,\cdot\rangle\}$ by $\mathcal{M}'$, where

$$\mathcal{M}' = \left\{ r : \mathcal{M} \to A \,\middle|\, r \text{ is } A\text{-linear and bounded} \right\}.$$

A Hilbert C*-module $\{\mathcal{M}, \langle\cdot,\cdot\rangle\}$ is *self-dual* if the standard isometric C*-linear embedding $x \in \mathcal{M} \to \langle\cdot,x\rangle \in \mathcal{M}'$ is surjective.

The class of (self-dual) Hilbert W*-modules is of special interest. Many pathologies can be avoided for them because the C*-valued inner product lifts always to the C*-dual Banach W*-module turning it into a self-dual Hilbert W*-module [15]. To each Hilbert C*-module $\{\mathcal{M}, \langle\cdot,\cdot\rangle\}$ over a C*-algebra $A$ one can assign a standard Hilbert W*-module over the bidual W*-algebra $A^{**}$ of $A$ in the following way (cf. [14: Definition 1.3] and [15: Section 4]): form the algebraic tensor product $A^{**} \otimes \mathcal{M}$ which becomes a (left) $A^{**}$-module defining the action of $A^{**}$ on its elementary tensors by the formula $ab \otimes x = a(b \otimes x)$ for $a,b \in A^{**}$ and $x \in \mathcal{M}$. Now, setting

$$\left[\sum_i a_i \otimes x_i, \sum_j b_j \otimes y_j\right] = \sum_{i,j} a_i \langle x_i, y_j\rangle b_j^*$$

on finite sums of elementary tensors we obtain a degenerate $A^{**}$-valued inner pre-product. The completion of the factorization of $A^{**} \otimes \mathcal{M}$ by the set $\{z \in A^{**} \otimes \mathcal{M} : [z,z] = 0\}$ gives a Hilbert $A^{**}$-module denoted by $\mathcal{M}^{\#}$ in the sequel. It contains $\mathcal{M}$ as an $A$-submodule. If $\mathcal{M}$ is self-dual, then $\mathcal{M}^{\#}$ is self-dual, too, but the converse conclusion is still an open problem. Every bounded $A$-linear operator $T$ on $\mathcal{M}$ has a unique extension to a bounded $A^{**}$-linear operator on $\mathcal{M}^{\#}$ preserving the operator norm.

In the following we want to consider several kinds of module operators on Hilbert C*-modules. An $A$-linear bounded operator $K$ on a Hilbert $A$-module $\{\mathcal{M}, \langle\cdot,\cdot\rangle\}$ is "compact" if it belongs to the norm-closed linear hull of the set of elementary operators

$$\left\{\theta_{x,y} \middle| \theta_{x,y}(z) = \langle z, x\rangle y \quad \text{for} \quad x, y \in \mathcal{M}\right\}$$

(see [10, 15]). The set of all "compact" operators on $\mathcal{M}$ is denoted by $\mathrm{K}_A(\mathcal{M})$. A bounded $A$-linear operator $T$ on a Hilbert C*-module $\mathcal{M}$ is *adjointable* if the operator $T^*$ defined by the formula $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for all $x, y \in \mathcal{M}$ is a bounded $A$-linear operator on $\mathcal{M}$. By [9, 10] the C*-algebra $\mathrm{K}_A(\mathcal{M})$ is a two-sided ideal of the set of all bounded, adjointable module operators $\mathrm{End}_A^*(\mathcal{M})$ on $\mathcal{M}$ which is $*$-isomorphic to its multiplier C*-algebra.

To characterize unitary isomorphisms of Hilbert C*-modules we use the following definition.

**Definition 1.** Let $A$ be a fixed C*-algebra. Two Hilbert $A$-modules $\{\mathcal{M}_1, \langle\cdot,\cdot\rangle_1\}$ and $\{\mathcal{M}_2, \langle\cdot,\cdot\rangle_2\}$ are said to be *isomorphic as Hilbert C*-modules* (or, equivalently, *unitarily isomorphic*) if there exists a linear bijective mapping $S : \mathcal{M}_1 \to \mathcal{M}_2$ such that the equalities $S(ax) = aS(x)$ and $\langle x, y\rangle_1 = \langle S(x), S(y)\rangle_2$ are valid for every $a \in A$ and every $x, y \in \mathcal{M}_1$.

The literature contains some results about the existence of such isomorphisms between Hilbert C*-modules: if a Hilbert $A$-module $\{\mathcal{M}, \langle\cdot,\cdot\rangle_1\}$ over a given C*-algebra $A$ is self-dual, then every $A$-valued inner product $\langle\cdot,\cdot\rangle_2$ on $\mathcal{M}$ inducing a norm equivalent to the given one fulfills the identity $\langle\cdot,\cdot\rangle_2 = \langle S(\cdot), S(\cdot)\rangle_1$ on $\mathcal{M} \times \mathcal{M}$ for a unique positive invertible bounded $A$-linear operator $S$ on $\mathcal{M}$ (cf. [5: Theorem 2.6]). Similarly, E. C. Lance proved for arbitrary Hilbert $A$-modules $\mathcal{M}_1$ and $\mathcal{M}_2$ over a fixed C*-algebra $A$ that in case of an existing bounded $A$-linear adjointable operator $T : \mathcal{M}_1 \to \mathcal{M}_2$ with dense ranges for both $T$ and $T^*$ there exists a unitary isomorphism of $\mathcal{M}_1$ and $\mathcal{M}_2$ [12: Proposition 3.8]. Countably generated Hilbert C*-modules are isomorphic as Banach C*-modules if and only if they are isometrically isomorphic as Banach C*-modules (compare [3: Corollary 4.8 and Theorem 4.9] and [6: Theorem 3.1]).

To explain what kind of general results one could obtain we prefer to rely on multiplier theory of C*-algebras. The fundamental result of H. Lin cited below appears to be very helpful. (In fact, it extends a well-known result of P. Green and G. G. Kasparov.)

**Proposition 2** ([13: Theorems 1.5 and 1.6]; cf. also [9, 10, 17]). *Let $A$ be a C*-algebra and $\{\mathcal{M}, \langle\cdot,\cdot\rangle\}$ be a Hilbert $A$-module. Then:*

**(i)** *The mapping $\phi$ defined by the formula*

$$\phi : \mathrm{End}_A(\mathcal{M}, \mathcal{M}') \longrightarrow \mathrm{QM}(\mathrm{K}_A(\mathcal{M}))$$
$$\theta_{x,y}\phi(T)\theta_{z,t} = \theta_{(T(t)(x))z,y} \quad (x, y, z, t \in \mathcal{M})$$

*is an isometric isomorphism of involutive Banach spaces.*

**(ii)** *The restriction of* $\phi$ *to* $\mathrm{End}_A(\mathcal{M})$ *induces an isometric algebraic isomorphism to the Banach algebra* $\mathrm{LM}(\mathrm{K}_A(\mathcal{M}))$.

**(iii)** *The restriction of* $\phi$ *to* $\mathrm{End}^*_A(\mathcal{M})$ *induces a* $*$*-isomorphism to the* $C^*$*-algebra* $\mathrm{M}(\mathrm{K}_A(\mathcal{M}))$.

Note that every left Hilbert $A$-module $\mathcal{M}$ can be considered as a right Hilbert $\mathrm{K}_A(\mathcal{M})$-module fixing another $\mathrm{K}_A(\mathcal{M})$-valued inner product $\langle x,y\rangle_{Op.} = \theta_{x,y}$. This point of view gives another interpretation of the left actions of $\mathrm{M}(A)$ and of $\mathrm{LM}(A)$ on full Hilbert $A$-modules $\mathcal{M}$ by Proposition 2.

## 3. The results

Our key observation is that every $A$-valued inner product $\langle\cdot,\cdot\rangle$ on a Hilbert C*-module $\mathcal{M}$ over a given C*-algebra $A$ defines a mapping $T$ from $\mathcal{M}$ into its $A$-dual Banach $A$-module $\mathcal{M}'$ by the formula $T : x \in \mathcal{M} \to \langle\cdot,x\rangle \in \mathcal{M}'$. The properties of these mappings $T$ in terms of multiplier C*-theory are the following ones.

**Proposition 3.** *Let $A$ be a $C^*$-algebra and let $\{\mathcal{M},\langle\cdot,\cdot\rangle_1\}$ be a Hilbert $A$-module. Denote by $\langle\cdot,\cdot\rangle_2$ a second $A$-valued inner product on $\mathcal{M}$ inducing a norm equivalent to the given one. Then the mapping $T : x \in \mathcal{M} \to \langle\cdot,x\rangle_2 \in \mathcal{M}'$ can be identified with a uniquely defined invertible positive element of* $\mathrm{QM}(\mathrm{K}_A(\mathcal{M})) \subset \mathrm{K}_A^{(1)}(\mathcal{M})^{**}$. *Conversely, every invertible positive element* $T' \in \mathrm{QM}(\mathrm{K}_A(\mathcal{M})) \subset \mathrm{K}_A^{(1)}(\mathcal{M})^{**}$ *induces an $A$-valued inner product and an equivalent norm on $\mathcal{M}$ via the formula $\langle x,y\rangle_2 = (\phi^{-1}(T')(y))(x)$ for $x,y \in \mathcal{M}$.*

**Proof.** Using the identifications of Proposition 2 made by the mapping $\phi$ one derives the equality

$$\theta^{(1)}_{x,y}\phi(T)\theta^{(1)}_{z,t} = \theta^{(1)}_{\langle x,t\rangle_2 z,y} \in \mathrm{K}_A^{(1)}(\mathcal{M})$$

which defines $\phi(T) \in \mathrm{QM}(\mathrm{K}_A(\mathcal{M}))$ by the right side of this equality. To show the positivity of the quasi-multiplier $\phi(T)$ one modifies the equality above setting $x = t$ and $y = z$. Making use of the identity $\theta_{x,y} = \theta^*_{y,x}$ valid for every $x,y \in \mathcal{M}$ one obtains

$$\left\langle\theta^{(1)}_{\langle x,x\rangle_2 t,t}(s),s\right\rangle_1 = \left\langle\langle s,t\rangle_1\langle z,z\rangle_2 t,s\right\rangle_1 = \langle s,t\rangle_1\langle z,z\rangle_2\langle t,s\rangle_1 \geq 0$$

for every $s \in \mathcal{M}$. Since $\phi(T) \in \mathrm{K}_A^{(1)}(\mathcal{M})^{**}$ by construction and since the linear span of the "compact" operators of type $\theta$ is norm dense inside $\mathrm{K}_A^{(1)}(\mathcal{M})$ the positivity of $\phi(T)$ as an element of the W*-algebra $\mathrm{K}_A^{(1)}(\mathcal{M})^{**}$ follows.

To show the invertibility of $\phi(T)$ inside $\mathrm{K}_A^{(1)}(\mathcal{M})^{**}$ we use a standard construction from the introduction. First, build the Hilbert $A^{**}$-module $\mathcal{M}^{\#}$ from $\mathcal{M}$. Both the $A$-valued inner products on $\mathcal{M}$ can be extended to $A^{**}$-valued inner products on $\mathcal{M}^{\#}$ in a unique way. Secondly, take the (self-dual) $A^{**}$-dual Hilbert $A^{**}$-module $(\mathcal{M}^{\#})'$ of $\mathcal{M}^{\#}$. Again, both the inner products can be continued (cf. [15: Theorems 3.2 and 3.6]), and their extensions are connected by an invertible positive operator $S$ as described in

[5: Theorem 2.6]. Obviously, the uniquely defined extension of the operator $T$ inside $\mathrm{End}_{A^{**}}((\mathcal{M}^{\#})')$ equals $S^*S$. Hence, the (real) spectrum of $T$ is deleted away from zero by a positive constant, and $\phi(T)$ is invertible.

Conversely, set $\langle x,y\rangle_2 = (\phi^{-1}(T')(y))(x)$ for $x,y \in \mathcal{M}$ and for a given invertible positive $T' \in \mathrm{QM}(\mathrm{K}_A(\mathcal{M})) \subset \mathrm{K}_A^{(1)}(\mathcal{M})^{**}$. As can be easily seen by considerations similar to that above $\langle\cdot,\cdot\rangle_2$ is an $A$-valued inner product on $\mathcal{M}$ inducing a norm equivalent to the given one ∎

**Example 4.** Let $A$ be a C*-algebra. Define the action of $A$ on itself by multiplication from the left. Then $A$ becomes a Hilbert $A$-module setting $\langle a,b\rangle_T = aTb^*$ for every $a,b \in A$ and a fixed positive invertible $T \in \mathrm{QM}(A)$. Vice versa, every $A$-valued inner product on $A$ arises in this manner. If $A$ is unital, then $T \in A \equiv \mathrm{QM}(A)$.

**Theorem 5** (E. C. Lance [11: Theorem] and D. P. Blecher [2: Theorems 3.1 and 3.2]). *Let $A$ be a C*-algebra and $\mathcal{M}$ be a left Banach $A$-module the norm of which is known to be generated by an $A$-valued inner product on $\mathcal{M}$ with unknown values. Then this $A$-valued inner product $\langle\cdot,\cdot\rangle$ on $\mathcal{M}$ is unique, and the values can be recovered by the formulae*

$$\langle x,x\rangle := \sup\left\{r(x)r(x)^* \,\middle|\, r \in \mathcal{M}' \ \text{ with } \ \|r\| \le 1\right\} \tag{1}$$

$$\langle x,y\rangle := \frac{1}{4}\sum_{k=0}^{3} \mathrm{i}^k\langle x+\mathrm{i}^k y, x+\mathrm{i}^k y\rangle$$

*for every $x,y \in \mathcal{M}$, where the right side of (1) uses the norm of the underlying Banach $A$-module only.*

*Consequently, every bijective isometric $A$-linear isomorphism of two Hilbert $A$-modules $S : \{\mathcal{M}_1,\langle\cdot,\cdot\rangle_1\} \to \{\mathcal{M}_2,\langle\cdot,\cdot\rangle_2\}$ identifies the two $A$-valued inner products by the formula $\langle\cdot,\cdot\rangle_1 \equiv \langle S(\cdot),S(\cdot)\rangle_2$, and vice versa.*

**Proof.** To show the estimation formula (1) of the values of the $A$-valued inner product and the uniqueness of the derived $A$-valued inner product $\langle\cdot,\cdot\rangle$ we recall that

$$r(x)r(x)^* \le \|r\|^2\langle x,x\rangle$$

for every $x \in \mathcal{M}$ and every $r \in \mathcal{M}'$ by [15: Theorem 2.8]. Since any suitable $A$-valued inner product $\langle\cdot,\cdot\rangle$ induces an isometric $A$-linear embedding of $\mathcal{M}$ into $\mathcal{M}'$ by the formula $x \to \langle\cdot,x\rangle$ one has only to indicate a sequence $\{r_n\}_{n\in\mathbb{N}}$ of bounded by one $A$-linear functionals on $\mathcal{M}$ of this special nature such that the set $\{r_n(x)r_n(x)^*\}_{n\in\mathbb{N}}$ converges to the value $\langle x,x\rangle$ in norm from below. This can be arranged by setting $r_n(\cdot) = \langle\cdot,(\langle x,x\rangle + \frac{1}{n}\cdot 1_A)^{-\frac{1}{2}}x\rangle$ for $n \in \mathbb{N}$. Consequently, the supremum really exists, it is unique and depends only on the norm given on $\mathcal{M}$. The second formula is obvious.

To give an alternative uniqueness argument we use multiplier theory. Let $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ be two $A$-valued inner products on $\mathcal{M}$ giving the norm. Applying again the standard construction from the introduction both the $A$-valued inner products can be continued to $A^{**}$-valued inner products on the self-dual Hilbert $A^{**}$-module $(\mathcal{M}^{\#})'$. Inside $\mathrm{End}_{A^{**}}((\mathcal{M}^{\#})')$ there exists a positive invertible operator $T$ such that the identity

$\langle\cdot,\cdot\rangle_2 \equiv \langle T(\cdot),\cdot\rangle_1$ holds for the continued $A^{**}$-valued inner products on $(\mathcal{M}^\#)' \times (\mathcal{M}^\#)'$ (cf. [5: Proposition 2.2]). By construction one has

$$\|x\| \equiv \|\langle T(x),x\rangle_1\|_A \equiv \|\langle x,x\rangle_1\|_A$$
$$\|x\| \equiv \|\langle T^{-1}(x),x\rangle_2\|_A \equiv \|\langle x,x\rangle_2\|_A$$

and by a theorem of W. L. Paschke [15: Theorem 2.8] one obtains

$$\|\langle T(x),x\rangle_1\|_A \leq \|T^{\frac{1}{2}}\|^2 \cdot \|\langle x,x\rangle_1\|_A$$
$$\|\langle T^{-1}(x),x\rangle_2\|_A \leq \|T^{-\frac{1}{2}}\|^2 \cdot \|\langle x,x\rangle_2\|_A.$$

This implies $\|T\| = \|T^{-1}\| = 1$, and by the positivity of $T$ and general spectral properties of elements of C*-algebras $T = \mathrm{id}_{\mathcal{M}}$ yields.

To show the last statement one has to consider the two $A$-valued inner products $\langle\cdot,\cdot\rangle_1$ and $\langle S(\cdot),S(\cdot)\rangle_2$ on the Hilbert $A$-module $\mathcal{M}_1$ inducing exactly the same norm for every element of $\mathcal{M}_1$. Therefore, they admit identically the same values by the first part of the proof ∎

**Proposition 6.** *Let $A$ be a C*-algebra and $\mathcal{M}$ be a Banach $A$-module possessing two $A$-valued inner products $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ inducing norms equivalent to the given one on $\mathcal{M}$. Suppose, $0 < C,D < \infty$ are the minimal real numbers for which the inequality $\|x\|_1 \leq C\,\|x\|_2 \leq D\,\|x\|_1$ is satisfied for every $x \in \mathcal{M}$. Then the inequality*

$$\langle x,x\rangle_1 \leq C^2\langle x,x\rangle_2 \leq D^2\langle x,x\rangle_1 \qquad (2)$$

*is valid for every $x \in \mathcal{M}$ and the same real numbers $C$ and $D$.*

**Proof.** We extend both the $A$-valued inner products to $A^{**}$-valued inner products to the self-dual Hilbert $A^{**}$-module $(\mathcal{M}^\#)'$ using the standard construction. Then there exists a positive invertible operator $T$ inside $\mathrm{End}_{A^{**}}((\mathcal{M}^\#)')$ such that the identity $\langle\cdot,\cdot\rangle_2 \equiv \langle T(\cdot),\cdot\rangle_1$ holds for the continued $A^{**}$-valued inner products on $(\mathcal{M}^\#)' \times (\mathcal{M}^\#)'$ (cf. [5]). Applying [15: Theorem 2.8] to both the operators $T$ and $T^{-1}$ on $(\mathcal{M}^\#)'$ one obtains the minimal real numbers $C^2 = \|T^{-1}\|^2$ and $D^2 = \|T\|^2\,\|T^{-1}\|^2$ for which the inequality (2) is valid, and these constants equal the squares of the minimal constants obtained in the comparison inequality of the two norms ∎

**Remark 7.** The expression (1) could make sense for more general Banach C*-modules than Hilbert C*-modules. However, if it would be well-defined for every element $x$ of a Banach, non-Hilbert C*-module $\mathcal{M}$, then it should be non-C*-linear and/or degenerated, anyway. In a manuscript of N. C. Phillips and N. Weaver entitled "*Modules with norms which take values in a C*-algebra*" (funct-an # 9612005) the following fact appeared: if a C*-algebra $A$ has no non-zero commutative ideals, then every $A$-module $\mathcal{M}$ which is equipped with a map $\rho : \mathcal{M} \to A_+$ into the positive cone $A_+$ of $A$ such that

(1) the map $\|\cdot\|_{\mathcal{M}} : x \to \|\rho(x)\|_A$ is a norm on the linear space $\mathcal{M}$

(2) $\rho(ax)^2 = a\,\rho(x)^2 a^*$ for every $a \in A$ and $x \in \mathcal{M}$

must be a pre-Hilbert C*-module. This shows the robustness of the concept of Hilbert C*-modules.

For more similar results we refer the reader to the work of D. P. Blecher who has treated Hilbert C*-modules as operator spaces and operator modules over (non-self-adjoint) operator algebras using mainly geometric notions like complete contractability and complete boundedness of mappings (see, for example, [1, 2]). The advantage of our approach comes to light in the following statements characterizing isometric isomorphisms of different C*-valued inner products on a fixed Banach C*-module in terms of $*$-isomorphisms of the related operator C*-algebras. Also we show the possibility to recover the values of the $A$-valued inner product on $\mathcal{M}$ from the dual norm on $\mathcal{M}'$.

**Theorem 8.** *Let $A$ be a $C^*$-algebra and let $\{\mathcal{M}, \langle\cdot,\cdot\rangle_1\}$ be a Hilbert $A$-module. Let $\langle\cdot,\cdot\rangle_2$ be another $A$-valued inner product on $\mathcal{M}$ inducing a norm equivalent to the given one. The following conditions are equivalent:*

**(i)** *The $A$-valued inner product $\langle\cdot,\cdot\rangle_2$ on $\mathcal{M}$ is generated by an invertible bounded $A$-linear operator $S$ on $\mathcal{M}$ satisfying the identity $\langle\cdot,\cdot\rangle_2 \equiv \langle S(\cdot), S(\cdot)\rangle_1$ on $\mathcal{M} \times \mathcal{M}$.*

**(ii)** *The positive invertible quasi-multiplier $T$ of $\mathrm{K}_A^{(1)}(\mathcal{M})$ corresponding to the $A$-valued inner product $\langle\cdot,\cdot\rangle_2$ by Proposition 3 is decomposable as $T = S^*S$ for at least one invertible left multiplier $S$ of $\mathrm{K}_A^{(1)}(\mathcal{M})$.*

**(iii)** *The $C^*$-algebra $\mathrm{K}_A^{(2)}(\mathcal{M})$ of "compact" operators on $\mathcal{M}$ corresponding to the $A$-valued inner product $\langle\cdot,\cdot\rangle_2$ is $*$-isomorphic to the original $C^*$-algebra of "compact" operators $\mathrm{K}_A^{(1)}(\mathcal{M})$.*

**(iv)** *The $C^*$-algebra $\mathrm{End}_A^{*,(2)}(\mathcal{M})$ of adjointable bounded $A$-linear operators on $\mathcal{M}$ corresponding to the $A$-valued inner product $\langle\cdot,\cdot\rangle_2$ is $*$-isomorphic to the original $C^*$-algebra of adjointable bounded $A$-linear operators $\mathrm{End}_A^{*,(1)}(\mathcal{M})$.*

**Proof.** The implications (i) $\Leftrightarrow$ (ii) follow from Proposition 2 together with the key Proposition 3. Keeping in mind Proposition 3 one adapts L. G. Brown's results on quasi-multipliers [3: Theorem 4.2 and Proposition 4.4] of (non-unital) C*-algebras to the C*-algebra $\mathrm{K}_A^{(1)}(\mathcal{M})$: For a positive invertible quasi-multiplier $T$ of $\mathrm{K}_A^{(1)}(\mathcal{M})$ the C*-subalgebra $T^{\frac{1}{2}}\mathrm{K}_A^{(1)}(\mathcal{M})T^{\frac{1}{2}}$ of the bidual W*-algebra $\mathrm{K}_A^{(1)}(\mathcal{M})^{**}$ is $*$-isomorphic to $\mathrm{K}_A^{(1)}(\mathcal{M})$ if and only if there exists a left multiplier $S$ of $\mathrm{K}_A^{(1)}(\mathcal{M})$ such that $T = S^*S$ inside $\mathrm{K}_A^{(1)}(\mathcal{M})^{**}$. Thus, one obtains the equivalence of the conditions (ii) and (iii) (cf. [4, 6]). The equivalence of the last two statements is shown in [7] ∎

**Theorem 9.** *Let $A$ be a $C^*$-algebra and $\mathcal{M}'$ be the $A$-dual Banach $A$-module of a Hilbert $A$-module $\mathcal{M}$. The norm on $\mathcal{M}'$ which is dual to a Hilbert norm on $\mathcal{M}$ determines the inducing $A$-valued inner product on $\mathcal{M}$ up to unitary equivalence.*

**Proof.** Let $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ be two $A$-valued inner products on $\mathcal{M}$ giving rise to the same dual norm on $\mathcal{M}'$. Following the same idea as in the proof of Theorem 5 we find a unique positive invertible operator $T \in \mathrm{End}_{A^{**}}((\mathcal{M}^{\#})')$ such that the identity $\langle\cdot,\cdot\rangle_2 \equiv \langle T(\cdot),\cdot\rangle_1$ holds on $(\mathcal{M}^{\#})' \times (\mathcal{M}^{\#})'$. By [15: Theorem 2.8] the inequalities

$$r(x)r(x)^* \leq \|r\|\langle x,x\rangle_1\,, \qquad r(x)r(x)^* \leq \|r\|\langle x,x\rangle_2 \leq \|r\|\|T\|\langle x,x\rangle_1$$

are valid for every $x \in \mathcal{M}$, $r \in \mathcal{M}'$, where $\|r\| = \inf\{C \in \mathbb{R} \,|\, r(x)r(x)^* \leq C\langle x,x\rangle_{1,2}\}$, $\|T\| = \inf\{C \in \mathbb{R} \,|\, \langle x,x\rangle_2 \leq C\langle x,x\rangle_1\}$. Consequently, $\|r\| = \|r\|\|T\|$ and $\|T\| = 1$, and also $\|T^{-1}\| = 1$ by the symmetry of the situation. Spectral theory forces $T = \mathrm{id}_{\mathcal{M}}$ ∎

**Example 10.** Every positive invertible quasi-multiplier $T$ of a (non-unital) C*-algebra $A$ is decomposable as $T = S^*S$ for at least one invertible left multiplier $S$ of $A$ if and only if every pair of $A$-valued inner products $\langle\cdot,\cdot\rangle_1$, $\langle\cdot,\cdot\rangle_2$ on $A$ inducing equivalent norms to the given C*-norm is connected by an isometric Banach $A$-module isomorphism $S$ of the two corresponding (left) Banach $A$-modules $\{A, \|\cdot\|_1\}$ and $\{A, \|\cdot\|_2\}$ (cf. [13: Example 2.3] for an example of a non-decomposable positive invertible quasi-multiplier).

Note that the equivalence of the conditions of Theorem 8 does not hold any longer if one considers C*-valued inner products on different Banach C*-modules and $*$-isomorphisms of corresponding operator C*-algebras, in general. A counterexample can be found in [6, 7]. The canonical question arising is whether the original Hilbert norm can be recovered from the C*-norm of the related operator C*-algebras, or not. The answer is given by the following statement.

**Proposition 11.** *Let $A$ be a C*-algebra and $\{\mathcal{M}, \langle\cdot,\cdot\rangle_1\}$ be a full Hilbert $A$-module possessing a second $A$-valued inner product $\langle\cdot,\cdot\rangle_2$ which induces a norm equivalent to the given one. Suppose, both the $A$-valued inner products define the same bounded $A$-linear operators on $\mathcal{M}$ to be "compact", and both they induce the same involution and C*-norm on this algebra of all "compact" $A$-linear operators. Then there exists an invertible positive element $a$ of the center of the multiplier C*-algebra $\mathrm{M}(A)$ of $A$ such that the identity $\langle\cdot,\cdot\rangle_1 \equiv a \cdot \langle\cdot,\cdot\rangle_2$ holds on $\mathcal{M} \times \mathcal{M}$.*

*If the center of $\mathrm{M}(A)$ is trivial, then the condition $\|x\| = 1$ for some fixed non-zero $x \in \mathcal{M}$ makes the choice of the $A$-valued inner product on $\mathcal{M}$ unique.*

**Proof.** Since both the related C*-algebras of "compact" operators coincide, i.e. they are $*$-isomorphic, Theorem 8 applies: The invertible positive quasi-multiplier $T$ corresponding to the $A$-valued inner product $\langle\cdot,\cdot\rangle_2$ is decomposable as $T = S^*S$ for an invertible left multiplier $S$ which can be considered as a bounded invertible $A$-linear operator on $\mathcal{M}$ by Proposition 3. In particular, the inequality

$$\langle K(x), x\rangle_2 = \langle (SK)(x), S(x)\rangle_1 \geq 0$$

holds for every positive "compact" operator $K$ and every $x \in \mathcal{M}$. Consequently, $S$ commutes with every positive "compact" operator and belongs to the center of the multiplier C*-algebra $\mathrm{End}_A^*(\mathcal{M})$ of $\mathrm{K}_A(\mathcal{M})$. However, $\mathrm{Z}(\mathrm{End}_A^*(\mathcal{M}))$ consists of the operators $\{a \cdot \mathrm{id}_{\mathcal{M}} : a \in \mathrm{Z}(\mathrm{M}(A))\}$, and it is $*$-isomorphic to $\mathrm{Z}(\mathrm{M}(A))$. No further restrictions apply to $S$ and $T = S^*S$ since $\|x\|_1 = \|a^{-\frac{1}{2}} \cdot x\|_2$, $\|K(x)\|_1 = \|K(a^{-\frac{1}{2}} \cdot x)\|_2$ for every $x \in \mathcal{M}$ and every $K \in \mathrm{K}_A(\mathcal{M})$ (where $T = a \in \mathrm{Z}(\mathrm{M}(A))$) ∎

**Acknowledgement.** The author is indebted to Nien-Tsu Shen for submitting a copy of her Ph.D. thesis to him. He is grateful to D. P. Blecher, L. G. Brown and E. C. Lance for sending copies of their preprints and for comments on the subject. The special net of C*-linear functionals in the proof of Proposition 5 was obtained during discussions with A. S. Mishchenko. A. Kasparek brought the Hilbert space case of Proposition 11 to the authors attention.

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Received 01.08.1996, in revised form 16.06.1997