

A Priori Estimates for the Solution of Convection-Diffusion Problems and Interpolation on Shishkin Meshes

M. Dobrowolski and H.-G. Roos

Abstract. The solution of singularly perturbed convection-diffusion problems can be split into a regular and a singular part containing the boundary layer terms. In dimensions $n = 1$ and $n = 2$, sharp estimates of the derivatives of both parts up to order 2 are given. The results are applied to estimate the interpolation error for the solution on Shishkin meshes for piecewise bilinear finite elements on rectangles and piecewise linear elements on triangles. Using the anisotropic interpolation theory it is proved that the interpolation problem on Shishkin meshes is quasi-optimal in L_∞ and in the energy norm.

Keywords: *Singularly perturbed problems, convection-diffusion problems, finite elements, Shishkin meshes*

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1. Introduction

We are interested in robust numerical methods for the singularly perturbed convection-diffusion problem

$$\left. \begin{aligned} Lu := -\varepsilon \Delta u - b \cdot \nabla u + cu &= f & \text{in } \Omega = (0, 1)^2 \\ u &= 0 & \text{on } \Gamma = \partial\Omega. \end{aligned} \right\} \quad (1.1)$$

The analysis of such methods – both exponentially fitted methods and specially designed mesh methods (see [6, 7]) requires sharp estimates for the exact solution. Depending on the method, information about derivatives of order 2, 3 or even 4 is desirable (it is not very realistic to look for robust higher-order methods because, in general, the solution of an elliptic problem in a non-smooth domain is not very smooth).

If smoothness conditions on the data and compatibility conditions guarantee that $u \in C^{k,\lambda}(\bar{\Omega})$, and in addition $\|u\| \leq C$ is known, then we have the rough estimate (see Theorem 4 in the Appendix of [6])

$$\|u\|_k \leq C \varepsilon^{-k} \quad (1.2)$$

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where $\|\cdot\|$ is the maximum norm and $\|\cdot\|_k$ the standard norm in $C^k(\bar{\Omega})$, while C is a generic constant independent of the perturbation parameter ε . Let us now additionally assume

$$b_1(x, y) \geq 2\beta_1 > 0 \quad \text{and} \quad b_2(x, y) \geq 2\beta_2 > 0. \quad (1.3)$$

Then the solution of problem (1.1) is characterized by exponential layers concentrated at $x = 0$ and $y = 0$ (see [7]), and we wish to have sharper estimates than (1.2) that better reflect the layer structure. For instance, we proved in [8] that

$$\left. \begin{aligned} |u_x(x, y)| &\leq C \left(1 + \frac{1}{\varepsilon} \exp\left(-\frac{\beta_1}{\varepsilon}x\right) \right) \\ |u_y(x, y)| &\leq C \left(1 + \frac{1}{\varepsilon} \exp\left(-\frac{\beta_2}{\varepsilon}y\right) \right) \end{aligned} \right\} \quad (1.4)$$

using the maximum principle for elliptic systems.

The first aim of this paper is to present precise conditions under which estimates similar to (1.4) also hold for second and third order derivatives. Such conditions have a technical character and can be justified on the asymptotic expansions which we can see in the paper. In the book [9], Shishkin presented estimates for the solution of problem (1.1) in very general situations. But unfortunately, the precise assumptions are hidden in the text. Moreover, sometimes smoothness of some components is simply assumed which makes this source inconvenient.

For analyzing upwind difference schemes on special meshes (see [6] or [9]) Shishkin introduced the following splitting of the exact solution:

$$u = G + E, \quad (1.5)$$

where the smooth part G satisfies $LG = f$, while the layer part (also called the *singular part* in [6]) satisfies $LE = 0$. Additionally, G and E and derivatives of G and E up to a certain order can be estimated precisely (we explain this later in detail). A splitting into the smooth component and the layer component allows a corresponding splitting in the analysis of discretization methods and is therefore extremely useful. Working with majorizing functions based on the discrete maximum principle, the additional property $LE = 0$ of a Shishkin decomposition simplifies the argumentation (see [6]). The second aim of this paper is to classify the relation between splittings based on standard asymptotic expansions and a Shishkin splitting.

The estimates that we want to derive for the derivatives are the result of a careful investigation of an asymptotic decomposition of the solution. For transparency we explain the basic approach in Section 2 for a one-dimensional problem. This is much easier than the elliptic two-dimensional problem studied in Section 3 because in the 1-dimensional case compatibility conditions do not play any role. In Section 4 we apply our a priori estimates for deriving sharp bounds for the interpolation error on Shishkin meshes. Such interpolation results are not trivial because a two-dimensional Shishkin mesh does not satisfy the standard assumptions of the finite element technology.

For simplicity of the representation, we assume $c \equiv 0$. Let us further assume that b and f are sufficiently smooth. It would be possible to specify these smoothness conditions with respect to the data, but we are mainly interested in the careful examination of the necessary compatibility conditions in the two-dimensional case.

2. A one-dimensional problem with an exponential layer

Let us consider the boundary value problem

$$\left. \begin{aligned} Lu &:= -\varepsilon u'' - bu' = f \\ u(0) &= u(1) = 0 \end{aligned} \right\} \tag{2.1}$$

with $b \geq 2\beta > 0$ and introduce the notation $L = \varepsilon L_1 + L_0$. A standard asymptotic expansion of u has the form

$$u = u_0 + \varepsilon u_1 + \dots + \varepsilon^k u_k + v_0 + \varepsilon = v_1 + \dots + \varepsilon^k v_k + \varepsilon^{k+1} R. \tag{2.2}$$

Here the u_ℓ are determined by

$$\begin{aligned} L_0 u_0 &= f, \quad u_0(1) = 0 \\ L_0 u_\ell &= -L_1 u_{\ell-1}, \quad u_\ell(1) = 0 \quad (\ell = 1, \dots, k). \end{aligned}$$

At $x = 0$ an exponential boundary layer exists. Introducing $\xi = \frac{1}{\varepsilon}x$ and considering a Taylor expansion of L_ℓ in the new variable, we see that $v_\ell(\xi)$ satisfy

$$\begin{aligned} \mathcal{L}_0 v_0 &= 0, \quad v_0(0) = -u_0(0) \\ \mathcal{L}_0 v_\ell &= \mathcal{L}^*(v_0, \dots, v_{\ell-1}), \quad v_\ell(0) = -u_\ell(0), \quad \lim_{\xi \rightarrow \infty} v_\ell(\xi) = 0 \quad (\ell = 1, \dots, k) \end{aligned}$$

with

$$\mathcal{L}_0 = -\frac{d^2}{d\xi^2} - b(0)\frac{d}{d\xi} \quad \text{and} \quad \mathcal{L}^*(v_0, \dots, v_{\ell-1}) = \sum_{\mu=1}^{\ell} \frac{b^{(\mu)}(0)}{\mu!} \frac{dv_{\ell-\mu}}{d\xi}.$$

Therefore, we have for all ℓ

$$\left| \frac{\partial^m v_\ell}{\partial x^m} \right| \leq C \varepsilon^{-m} e^{-\frac{\beta}{\varepsilon}x}.$$

We obtain the following result.

Lemma 2.1 (Splitting based on the asymptotic expansion). *The solution u of problem (2.1) admits the splitting $u = G + E$, where the smooth part G satisfies for any prescribed finite order q*

$$|G^{(\ell)}| \leq C \quad \text{for } 0 \leq \ell \leq q, \tag{2.3}$$

while the layer part E satisfies $|E(0)| \leq C$, $|E(1)| \leq C e^{-\frac{\beta}{\varepsilon}}$ and

$$|E^{(\ell)}| \leq C \varepsilon^{-\ell} e^{-\frac{\beta}{\varepsilon}x} \quad \text{for } 0 \leq \ell \leq q. \tag{2.4}$$

Proof. The use of a suitable barrier function (inspecting the boundary value problem for the remainder R) leads to $\|R\| \leq C$. Then the rough estimate (1.2) yields $\|\frac{d^m R}{dx}\| \leq C \varepsilon^{-m}$. Setting

$$\left. \begin{aligned} G &= u_0 + \varepsilon u_1 + \dots + \varepsilon^k u_k + \varepsilon^{k+1} R \\ E &= v_0 + \varepsilon v_1 + \dots + \varepsilon^k v_k \end{aligned} \right\}$$

Lemma 2.1 follows immediately ■

Remark 2.2. It seems to us that Lemma 2.1 is well-known, but we do not know its origin. It is significant that the analysis of numerical methods using a decomposition that satisfies (2.3) and (2.4) is simpler than an analysis based on the bounds $|u^{(k)}(x)| \leq C(1 + \varepsilon^{-k} e^{-\frac{\ell}{\varepsilon}x})$ obtained in Kellogg et al. [4].

In the next step we modify (2.2) to obtain a Shishkin decomposition. Let us introduce, instead of (2.2),

$$u = u_0 + \varepsilon u_1 + \dots + \varepsilon^k u_k + \varepsilon^{k+1} u_{k+1}^* + v_0 + \varepsilon v_1 + \dots + \varepsilon^k v_k + \varepsilon^{k+1} v_{k+1}^* \quad (2.5)$$

where u_0, \dots, u_k and v_0, \dots, v_k are the standard terms of an asymptotic expansion. Now we define u_{k+1}^* and v_{k+1}^* by

$$\left. \begin{aligned} Lu_{k+1}^* &= -L_1 u_k \\ u_{k+1}^*(0) &= u_{k+1}^*(1) = 0 \end{aligned} \right\} \quad (2.6)$$

and

$$\left. \begin{aligned} Lv_{k+1}^* &= -\varepsilon^{-1} L(v_0 + \varepsilon v_1 + \dots + \varepsilon^k v_k) \\ v_{k+1}^*(0) &= 0, \quad v_{k+1}^*(1) = -(v_0 + \varepsilon v_1 + \dots + \varepsilon^k v_k)(1). \end{aligned} \right\} \quad (2.7)$$

If we introduce

$$\left. \begin{aligned} G^* &= u_0 + \varepsilon u_1 + \dots + \varepsilon^k u_k + \varepsilon^{k+1} u_{k+1}^* \\ E^* &= v_0 + \varepsilon v_1 + \dots + \varepsilon^{k+1} v_{k+1}^* \end{aligned} \right\}$$

we obtain the following assertion.

Lemma 2.3 (Shishkin decomposition.) *The solution of the boundary value problem (2.1) can be decomposed as $u = G^* + E^*$, where for any prescribed order q the smooth part G^* satisfies $LG^* = f$ and*

$$|G^{*(\ell)}| \leq C \quad \text{for } 0 \leq \ell \leq q$$

while the layer part E^* satisfies $LE^* = 0$, $|E^*(0)| \leq C$, $|E^*(1)| \leq C e^{-\frac{\ell}{\varepsilon}}$ and

$$|E^{*(\ell)}(x)| \leq C \varepsilon^{-\ell} e^{-\frac{\ell}{\varepsilon}x} \quad \text{for } 0 \leq \ell \leq q.$$

Remark 2.4. Lemma 2.3 generalizes Theorem 2 in Chapter 8 of [6]. But while this generalization is not difficult to obtain, we hope that, comparing the proofs of Lemma 2.1 and Lemma 2.3, the reader clearly recognizes both the close similarity between a decomposition based on a standard asymptotic expansion and a Shishkin decomposition as well as the differences between these two constructions.

Both the standard asymptotic decomposition and the Shishkin decomposition are useful in the analysis of discretization methods. The Shishkin decomposition (if available, see Section 3) is preferable if the method used can take advantage of the property that the layer part satisfies a *homogeneous* equation.

3. The two-dimensional problem with exponential layers

In this section we shall study the boundary value problem

$$\left. \begin{aligned} -\varepsilon\Delta u - b\nabla u &= f && \text{in } \Omega = (0, 1)^2 \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \right\} \tag{3.1}$$

assuming again (1.3). The corners $T_1 = (0, 0)$, $T_2 = (0, 1)$, $T_3 = (1, 1)$ and $T_4 = (1, 0)$ play an essential role in our considerations. Let us first state from [2] the following fact.

Lemma 3.1. *Let the smooth data of the boundary value problem*

$$\left. \begin{aligned} -\varepsilon\Delta u - b\nabla u &= f && \text{in } \Omega = (0, 1)^2 \\ u(s, 0) &= g_s(s), \quad u(s, 1) = g_n(s), \quad u(0, s) = g_w(s), \quad u(1, s) = g_e(s) && (s \in [0, 1]) \end{aligned} \right\}$$

satisfy the first-order compatibility conditions. Then the given boundary value problem has a unique solution $\bar{u} \in C^{3,\alpha}(\bar{\Omega})$.

We describe the concrete form of the compatibility conditions of Lemma 3.1 at the corner $T_1 = (0, 0)$, for instance. The compatibility condition of order zero requires

$$g_s(0) - g_w(0) = 0 \tag{C_{1,0}}$$

while the first order condition additionally requires

$$\varepsilon g_s''(0) + b_1(0, 0)g_s'(0) + \varepsilon g_w''(0) + b_2(0, 0)g_w'(0) + f(0, 0) = 0. \tag{C_{1,1}}$$

At the moment we need Lemma 3.1 for problem (3.1) with homogeneous boundary conditions (later it will be invoked in a different situation).

Corollary 3.2. *If b and f are sufficiently smooth and f satisfies the compatibility conditions*

$$f(T_i) = 0 \quad (i = 1, 2, 3, 4), \tag{C 1}$$

then problem (3.1) admits a unique solution $u \in C^{3,\alpha}(\bar{\Omega})$.

An asymptotic approximation of u is well known (compare [7: Chapter III/Example 1.15]):

$$\begin{aligned} u &= u_0 + u_0(0, y) \exp\left(-\frac{b_1(0, y)}{\varepsilon}x\right) - u_0(x, 0) \exp\left(-\frac{b_2(x, 0)}{\varepsilon}y\right) \\ &+ u_0(0, 0) \exp\left(-\frac{b_1(0, 0)}{\varepsilon}x\right) \exp\left(-\frac{b_2(0, 0)}{\varepsilon}y\right) + \varepsilon R. \end{aligned} \tag{3.2}$$

Here u_0 – the solution of the reduced problem – satisfies

$$\left. \begin{aligned} -b\nabla u_0 &= f \\ u_0|_{x=1} &= u_0|_{y=1} = 0. \end{aligned} \right\} \tag{3.3}$$

To guarantee smoothness of u_0 (at present, we need $u_0 \in C^2(\bar{\Omega})$, but later we shall want higher-order smoothness) we need compatibility conditions at the corner T_3 . Thus, let us additionally assume

$$f_x(T_3) = f_y(T_3) = 0. \tag{C2_a}$$

Then the remainder R can be estimated using a suitable barrier function. With this estimate $\|R\| \leq C$ and (1.2) then yields information on derivatives of R . We set

$$G = u_0 + \varepsilon R \quad \text{and} \quad E = E_1 + E_2 + E_3$$

(the E_i are the layer terms in (3.2)). Thus, we have the following assertion.

Lemma 3.3. (Splitting based on the asymptotic expansion). *Let b and f be sufficiently smooth and assume that f satisfies the compatibility conditions (C1) and (C2_a). Then the solution u of problem (3.1) has the representation*

$$u = G + E_1 + E_2 + E_3$$

where the smooth part satisfies

$$\|G\| \leq C, \quad \|G\|_1 \leq C, \quad \|G\|_2 \leq C\epsilon^{-1} \tag{3.5}$$

while

$$\left. \begin{aligned} \left| \frac{\partial^{i+j} E_1}{\partial x^i \partial y^j} \right| &\leq C\epsilon^{-i} e^{-\frac{\beta_1}{\epsilon} x} \\ \left| \frac{\partial^{i+j} E_2}{\partial x^i \partial y^j} \right| &\leq C\epsilon^{-j} e^{-\frac{\beta_2}{\epsilon} y} \\ \left| \frac{\partial^{i+j} E_3}{\partial x^i \partial y^j} \right| &\leq C\epsilon^{-(i+j)} e^{-\frac{\beta_1}{\epsilon} x} e^{-\frac{\beta_2}{\epsilon} y} \end{aligned} \right\} \quad (i + j \leq 2). \tag{3.6}$$

Remark 3.4. In the analysis of an exponentially fitted finite element method [8] we needed the estimates

$$\begin{aligned} |u_x| &\leq C \left(1 + \frac{1}{\epsilon} \exp\left(-\frac{\beta_1}{\epsilon} x\right) \right) \\ |u_y| &\leq C \left(1 + \frac{1}{\epsilon} \exp\left(-\frac{\beta_2}{\epsilon} y\right) \right) \end{aligned}$$

and

$$\max \left(|-\epsilon u_{xx} - b_1 u_x|, |-\epsilon u_{yy} - b_2 u_y| \right) \leq C.$$

These estimates are simple consequences of the decomposition (3.4). Thus, the results of [8] hold true without the assumptions (L) and (I) used in that paper.

If one needs better bounds than (3.5) for the smooth part of the splitting, it is necessary to analyse the next terms in an asymptotic expansion. Thus, let us set

$$u = u_0 + \epsilon u_1 + v_0 + \epsilon v_1 + w_0 + \epsilon w_1 + z_0 + \epsilon z_1 + \epsilon^2 R. \tag{3.7}$$

Now u_1 satisfies

$$\left. \begin{aligned} -b \nabla u_1 &= \Delta u_0 \\ u_1|_{x=1} &= u_1|_{y=1} = 0. \end{aligned} \right\} \tag{3.8}$$

We wish to have $u_0 \in C^2(\bar{\Omega})$ and $u_1 \in C^2(\bar{\Omega})$. Therefore, we require

$$f_{xx}(T_3) = f_{xy}(T_3) = f_{yy}(T_3) = 0. \tag{C2b}$$

Now v_1 and w_1 are the next terms in the ordinary layers at $x = 0$ and $y = 0$, respectively. With $\xi \equiv \frac{1}{\epsilon} x$, v_1 for instance satisfies

$$\left. \begin{aligned} \frac{d^2 v_1}{d\xi^2} + b_1(0, y) \frac{dv_1}{d\xi} &= -b_2(0, y) \frac{\partial v_0}{\partial y} - \frac{\partial b_1}{\partial x}(0, y) \xi \frac{\partial v_0}{\partial \xi} \\ v_1(0, y) &= -u_1(0, y). \end{aligned} \right\} \tag{3.9}$$

Thus v_1 has the structure

$$v_1(\xi, y) = \xi(A(y) + B(y)\xi)e^{-b_1(0,y)\xi} - u_1(0, y)e^{-b_1(0,y)\xi}. \tag{3.10}_a$$

Analogously, with $\eta = \frac{1}{\varepsilon}y$, we obtain

$$w_1(x, \eta) = [-u_1(x, 0) + \eta(C(x) + D(x)\eta)] e^{-b_2(x,0)\eta}. \tag{3.10}_b$$

The corner layer term z_1 ensures that the boundary conditions at $x = 0$ and $y = 1$ are satisfied. With $\xi = \frac{x}{\varepsilon}$ and $\eta = \frac{y}{\varepsilon}$ we obtain the corner layer equation

$$\frac{\partial^2 z_1}{\partial \xi^2} + \frac{\partial^2 z_1}{\partial \eta^2} - b_1(0, 0) \frac{\partial z_1}{\partial \xi} - b_2(0, 0) \frac{\partial z_1}{\partial \eta} = \frac{\partial b_1}{\partial x}(0, 0)\xi \frac{\partial z_0}{\partial \xi} + \frac{\partial b_2}{\partial \eta}(0, 0)\eta \frac{\partial z_0}{\partial \eta}$$

with the boundary conditions

$$\left. \begin{aligned} z_1|_{\xi=0} &= -w_1|_{x=0} = [u_1(0, 0) - \eta(C(0) + D(0)\eta)] e^{-b_2(0,0)\eta} \\ z_1|_{\eta=0} &= -v_1|_{y=0} = [u_1(0, 0) - \xi(A(0) + B(0)\xi)] e^{-b_1(0,0)\xi}. \end{aligned} \right\}$$

The crucial question is: is z_1 smooth at $(0,0)$ (or equivalently: are the first-order compatibility conditions of Lemma 3.1 satisfied)? A simple computation shows that for our z_1 -problem the compatibility condition is

$$A(0)b_1(0, 0) - 2B(0) + C(0)b_2(0, 0) - 2D(0) = 0. \tag{3.11}$$

But the solution of the differential equation (3.9) yields

$$A(y)b_1(0, y) - 2B(y) = b_2(0, y) \frac{\partial u_0(0, y)}{\partial y},$$

and analogously

$$C(x)b_2(x, 0) - 2D(x) = b_1(x, 0) \frac{\partial u_0(x, 0)}{\partial x}.$$

Thus (3.11) is equivalent to

$$b_1(0, 0) \frac{\partial u_0(0, 0)}{\partial x} + b_2(0, 0) \frac{\partial u_0(0, 0)}{\partial y} = 0.$$

But u_0 satisfies (3.3) and is smooth, and $f(0, 0) = 0$, so the compatibility condition (3.12) is automatically satisfied! Our standard arguments with respect to the remainder R in (3.7) and the definition

$$\left. \begin{aligned} G &= u_0 + \varepsilon u_1 + \varepsilon^2 R \\ E &= E_1 + E_2 + E_3 \end{aligned} \right\} \tag{3.13}$$

with

$$E_1 = v_0 + \varepsilon v_1, \quad E_2 = w_0 + \varepsilon w_1, \quad E_3 = z_0 + \varepsilon z_1$$

result in the following main outcome of this section.

Theorem 3.5 (Splitting based on the asymptotic expansion). *Let b and f be sufficiently smooth. Further, we assume that the compatibility conditions $(C1)$, $(C2_a)$ and $(C2_b)$ are satisfied. Then the smooth part G of the decomposition (3.13) can be bounded by*

$$\|G\|_\ell \leq C \quad \text{for } \ell = 0, 1, 2, \tag{3.14}$$

while E_1, E_2, E_3 satisfy again (3.6) for $i + j \leq 2$.

If we additionally assume that $u_1 \in C^3(\bar{\Omega})$, we also have $\|G\|_3 \leq C\epsilon^{-1}$ and (3.6) holds true for $i + j \leq 3$.

In the two-dimensional case the derivation of a "smooth" Shishkin decomposition ($u = G^* + E^*$ with $LG^* = f$ and information about bounded derivatives of G^* and E^* up to a certain order) seems to be complicated. If, analogously to Section 2, we try to introduce u_2^* (in $G^* = u_0 + \epsilon u_1 + \epsilon^2 u_2^*$) by $Lu_2^* = \Delta u_1$ to guarantee $LG^* = f$, then compatibility at the corners T_1, T_2, T_4 is a problem (in [6] the authors require additionally $u_2^* = 0$ on Γ , but then the estimate $\|u_2^*\| \leq C\epsilon^{-1}$ used in the proof of Theorem 3, Chapter 12 cannot be guaranteed because Δu_1 is not zero in T_1, T_2, T_4 , in general; thus the conditions in [6] are not constructive).

In [9: p. 203], quite correctly a corner singularity term in the estimate of u_2^* appears. But then we obtain only

$$G^* = u_0 + \epsilon u_1 + \epsilon^2 u_2^* \tag{3.15}$$

with $\|u_2^*\| \leq C$, while derivatives of u_2^* are unbounded. In the splitting of E^* a similar term arises. We call such a splitting a *perturbed* Shishkin decomposition (for such a decomposition the necessary compatibility conditions can easier be justified). As a consequence, Shishkin's analysis based on "perturbed" decomposition leads to very low convergence rates (compare, for instance, [9: Chapter III/Theorem 2.3] where the rates are $\frac{1}{4}$ or $\frac{1}{8}$ depending on the precise assumptions) because he balances the classical error terms (the result of a standard analysis for fixed ϵ which is useless for extremely small ϵ) with the perturbation terms of order $O(\epsilon^q)$ in the decomposition. Thus the exotic rates seem to be the result of the method used to prove uniform convergence. Unfortunately, better proofs in general situations are not available yet.

Remark 3.6. Reaction-diffusion problems in a square were already carefully analyzed in [2], see Theorem 4.2. In [10] the corresponding results – especially the fact that existing corner singularities arise only in the corner layers and not in the remainder term – were used to create special meshes adapting to both the layers and the corner singularities.

In a further paper [5] the analysis of [2] is extended to the case of a general polygon. It turns out that a small adjustment of the asymptotic expansion for angles different from $\frac{\pi}{2}$ also leads to an expansion which can be term-wise differentiated.

It is possible to analyse problems with a parabolic layer in a similar way. See [3: Chapter IV/Section 1] for a detailed discussion of the terms arising in an asymptotic expansion for the problem

$$\left. \begin{aligned} -\epsilon \Delta u - bu_y &= f & \text{in } \Omega = (0, 1)^2 \\ u &= 0 & \text{on } \partial\Omega. \end{aligned} \right\}$$

In the Appendix D of [9], Shishkin discusses a more general class of equations, namely $\epsilon L_2 u + L_1 u = f$ with a general second order elliptic L_2 and $L_1 u := \sum_1^n b_s(x) \frac{\partial u}{\partial x_s} + c(x)u$ with $b_1, \dots, b_p > 0$ and $b_{p+1}, \dots, b_n \equiv 0$. But the results – "smooth" and "perturbed" decompositions – are very briefly described and the precise assumptions made are hidden in the text or not stated precisely.

4. Interpolation error estimates on a two-dimensional

Shishkin mesh

Let us start from the decomposition of a given function u on $\Omega = (0, 1)^2$ in the sense of Theorem 3.5: Let u admit the representation

$$u = G + E_1 + E_2 + E_3, \tag{4.1}$$

where the smooth part satisfies $\|G\| \leq C$, $\|G\|_1 \leq C$ and $\|G\|_2 \leq C$ while the layer terms can be estimated by

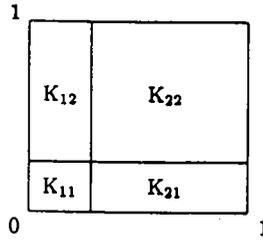
$$\begin{aligned} \left| \frac{\partial^{i+j} E_1}{\partial x^i \partial y^j} \right| &\leq C \epsilon^{-i} e^{-\frac{\beta_1}{\epsilon} x} \\ \left| \frac{\partial^{i+j} E_2}{\partial x^i \partial y^j} \right| &\leq C \epsilon^{-j} e^{-\frac{\beta_2}{\epsilon} y} \\ \left| \frac{\partial^{i+j} E_3}{\partial x^i \partial y^j} \right| &\leq C \epsilon^{-(i+j)} e^{-\frac{\beta_1}{\epsilon} x} e^{-\frac{\beta_2}{\epsilon} y} \end{aligned} \quad (i + j \leq 2). \tag{4.2}$$

To simplify the representation we additionally assume $\beta_1 = \beta_2 = 1$. Otherwise a simple scaling leads to corresponding results.

In the following, we will describe a slight generalization of an anisotropic mesh already introduced by Shishkin [9] which can also be used for non-rectangular domains. Our elements are anisotropic rectangles or general triangles with local step sizes depending on their positions in the unit square $(0, 1)^2$. Let $N \in \mathbb{N}$ and set $\tau = \min \{ \frac{1}{2}, 2\epsilon \ln N \}$. Since ϵ is considered to be very small we assume in the following that $\tau = 2\epsilon \ln N$. The small and large local step sizes are given by $h_1 = h = \frac{2\tau}{N}$ and $h_2 = H = \frac{2(1-\tau)}{N}$. Let $K_{11}, K_{12}, K_{21}, K_{22}$ be closed polygonal subsets of $\bar{\Omega}$ with disjoint interiors satisfying the following conditions:

- (i) K_{11} covers $(0, \tau)^2$ and is contained in $[0, c\tau]^2$.
- (ii) $K_{11} \cup K_{12}$ covers $(0, \tau) \times (0, 1)$ and is contained in $[0, c\tau] \times [0, 1]$.
- (iii) $K_{11} \cup K_{21}$ covers $(0, 1) \times (0, \tau)$ and is contained in $[0, 1] \times [0, c\tau]$.

(iv) $K_{11} \cup K_{12} \cup K_{21} \cup K_{22}$ covers $\bar{\Omega} = [0, 1]^2$.



For the elements in K_{ij} we state the following condition:

Each element Λ of K_{ij} is contained in a rectangle with side lengths (h_i, h_j) and contains a rectangle with side lengths $c_R(h_j, h_j)$ ($i, j = 1, 2$). The interior angles of Λ are bounded by $\theta < \pi$.

By this condition, each element Λ is characterized by a local step size $h = h(\Lambda) = (h_i, h_j)$ ($\Lambda \subset K_{ij}$). For $i \neq j$ the corresponding elements are long and thin which cause trouble when the classical interpolation theory is used.

In order to state the results of the anisotropic interpolation theory, some notations are required. For a multi-index $\alpha \in \mathbb{N}_0^2$ we use

$$h^\alpha = h(\Lambda)^\alpha = h_i^{\alpha_1} h_j^{\alpha_2} \quad (\Lambda \subset K_{ij}) \quad \text{and} \quad D^\alpha u = \frac{\partial^{\alpha_1}}{\partial x^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial y^{\alpha_2}} u.$$

The space of continuous and piecewise linear (bilinear) functions that vanish on the boundary $\partial\Omega$ is denoted by S_0 . Let $u^I \in S_0$ be the nodal interpolate of the continuous function u . Denoting the L_2 -norm on an element Λ by $\|\cdot\|_{0,\Lambda}$ we have the following interpolation estimates (see [1]):

$$\|u - u^I\|_{0,\Lambda}^2 \leq c \sum_{|\alpha| \leq 2} h^{2\alpha} \|D^\alpha u\|_{0,\Lambda}^2 \tag{4.3}$$

$$\|\partial_x(u - u^I)\|_{0,\Lambda}^2 \leq c \sum_{|\alpha|=1} h^{2\alpha} \|D^\alpha \partial_x u\|_{0,\Lambda}^2 \tag{4.4}$$

$$\|\partial_y(u - u^I)\|_{0,\Lambda}^2 \leq c \sum_{|\alpha|=1} h^{2\alpha} \|D^\alpha \partial_y u\|_{0,\Lambda}^2. \tag{4.5}$$

These estimates do not hold on general anisotropic quadrilaterals (see [12] for some results) and have enforced us to use an orthogonal mesh for bilinear elements.

By transforming the standard inverse estimate to anisotropic elements we obtain for $\Lambda \subset K_{ij}$

$$\|\partial_x u_h\|_{0,\Lambda}^2 \leq c h_i^{-2} \|u_h\|_{0,\Lambda}^2 \quad \text{for all } u_h \in S_0 \tag{4.6}$$

$$\|\partial_y u_h\|_{0,\Lambda}^2 \leq c h_j^{-2} \|u_h\|_{0,\Lambda}^2 \quad \text{for all } u_h \in S_0. \tag{4.7}$$

We wish to estimate the interpolation error in the L_∞ -norm $\|\cdot\|$, the L_2 -norm, and the ε -weighted H^1 -norm given by $\|v\|_\varepsilon^2 = \{\varepsilon \|\nabla v\|_0^2 + \|v\|_0^2\}^{\frac{1}{2}}$.

Theorem 4.1. *We have*

$$\|u - u^I\|_{K_{22}} \leq CN^{-2} \quad \text{and} \quad \|u - u^I\|_{\Omega \setminus K_{22}} \leq CN^{-2} \ln^2 N \quad (4.8)$$

and, if $\varepsilon^{\frac{1}{2}} \ln^2 N \leq C$ (which is practically not restrictive),

$$\|u - u^I\|_0 \leq CN^{-2}. \quad (4.9)$$

Proof. Assertion (4.8) is well known [11], we stated that estimate for completeness. To prove (4.9), we use the splitting $u = G + E$ of (4.1) and the corresponding decomposition $u^I = G^I + E^I$. The interpolation error estimate with respect to G follows from the standard theory.

Now let us consider the layer terms, for instance $E_1 - E_1^I$. On the fine mesh K_{11} the standard estimate yields

$$\|E_1 - E_1^I\|_{0,K_{11}} \leq Ch^2 \|\nabla^2 E_1\|_{0,K_{11}} \leq C\varepsilon^{\frac{1}{2}} N^{-2} \ln^2 N.$$

On $K_{21} \cup K_{22}$ we have by definition of the sets K_{ij}

$$\begin{aligned} \|E_1 - E_1^I\|_{0,K_{21} \cup K_{22}} &\leq \|E_1\|_{0,K_{21} \cup K_{22}} + \|E_1^I\|_{0,K_{21} \cup K_{22}} \\ &\leq \|E_1\|_{0,K_{21} \cup K_{22}} + \|E_1\|_{K_{21} \cup K_{22}} \\ &\leq CN^{-2}. \end{aligned}$$

Finally, on K_{12} we use the anisotropic estimate (4.3) to get

$$\|E_1 - E_1^I\|_{0,K_{12}}^2 \leq Ch^4 \|\partial_{xx} E_1\|_{0,K_{12}}^2 + Ch^2 H^2 \|\partial_{xy} E_1\|_{0,K_{12}}^2 + CH^4 \|\partial_{yy} E_1\|_{0,K_{12}}^2.$$

Inserting the estimates for the derivatives which follow from (4.2) we obtain

$$\|E_1 - E_1^I\|_{0,K_{12}}^2 \leq C\varepsilon N^{-4} \ln^4 N + C\varepsilon N^{-4} \ln^2 N + CN^{-4}.$$

Thus Theorem 4.1 is proved ■

In the next step we also estimate the gradient of the interpolation error.

Theorem 4.2. *The gradient of the interpolation error can be estimated by*

$$\varepsilon^{\frac{1}{2}} \|\nabla(u - u^I)\|_{0,K_{22}} \leq CN^{-1} \quad \text{and} \quad \varepsilon^{\frac{1}{2}} \|\nabla(u - u^I)\|_{0,\Omega \setminus K_{22}} \leq CN^{-1} \ln N.$$

Proof. Again we consider only the term $E_1 - E_1^I$ because the other ones can be estimated in a simpler or similar way. On the fine mesh K_{11} again the standard estimate works:

$$\|\nabla(E_1 - E_1^I)\|_{0,K_{11}} \leq Ch \|\nabla^2 E_1\|_{0,K_{11}} \leq C\varepsilon^{-\frac{1}{2}} N^{-1} \ln N.$$

On the subdomain $K_{21} \cup K_{22}$ we have

$$\|\partial_x(E_1 - E_1^I)\|_{0,K_{21} \cup K_{22}} \leq \|\partial_x E_1\|_{0,K_{21} \cup K_{22}} + \|\partial_x E_1^I\|_{0,K_{21} \cup K_{22}}.$$

From the inverse estimate (4.6) it follows that

$$\begin{aligned} \|\partial_x(E_1 - E_1^I)\|_{0, K_{21} \cup K_{22}} &\leq \|\partial_x E_1\|_{0, K_{21} \cup K_{22}} + CH^{-1} \|E_1^I\|_{0, K_{21} \cup K_{22}} \\ &\leq \|\partial_x E_1\|_{0, K_{21} \cup K_{22}} + CH^{-1} \|E_1\|_{K_{21} \cup K_{22}} \\ &\leq C\epsilon^{-\frac{1}{2}} N^{-2} + CN^{-1}. \end{aligned}$$

Finally, we apply on K_{12} the anisotropic estimate (4.4):

$$\|\partial_x(E_1 - E_1^I)\|_{0, K_{12}} \leq Ch \|\partial_{xx} E_1\|_{0, K_{12}} + ChH \|\partial_{xy} E_1\|_{0, K_{12}}.$$

Taking into account the corresponding estimates for the derivatives of E_1 it follows

$$\|\partial_x(E_1 - E_1^I)\|_{0, K_{12}} \leq C\epsilon^{-\frac{1}{2}} N^{-1} \ln N + C\epsilon^{-\frac{1}{2}} N^{-1}.$$

Thus Theorem 4.2 is proved ■

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