

A Product for Special Classes of Monogenic Functions and Tensors

F. Sommen and P. Van Lancker

Abstract. In this paper we define a product for monogenic functions derived from Fischer's decomposition. Although this product can be defined for all monogenic polynomials, it can be canonically defined only for a special subclass of polynomials. The definition of the product can be generalized to the setting of monogenic tensors.

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0. Introduction

Let R_m be the Clifford algebra generated by the basis $\{e_1, \dots, e_m\}$ of R^m and the defining relations $e_i e_j + e_j e_i = -2\delta_{ij}$. Then the R_m -valued solutions $f(x)$ of $\partial_x f(x) = 0$, with $\partial_x = \sum e_j \partial_{x_j}$ being the Dirac operator, are called *left monogenic functions*. The same name is used for null-solutions of the operator $\partial_{x_0} + \partial_x$ which is also called *generalized Cauchy Riemann operator* since it generalizes the Cauchy-Riemann operator in the plane. This is true except for the important fact that the product $f(x_0, x) g(x_0, x)$ of two left monogenic functions is no longer left monogenic. So the question is: Do there exist "product-like operations" for monogenic functions? One such operation is the so-called *Cauchy-Kowalewski product* (see [1, 5]) which is based on the uniqueness of the left monogenic extension $f(x_0, x)$ of a given real analytic function $f(x) = f(0, x)$. The idea works as follows: Given two left monogenic functions f and g , consider the product of real analytic functions $f(0, x) g(0, x)$ and define the Cauchy-Kowalewski product to be the monogenic extension $f \cdot g$ of this. Although well defined, the Cauchy-Kowalewski product is very hard to work with and, except for strip domains, there are no examples of domains for which the space of monogenic functions becomes an algebra. Moreover, the Cauchy-Kowalewski product is quite unnatural to be defined for all monogenic functions. Indeed, the space of left monogenic functions is a right R_m -module which means that one has to consider Cauchy-Kowalewski extensions of products of restrictions of monogenic functions of the form $(f(0, x)e_1) g(0, x) = f(0, x)(e_1 g(0, x))$, but the function $e_1 g(x_0, x)$ is usually not left monogenic. Hence, in general there is conceptually something wrong with this product which again does not mean that it is entirely meaningless; it may be

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an interesting and naturally defined product structure for special types of monogenic functions. Moreover, the Cauchy-Kowalewski theorem may be formulated for general analytic surfaces (see, e.g., [6]) which means that a Cauchy-Kowalewski product may be defined with respect to general analytic surfaces such as the unit sphere.

But there is still another theorem that may give rise to a meaningful product, namely the Fischer decomposition of homogeneous polynomials which can be obtained for general differential operators such as the Laplacian or the Dirac operator (see, e.g., [2]). For the Laplacian we have a Fischer decomposition of the form

$$R_k(x) = \sum_s |x|^{2s} H_{k-2s}(x),$$

whereby R_k is a given homogeneous polynomial of degree k and H_{k-2s} are uniquely determined spherical harmonics of degree $k - 2s$. This theorem leads to the following well known product structure defined for spherical harmonics (see also [12]) :

Given two spherical harmonics H_k and H_l one may form the usual product $H_k H_l$ being a homogeneous polynomial R_{k+l} of degree $k + l$ and then define the harmonic product $H_k \circ H_l$ to be the highest piece in the Fischer decomposition of R_{k+l} .

In Section 1 of this paper we define a Fischer product for spherical monogenics of a special type. This restriction again has to do with the incompatibility of the Fischer product and the right module structure of monogenic functions. But the algebra we obtain still contains the reproducing kernels for the spherical monogenics. In Sections 2 and 3 we extend the definition of the Fischer product to the setting of monogenic tensors and forms (see also [8, 9]).

1. The Fischer product for spherical monogenics

Let \mathcal{P}_k denote the right module of R_m -valued homogeneous polynomials of degree k . Then the Fischer inner product on \mathcal{P}_k is given by

$$(R_k(x), S_k(x)) = \frac{1}{k!} \bar{R}_k(\partial_x) S_k(x).$$

Hereby the map $R_k(x) \rightarrow R_k(\partial_x)$ stands for replacing the x_j -variables by partial derivatives and the map $a \rightarrow \bar{a}$ is the main anti-involution on R_m determined by $\overline{ab} = \bar{b}\bar{a}$ and $\bar{e}_j = -e_j$. The above inner product is compatible with the right module structure and the scalar part of it determines a positive definite inner product on the space of all homogeneous polynomials (in fact on the space of all Clifford polynomials). Hence we may consider the subspace $x\mathcal{P}_{k-1}$ of \mathcal{P}_k of polynomials of the form $x R_{k-1}(x)$ and the orthogonal complement of this subspace is clearly the space \mathcal{M}_k of spherical monogenics of degree k , i.e. of homogeneous polynomials P_k satisfying $\partial_x P_k(x) = 0$. This means that every homogeneous Clifford polynomial R_k admits a unique orthogonal decomposition of the form

$$R_k(x) = P_k(x) + x R_{k-1}(x) \quad (R_{k-1} \in \mathcal{P}_{k-1}, \partial_x P_k(x) = 0)$$

called *Fischer decomposition* and by recursive application of this decomposition one arrives at the complete Fischer decomposition

$$R_k(x) = \sum_s x^s P_{k-s}(x) \quad (P_{k-s} \in \mathcal{M}_{k-s}).$$

This orthogonal decomposition refines the "harmonic decomposition" of polynomials and in principle there is nothing in the way of defining a "Fischer product" for spherical monogenics in much the same way as for spherical harmonics. Indeed, given two spherical monogenics $P_k \in \mathcal{M}_k$ and $P_l \in \mathcal{M}_l$ one may consider the product of polynomials $R_{k+l} = P_k P_l \in \mathcal{P}_{k+l}$ and then define the "Fischer product" $P_k \circ P_l$ to be the highest piece in the Fischer decomposition of the polynomials R_{k+l} . But again this product conflicts with the right module structure for spherical monogenics because again one may consider products of monogenic polynomials of the form $(P_k(x) e_1) P_l(x) = P_k(x) (e_1 P_l(x))$ but the polynomial $e_1 P_l$ is no longer spherical monogenic; it is only spherical harmonic.

This already suggest that the Fischer product for spherical harmonics may even be more natural than that for spherical monogenics and this is indeed true because the Fischer product of spherical harmonics can be defined on the abstract level of ring theory. To that end, let R be a ring and let I be a two-sided ideal in R . Then the quotient space R/I inherits the ring structure from R and the product is given by $(a + I)(b + I) = ab + I$.

In case where $R = \mathcal{P}$, \mathcal{P} being the ring of all polynomials, we may consider the subspace of polynomials of the form $I = |x|^2 \mathcal{P}$ which is clearly a two-sided ideal. Moreover, the quotient space R/I is canonically isomorphic to the space of all harmonic polynomials. What goes wrong in Clifford analysis is that the space of Clifford polynomials $I = x \mathcal{P}$, is not a two-sided ideal of the ring \mathcal{P} of all Clifford polynomials; it is only a right ideal. However, it is possible to define a subring $\mathcal{P}b$ of \mathcal{P} for which $I = x \mathcal{P}$ becomes a two-sided ideal.

Definition 1. $\mathcal{P}b$ is the ring of all R_m -valued polynomials R which are intertwenable with the polynomial $x = \sum_j x_j e_j$, i.e. there exists a polynomial S for which $x R(x) = S(x)x$.

It is readily seen that in case $x R(x) = S(x)x$ we also have $x S(x)x = x^2 R(x) = R(x)x^2$ and, as $f(x) = x$ is invertible outside the origin, we also have $x S(x) = R(x)x$ so that R and S both belong to $\mathcal{P}b$. Hence, intertwining with x from the left implies intertwining with x from the right; there is no left - right distinction in Definition 1.

Examples.

(1) The polynomial e_1 does not belong to the space $\mathcal{P}b$ for $x e_1$ can never be written as $a x$ for some Clifford number a .

(2) Any scalar polynomial belongs to $\mathcal{P}b$.

(3) The wedge products $x \wedge u_1 \wedge \dots \wedge u_k$ belong to $\mathcal{P}b$ because

$$x(x \wedge u_1 \wedge \dots \wedge u_k) = (-1)^k (x \wedge u_1 \wedge \dots \wedge u_k)x.$$

(4) The zonal spherical monogenics $Z_k(x, u)$ may be defined as the reproducing kernels for the spaces of spherical monogenics \mathcal{M}_k , i.e. $Z_k(x, \partial_u) R_k(u) = P_k(x)$ in case $R_k(x) = P_k(x) + x R_{k-1}(x)$ ($P_k \in \mathcal{M}_k$) (see also [10]). As they are of the form $A(x, u) + x \wedge u B(x, u)$ for scalar polynomials A and B they belong to $\mathcal{P}b$.

Lemma 1. *The space of Clifford polynomials of the form xP is a two-sided ideal inside \mathcal{Pb} .*

Proof. Note that any polynomial of the form $xR(x)$ satisfies the intertwining relation $x(xR(x)) = (R(x)x)x$ and hence belongs to \mathcal{Pb} . That xP is a right ideal is clear while for any $R \in \mathcal{Pb}$ with $xR(x) = S(x)x$ we also have that $xS(x) = R(x)x$ so that $R(x)xP = xS(x)P$ ■

It follows from the above lemma that the quotient space \mathcal{Pb}/xP has a natural ring structure inherited from \mathcal{Pb} . Moreover, we have the following

Lemma 2. *The space \mathcal{Pb}/xP is canonically isomorphic to the space of monogenic polynomials $P \in \mathcal{Pb}$, i.e. $P \in \mathcal{Pb}$ and $\partial_x P(x) = 0$.*

Proof. The Fischer inner product remains positive definite after restriction to the spaces \mathcal{Pb}_k of homogeneous polynomials of degree k in \mathcal{Pb} . Hence \mathcal{Pb}_k/xP_{k-1} is canonically isomorphic to the orthogonal complement of xP_{k-1} inside \mathcal{Pb}_k which is the space of polynomials $P_k \in \mathcal{Pb}_k$ for which $[\bar{R}_{k-1}(\partial_x)\partial_x P_k(x)]_0 = 0$ for all $R_{k-1} \in P_{k-1}$, where $[a]_0$ denotes the scalar part of $a \in R_m$. This means that $P_k \in \mathcal{Pb}_k$ and $\partial_x P_k(x) = 0$ ■

Hence there is a canonical product for monogenic polynomials in \mathcal{Pb} .

We conclude this section with the following two remarks:

1. First of all we have to verify that the Fischer product indeed generalizes the product of holomorphic polynomials in the plane $z^k z^l = z^{(k+l)}$ ($z = x + iy$). To that end we interpret the complex number i as the bivector $-e_1 e_2$ so that $(x - e_1 e_2 y)^k$ is indeed a solution of the Dirac equation

$$(e_1 \partial_{x_1} + e_2 \partial_{x_2}) f(x_1, x_2) = e_1(\partial_{x_1} + i \partial_{x_2}) f(x_1, x_2) = 0.$$

Of course, the pointwise product $z^k z^l$ is also a Fischer product because the result $z^{(k+l)}$ is monogenic; no further monogenic projection is needed. Finally, the variable $x - e_1 e_2 y$ satisfies

$$(x e_1 + y e_2)(x - e_1 e_2 y) = (x + e_1 e_2 y)(x e_1 + y e_2)$$

so that it intertwines with the vector variable $x e_1 + y e_2$, i.e. it belongs to the class \mathcal{Pb} and so does any complex number $a + ib$. The Fischer product in the quotient ring \mathcal{Pb}/xP hence fully generalizes the product of holomorphic functions.

2. Yet the class of polynomials \mathcal{Pb} is not the only class for which the Fischer product can be meaningfully defined: it can in principle be defined for all Clifford polynomials, only it is in most cases not very natural to consider this product. But worse than this, the class of polynomials \mathcal{Pb} is not the only class for which the Fischer product is naturally defined. Indeed, let $m = 2n$ or $m = 2n + 1$. Then for any holomorphic function $f(x_1 + i x_2, \dots, x_{2n-1} + i x_{2n})$ the function $I f(x)$, with $I = (1 + i e_1 e_2) \cdots (1 + i e_{2n-1} e_{2n})$ satisfies the Dirac equation. Conversely, if for a scalar function f , $I f$ satisfies the Dirac equation, then f is holomorphic of several complex variables. Moreover, in case f and g are scalar holomorphic functions of several complex variables, then we have that $I f I g = I f g$, i.e. the pointwise product of holomorphic functions is another natural example of a Fischer product. We may thus conclude that we have two complementary examples of natural Fischer products.

2. The Fischer product defined for monogenic tensors

A Clifford k -tensor is simply a multilinear function $F(u_1, \dots, u_k)$ of k different vector variables $u_j = \sum_l u_{j,l} e_l$ (see also [4]). One can extend Clifford tensors as to include covariant and contravariant tensors or one can also consider Clifford tensors of several variables (bitensors) as in [8], but for this paper the above definition is sufficient. A k -tensor $M(u_1, \dots, u_k)$ is called (left) *monogenic* if it is monogenic in each vector variable, i.e. if

$$\partial_{u_j} M(u_1, \dots, u_k) = 0 \quad (\partial_{u_j} = \sum \partial_{u_{j,l}} e_l \quad (j = 1, \dots, k))$$

(see also [9]). Next, let \mathcal{T}_k be the right R_m -module of Clifford tensors $F(u_1, \dots, u_k)$. Then the Fischer inner product on \mathcal{T}_k is given by

$$(G, F) = \bar{G}(\partial_{u_1}, \dots, \partial_{u_k}) F(u_1, \dots, u_k),$$

like for Clifford polynomials. Moreover, the scalar part $[(G, F)]_0$ is positive definite on the space of Clifford tensors and with respect to this inner product every Clifford tensor admits a unique orthogonal decomposition of the form (the monogenic decomposition)

$$F(u_1, \dots, u_k) = M(F)(u_1, \dots, u_k) + \sum_{j=1}^k u_j F_j(u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_k)$$

whereby $M(F)(u_1, \dots, u_k)$ is a monogenic tensor.

The importance of this decomposition is that it generalized the Fischer decomposition (monogenic decomposition) for homogeneous Clifford polynomials $R_k(u)$ of one vector variable, for homogeneous Clifford polynomials of several vector variables $R_{k_1, \dots, k_l}(u_1, \dots, u_l)$ for Clifford algebra-valued differential forms (see also [2]) as well as for lots of other "tensor types" coming from the application of suitable symmetrization and anti-symmetrization operators, which as we pointed out in [9], commute with the monogenic decomposition. In particular, by using Young symmetry operators together with the monogenic decomposition, one arrives at models for all irreducible finite-dimensional representations of $\text{Spin}(m)$. All this clearly stresses the importance of monogenic tensors and it would be more than important to have a good product for monogenic tensors which would be at the basis of possible products for monogenic polynomials of one and several vector variables as well as for monogenic differential forms. The fundamental product to start from is the product of Clifford tensors $F \in \mathcal{T}_k$ and $G \in \mathcal{T}_l$ given by

$$F G(u_1, \dots, u_{k+l}) = F(u_1, \dots, u_k) G(u_{k+1}, \dots, u_{k+l}).$$

This product does not disturb possible symmetry properties of the individual tensors F and G and it is hence at the basis of the definition of products for all special Clifford tensor types (i.e. polynomials, differential forms, etc.). Moreover, for this product the direct sum $\mathcal{T} = R_m + \mathcal{T}_1 + \mathcal{T}_2 + \dots$ of tensor spaces becomes a ring. Moreover, the subspace $M(\mathcal{T}) = R_m + M(\mathcal{T}_1) + M(\mathcal{T}_2) + \dots$ of monogenic tensors is a right R_m -submodule of \mathcal{T} and already at this point one could define the Fischer product for

monogenic tensors $M \in M(T)$ and $M' \in M(T)$ to be the monogenic piece $M(M M')$ in the Fischer decomposition of the tensor product $M M'$ of M and M' . But again this product is incompatible with the right module structure of $M(T)$ again because $M e_1 \in M(T)$ and $(M e_1)M' = M(e_1 M')$ while $e_1 M'$ is no longer monogenic. Like in the previous section we have to construct a suitable restriction $T\mathfrak{b}$ of the ring T .

Definition 2. $T\mathfrak{b}_k$ is the space of all Clifford tensors $F(u_1, \dots, u_k)$ such that for any extra vector variable u_{k+1} , the tensor product $F(u_1, \dots, u_k)u_{k+1}$ has the form

$$\sum_{j=1}^{k+1} u_j F_j(u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_{k+1}).$$

By $T\mathfrak{b}$ we denote the direct sum of all spaces $T\mathfrak{b}_k$.

Examples.

(1) The tensor $e_1 \in R_m$ does not belong to $T\mathfrak{b}$ for $e_1 u, u = \sum e_j u_j$ can never be written in the form $u a$ with $a \in R_m$.

(2) Any scalar-valued tensor belongs to $T\mathfrak{b}$ and in particular also the bitensor $K(u_1, \dots, u_k; v_1, \dots, v_k) = \langle u_1, v_1 \rangle \cdots \langle u_k, v_k \rangle$ with $\langle x, y \rangle = \sum x_j y_j$, which is the reproducing kernel of T_k for the Fischer inner product.

It is left as an exercise to the reader to show that $T\mathfrak{b}$ is a subring of the ring of Clifford tensors T , i.e. the tensor product FG of $F \in T\mathfrak{b}$ and $G \in T\mathfrak{b}$ still belongs to $T\mathfrak{b}$. Next consider for each $k \in \mathbb{N}$ the orthogonal complement $M^\#(T_k)$ of the subspace $M(T_k)$ of T_k for the Fischer inner product, i.e. the space of Clifford tensors of the form

$$\sum_{j=1}^k u_j F_j(u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_k).$$

Then we may consider the direct sum $M^\#(T)$ of these right R_m -modules $M^\#(T_k)$ and it is clear that $M^\#(T)$ is even a right ideal of the ring T in the same way as $x\mathcal{P}$ is a right ideal of \mathcal{P} . But again $M^\#(T)$ is not a two-sided ideal of T which is the obstruction for a natural Fischer product for all Clifford tensors. However, we have

Lemma 3. *The ring $M^\#(T)$ is a two-sided ideal of the ring $T\mathfrak{b}$.*

Proof. It is easy to see that for any tensor $F(u_1, \dots, u_k)$ of the form

$$\sum u_j F_j(u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_k),$$

the product $F(u_1, \dots, u_k)u_{k+1}$ is of the form

$$\sum_{j=1}^k u_j G_j(u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_{k+1})$$

and hence of the special form required for elements of $T\mathfrak{b}$, i.e. $M^\#(T)$ is a subspace of $T\mathfrak{b}$ and hence also a right ideal. That $M^\#(T)$ is also a left ideal of $T\mathfrak{b}$ follows from Definition

2 which implies that the tensor product FG of $F \in T\mathfrak{b}_k$ and $G = \sum u_j G_j \in M^\#(T_l)$ is most certainly of the form

$$FG = \sum_{j=1}^{k+l} u_j H_j(u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_{k+l})$$

so that FG belongs to $M^\#(T_{k+l})$ ■

Corollary. *The monogenic piece $M(F)$ of any scalar-valued tensor belongs to $T\mathfrak{b}$. In particular, the reproducing kernel for the space of monogenic k -tensors $M(T_k)$, which is the monogenic projection $M(u_1, \dots, u_k; v_1, \dots, v_k) = M(\langle u_1, v_1 \rangle \dots \langle u_k, v_k \rangle)$ of the scalar bitensor $K(u_1, \dots, u_k; v_1, \dots, v_k) = \langle u_1, v_1 \rangle \dots \langle u_k, v_k \rangle$ belongs to $T\mathfrak{b}$.*

Proof. For any scalar tensor $F \in T_k$ the monogenic projection $M(T)$ is a sum of the form $F + \sum u_j F_j(u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_k)$ for suitable tensors F_j , and by Lemma 3, this clearly belongs to $T\mathfrak{b}$ ■

Hence the most fundamental monogenic tensor belongs to $T\mathfrak{b}$. Due to Lemma 2, the quotient space $T\mathfrak{b}/M^\#(T)$ has a natural ring structure whereby the product is abstractly defined by the formula $(F + M^\#(T))(G + M^\#(T)) = FG + M^\#(T)$. To complete the discussion we still prove the following

Lemma 4. *The quotient space $T\mathfrak{b}/M^\#(T)$ is naturally isomorphic to the space of monogenic tensors belonging to $T\mathfrak{b}$, i.e. $T\mathfrak{b} \cap M(T)$, which is also the monogenic projection $M(T\mathfrak{b})$ of $T\mathfrak{b}$.*

Proof. On the one hand, similar to Lemma 3 one can prove that the monogenic projection $M(F)$ of $F \in T\mathfrak{b}$ still belongs to $T\mathfrak{b}$. On the other hand, the Fischer inner product remains positive definite after restriction to $T\mathfrak{b}$ so that in fact $T\mathfrak{b}/M^\#(T)$ is a vector space isomorphic to the orthogonal complement of $M^\#(T)$ inside $T\mathfrak{b}$ with respect to the Fischer inner product, which is the intersection $T\mathfrak{b} \cap M(T)$ ■

Note that hence the monogenic projection $F \rightarrow M(F)$ ($F \in T\mathfrak{b}$) drops to the isomorphism $F + M^\#(T) \rightarrow M(F)$ between the quotient ring $T\mathfrak{b}/M^\#(T)$ and the space $M(T\mathfrak{b})$ and under this isomorphism the product in the quotient ring $T\mathfrak{b}/M^\#(T)$ corresponds to the binary operation

$$(M(F), M(G)) \rightarrow M(M(F)M(G)) \quad (F, G \in T\mathfrak{b}).$$

For example, denote by M_k the reproducing kernel for the space of monogenic k -tensors: $M_k = M(\langle u_1, v_1 \rangle \dots \langle u_k, v_k \rangle)$. Then it is readily verified that the Fischer product of M_k and M_l is just the reproducing kernel M_{k+l} of $M(T_{k+l})$. In other words, the Fischer product reduces to a mere shift in the index of the reproducing kernel.

Note that the class $T\mathfrak{b}$ is in fact the maximal class of tensors for which $M^\#(T)$ is a two-sided ideal; the condition to be satisfied by the elements of $T\mathfrak{b}$ indeed expresses the fact that the tensor product FG of $F \in T\mathfrak{b}$ and $G \in M^\#(T)$ again belongs to $M^\#(T)$. In the same way, the class of polynomials $\mathcal{P}\mathfrak{b}$ was defined as the maximal class

of polynomials for which $x\mathcal{P}$ is a two-sided ideal. Hence one may wonder that, under the process of total symmetrization

$$F(u_1, \dots, u_k) \rightarrow \frac{1}{k!} \sum_{\pi} F(u_{\pi(1)}, \dots, u_{\pi(k)})$$

and after identifying totally symmetric tensors with polynomials via the isomorphism $F(u_1, \dots, u_k) \rightarrow F(u, \dots, u)$ the class of tensors \mathcal{Tb} projects onto \mathcal{Pb} . To answer this, first note that the above map $\mathcal{T} \rightarrow \mathcal{P}$ is simply given by the replacement $F(u_1, \dots, u_k) \rightarrow F(u, \dots, u)$. Hence in case $F(u_1, \dots, u_k)$ satisfies the condition for \mathcal{Tb}

$$F(u_1, \dots, u_k) u_{k+1} = \sum_{j=1}^{k+1} u_j F_j(u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_{k+1}),$$

then the polynomial $F(u, \dots, u)$ satisfies

$$F(u, \dots, u)u = u \sum_{j=1}^{k+1} F_j(u, \dots, u)$$

and therefore belongs to \mathcal{Pb} , i.e. \mathcal{Tb} projects to a subring of \mathcal{Pb} . But \mathcal{Tb} projects to a proper subring of \mathcal{Pb} because the linear function $x \rightarrow x \wedge u \wedge v$ belongs to \mathcal{Pb} and can at the same time be shown not to belong to \mathcal{Tb} , i.e. the product $(x \wedge u \wedge v)y$ is not of the form $x F(y) + y G(x)$ for certain linear functions F and G . At first this may look surprising because tensor calculus is really at the basis of polynomial and differential form calculus. But on the other hand, the tensor product, even for symmetric tensors, is much more than just the tensor formulation of a polynomial product. For example, a linear function F may be interpreted both as polynomial and as a one-form, but the polynomial product of linear functions is quite different from the exterior product of one-forms. The tensor product of linear functions includes both products in the sense that polynomial and exterior product are expressible in terms of the tensor product and the class \mathcal{Tb} was defined for the tensor product. We finish this section with the complete characterization of the class of tensors \mathcal{Tb} in the following

Theorem 1. Any tensor $F \in \mathcal{Tb}$ may be written (not in a unique way) as a sum

$$F(u) = F_s(u) + \sum u_j F_j(u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_k)$$

of a tensor of the form $F_s(u) = G_s(u) + e_1 \dots e_m H_s(u)$ with G_s, H_s scalar tensors and a tensor belonging to the class $M^\#(\mathcal{T})$.

Proof. For tensors of length zero, i.e. $F \in \mathcal{Tb}_0 = R_m$ the condition of Definition 2 simply means that F satisfies the intertwining relation $Fu = uG$ for some $G \in R_m$ from which it follow that, for all $j = 1, \dots, m$, $F = -e_j G e_j$ and hence also that $e_j G e_j = e_k G e_k$. From this last condition it follows that G has the form "scalar $+ e_1 \dots e_m$ scalar", and as the pseudoscalar commutes or anticommutes with all e_j 's, it follows that $F = \text{scalar} + e_1 \dots e_m \text{ scalar}$.

The general property can be proved by induction on the length of the tensor. As it is rather complicated we suffice by proving the result for tensors of length one, i.e. for linear function $F(u) = \sum F_j u_j$ ($F_j \in R_m$). The condition $F(u)v = uG(v) + vH(u)$ from Definition 2 may be rewritten as a system of equations for Clifford numbers $F_j e_k = e_j G_k + e_k H_j$ from which we obtain that $e_j F_j = G_k e_k + e_j e_k H_j e_k$. Hence we also obtain that $e_i F_i - e_j F_j = e_i e_k H_i e_k - e_j e_k H_j e_k$ which leads to the relations for the elements $H_j \in R_m$

$$e_i e_k H_i e_k - e_j e_k H_j e_k = \text{ibid}(k \rightarrow l)$$

from which the Theorem is to be obtained.

First let us assume these last relations for all $k \neq i, j$. Then $e_k(e_i H_i - e_j H_j)e_k = \text{ibid}(k \rightarrow l)$ from which it readily follows that the expression $e_i H_i - e_j H_j$ has the form

$$A_{ij} + e_i B_{ij} + e_j C_{ij} + e_{ij} D_{ij} + e_1 \dots e_m \text{ dito}$$

whereby A_{ij}, B_{ij}, C_{ij} and D_{ij} are scalars. Next we can write down the identity for $k = i$ and any $l \neq i, j$

$$-H_i e_i - e_j e_i H_j e_i = A_{ij} - e_i B_{ij} - e_j C_{ij} + e_{ij} D_{ij} + e_1 \dots e_m \text{ dito.}$$

Hence the expressions $H_i + e_i e_j H_j$ and $H_i + e_j e_i H_j$ are given by

$$\left. \begin{aligned} H_i + e_i e_j H_j &= -e_j A_{ij} + B_{ij} - e_{ij} C_{ij} + e_j D_{ij} + e_1 \dots e_m \text{ dito} \\ H_i + e_j e_i H_j &= e_i A_{ij} + B_{ij} + e_{ij} C_{ij} + e_j D_{ij} + e_1 \dots e_m \text{ dito} \end{aligned} \right\}$$

so that, taking the sum, $H_i = B_{ij} + e_j D_{ij} + e_1 \dots e_m \text{ dito}$. Moreover, as this is true for all j , we must have that $H_i = B_i + e_1 \dots e_m \text{ dito}$ ($B_i \in R$). Hence we arrive at $F_j = -e_j G_k e_k + B_j + e_1 \dots e_m \text{ dito}$ so that also (take, e.g., $k = 1$) $F(u) = uG'(u) + \sum B_j u_j$ ■

The previous theorem in fact characterizes the monogenic product for tensors completely because the quotient ring $\mathcal{T}\mathfrak{b}/M^\#(\mathcal{T})$ is clearly isomorphic to the space $M(\mathcal{T}_s + e_1 \dots e_m \mathcal{T}_s)$ of monogenic projections of scalar and pseudoscalar tensors. This space is spanned (over the set of numbers of the form $a + e_1 \dots e_m b$ ($a, b \in R$)) by the collection of fundamental tensors $M(u_{j_1} \dots u_{j_k}) = M(\langle u, e_{j_1} \rangle \dots \langle u, e_{j_k} \rangle)$ for which we have already evaluated the Fischer product; it was merely a matter of shifting the indices. We may thus conclude that we have fully evaluated the product in the ring $\mathcal{T}\mathfrak{b}/M^\#(\mathcal{T})$ and as $\mathcal{T}\mathfrak{b}$ was in fact defined as the largest ring of tensors for which the above quotient is still a ring, we have come to the end of a certain line of thinking.

3. Products for special tensor types

Polynomials of one or several vector variables and Clifford differential forms are basic ingredients of what one could call the "Clifford analysis toolkit", i.e. they are natural objects to work with from a calculus point of view. The algebra of these objects is generated by

(i) the basic Clifford numbers e_1, \dots, e_m generating the Clifford algebra R_m

(ii) any number of vector variables $u_j = \sum u_{j_k} e_k$ whereby the coordinates u_{j_k} form a polynomial algebra

(iii) any number of vector differentials $du_j = \sum du_{j_k} e_k$ whereby the elements du_{j_k} multiply like generators of a Grassmann algebra.

An element of this algebra is a Clifford differential form with polynomial coefficients and algebras of Clifford differential forms were studied to some extent in [3, 7, 10]. From the above, it is clear how the multiplication of Clifford differential forms with polynomial coefficient $F(u_1, \dots, u_k; du_1, \dots, du_k)$ works; it is simply a mixture of Clifford multiplication, polynomial multiplication and differential form multiplication.

Of course, the basic toolkit for Clifford analysis is not complete with the introduction of this algebra; one also needs various endomorphisms on this algebra of which we mention the following ones:

(i) Dirac operators (or vector derivatives) $\partial_{u_j} = \sum \partial_{u_{j_k}} e_k$ which may act from the left or from the right on Clifford differential forms: $\partial_{u_j} F = \sum e_k \partial_{u_{j_k}} F$ and $F \partial_{u_j} = \sum \partial_{u_{j_k}} F e_k$.

(ii) Left and right contraction operators $\partial_{u_j} | F = \sum e_k \partial_{u_{j_k}} | F$ and $F | \partial_{u_j} = \sum \partial_{u_{j_k}} | F e_k$.

(iii) The exterior derivative $d : F \rightarrow dF$ which is the sum $d = d_{u_1} + \dots + d_{u_k}$ of all exterior derivatives $d_{u_j} = \sum du_{j_k} \partial_{u_{j_k}}$ with respect to all vector variables u_1, \dots, u_k under consideration.

(iv) Dual to the exterior derivative d , which transforms commuting variables u_{j_k} into anti-commuting variables du_{j_k} , one could even consider something like an operator " | " transforming commuting derivatives $\partial_{u_{j_k}}$ into anti-commuting derivatives $\partial_{u_{j_k}} |$, which are the basic contraction operators. Note hereby that the contraction operators $\partial_{u_{j_k}} |$ indeed behave like anti-commuting derivatives; they are determined by $\partial_{u_{j_k}} | du_{i_n} = \delta_{j_i} \delta_{k_n}$ together with the product rule

$$\partial_{u_{j_k}} |(FG) = (\partial_{u_{j_k}} | F)G + (-1)^k F(\partial_{u_{j_k}} | G),$$

F being a scalar k -form.

The formal calculus thus obtained is self-consistent and its rules are clear without any further need to relate things to Clifford tensors in order to give a mathematical explanation for differential forms. Everything follows from the axiomatic calculus of commuting and anti-commuting variables together with Clifford numbers, also the

integration of differential forms along oriented surfaces. Indeed, let $F(x, dx)$ ($x = \sum e_j x_j, dx = \sum dx_j e_j$) be a Clifford k -form and consider a piece of oriented surface given by a mapping

$$(u_1, \dots, u_k) \in O \subset R^k \rightarrow x(u_1, \dots, u_k).$$

Then one applies the chain rule $dx(u_1, \dots, u_k) = \sum \frac{\partial x_i}{\partial u_l} e_j du_l$ to arrive at a differential form $F(x(u), dx(u)) = G(u) du_1 \cdots du_k$ which can be integrated over the domain O the result being the integral of F over the surface $x(O)$. Note that the oriented surface measure on $x(O)$ corresponds to the differential form dx^k .

We now define a *conjugation* for polynomial Clifford forms as follows:

- (i) For $a \in R_m$ the conjugate is given by the main anti-involution $a \rightarrow \bar{a}$, i.e. $\bar{e}_j = -e_j$ and $\overline{ab} = ba$.
- (ii) For a scalar variable u_{j_k} , the conjugate is the derivative $\partial_{u_{j_k}}$ with respect to that variable.
- (iii) The conjugate of a differential du_{j_k} is the contraction operator $\partial_{u_{j_k}} |$ corresponding to that differential.
- (iv) The conjugation is denoted formally as a map

$$+ : F(u_1, \dots, u_n; du_1, \dots, du_n) \rightarrow \bar{F}(\partial_{u_1}, \dots, \partial_{u_n}; \partial_{u_1} |, \dots, \partial_{u_n} |)$$

and behaves like an anti-involution, i.e. $(FG)^+ = G^+ F^+$. In particular, for a product $du_{j_1 t_1} \cdots du_{j_k t_k}$ the conjugate is the operator of contraction $\partial_{u_{j_k t_k}} | \cdots \partial_{u_{j_1 t_1}} |$ (note that the conjugation is well defined in this way and can be extended to an actual anti-involution in a unique way).

Next we consider polynomial Clifford forms $F(u_1, \dots, u_n; du_1, \dots, du_n)$ which are homogeneous of degree k_1, \dots, k_n in the variables u_1, \dots, u_n and of degree l_1, \dots, l_n in the variables du_1, \dots, du_n . Then for two such polynomial Clifford forms, the action

$$G^+ F = \bar{G}(\partial_{u_1}, \dots, \partial_{u_n}; \partial_{u_1} |, \dots, \partial_{u_n} |) F(u_1, \dots, u_n; du_1, \dots, du_n)$$

is a Clifford number leading to the Fischer inner product

$$(G, F) = \frac{1}{k_1!} \cdots \frac{1}{k_n!} \frac{1}{l_1!} \cdots \frac{1}{l_n!} G^+ F$$

defined on the right R_m -module of homogeneous polynomial Clifford forms and hence extendable to the ring of all polynomial Clifford forms Φ . It is easy to see that the scalar part $[(G, F)]_0$ of the Fischer inner product defines a positive definite scalar inner product on each homogeneous subspace $\Phi_{k_1, \dots, k_n; l_1, \dots, l_n}$ of the ring Φ and hence also on the ring Φ itself.

Next we introduce the monogenic decomposition. First we have the following

Definition 3. A polynomial Clifford form $F(u_1, \dots, u_n; du_1, \dots, du_n)$ is called (left) *monogenic* if it satisfies the system

$$\left. \begin{aligned} \partial_{u_j} F(u_1, \dots, u_n; du_1, \dots, du_n) &= 0 \\ \partial_{u_j} |F(u_1, \dots, u_n; du_1, \dots, du_n) &= 0 \end{aligned} \right\} \quad (j = 1, \dots, n).$$

The subspace of monogenic elements in Φ is denoted by $M(\Phi)$.

Remarks.

(1) From the previous definition it is clear what are monogenic forms $F(u_1, \dots, u_n; du_1, \dots, du_n)$ in general (not only polynomial ones) – they are the solutions of $\partial_{u_j} F = \partial_{u_j} |F = 0$ ($j = 1, \dots, n$).

(2) From the relation $\partial_{u_j} = d\partial_{u_j} + \partial_{u_j}|d$ it follows that for any monogenic form F , the exterior derivative dF is still monogenic. In this way one arrives at a "complex of monogenic forms" which generalizes the complex of holomorphic forms $F(z_j, dz_j)$ of several complex variables (i.e. the holomorphic de Rham complex, not the Dolbeault complex).

Next consider the space $M^\#(\Phi)$ of polynomial Clifford forms of the form

$$F = \sum u_j F_j + \sum du_j F'_j \quad (F_j, F'_j \in \Phi).$$

Then from the previous definitions one easily obtains the following

Theorem 2 (Fischer decomposition). *Every polynomial Clifford form $F \in \Phi$ admits an orthogonal decomposition of the form $F = M(F) + M^\#(F)$, $M(F) \in M(\Phi)$ being the monogenic projection of F and $M^\#(F) \in M^\#(\Phi)$.*

By recursive application of this theorem one arrives at a complete and unique Fischer decomposition which is not orthogonal but which can nevertheless be decomposed into orthogonal pieces :

$$F = M_0(F) + M_1(F) + \dots + M_{k+l}(F)$$

whereby $M_0(F) \in M_0(\Phi) = M(\Phi)$ is the monogenic piece and $M_j(F) \in M_j(\Phi)$ has the form $M_j(F) = \sum u_i F_i + \sum du_i F'_i$ ($F_i \in M_{j-1}(\Phi)$). This can be proved directly or obtained from [9] after translating differential forms into tensors. In the definition of $M^\#(\Phi)$ and $M_k(\Phi)$ it is a little strange that vector variables u_j and vector differentials du_j are mixed up. It is possible to separate the vector variables u_j and the vector differentials du_j in the above Fischer decomposition. Indeed, one can decompose the spaces $M_k(\Phi) = M_{k,0}(\Phi) + \dots + M_{k,k}(\Phi)$ whereby $M_{0,0}(\Phi) = M_0(\Phi)$, $F \in M_{k,0}(\Phi)$ has the form $\sum du_j F_j$ with $F_j \in M_{k-1,0}(\Phi)$ and whereby $F \in M_{k,j}(\Phi)$ has the form $\sum u_j F_j + \sum du_j G_j$ with $F_j \in M_{k-1,j-1}(\Phi)$ and $G_j \in M_{k-1,j}(\Phi)$.

The above refinement of the Fischer decomposition can be shown to be a direct sum decomposition for $m \geq 3$ (see also [9]). But it is no orthogonal decomposition for $m = 2$ not even a direct sum (see also [9: Lemma 1]). For example, we have that

$$\left. \begin{aligned} (u_1 e_1 + u_2 e_2)(du_1 e_1 - du_2 e_2) &\in M_{1,1}(\Phi) \\ (du_1 e_1 + du_2 e_2)(u_1 e_1 - u_2 e_2) &\in M_{1,0}(\Phi) \end{aligned} \right\}$$

and both forms are equal.

Note that $d : M_{k,0}(\Phi) \rightarrow M_{k,0}(\Phi)$, which leads to a set of new complexes including the above mentioned complex $d : M_0(\Phi) \rightarrow M_0(\Phi)$. We also have that $d : M_{k,j}(\Phi) \rightarrow M_{k,j}(\Phi) + M_{k,j-1}(\Phi)$, which may give rise to even more complexes, but only $d : M_{k,0}(\Phi) \rightarrow M_{k,0}(\Phi)$ is a translation invariant complex defined in domains of $(R^m)^n$. Now let us turn to the definition of algebra of monogenic forms. To study these one does not need the complete Fischer decomposition; the statement of Theorem 2 is sufficient. The treatment of the product of monogenic forms is similar to the cases studied before: first we consider a subring $\Phi\mathfrak{b}$ of Φ consisting of those elements $F \in \Phi$ satisfying the two (or rather $2n$) conditions

$$\left. \begin{aligned} Fu_j &= \sum u_i F_i + \sum du_i F'_i \\ Fdu_j &= \sum u_i F_i + \sum du_i F'_i \end{aligned} \right\} \quad (j = 1, \dots, n; \text{ for some } F_j, F'_j \in \Phi).$$

Next, it is easily seen that the ring $M^\#(\Phi)$ is a two-sided ideal of the ring $\Phi\mathfrak{b}$ and that, moreover, due to Theorem 2, the quotient ring $\Phi\mathfrak{b}/M^\#(\Phi)$ is canonically isomorphic to the space $M(\Phi\mathfrak{b})$ of monogenic projections of elements in $\Phi\mathfrak{b}$. In this way the Fischer product for monogenic differential forms belonging to $\Phi\mathfrak{b}$ is canonically defined and $\Phi\mathfrak{b}$ is the largest subring of Φ which still contains $M^\#(\Phi)$ as a two-sided ideal. Hence again we have come to the end of a certain line of thinking. It is important to remark that the exterior derivative "d" leaves the ring $\Phi\mathfrak{b}$ invariant due to the fact that $d(F)u_j = d(Fu_j) - (-1)^{\text{deg } F} Fdu_j \in M^\#(\Phi)$ and $d(F)du_j = d(Fdu_j) \in M^\#(\Phi)$. Hence $d : M(\Phi\mathfrak{b}) \rightarrow M(\Phi\mathfrak{b})$.

We conclude this section with the following two remarks.

1. The first one has to do with the fact that we defined all operations in this section for differential forms with polynomial coefficients. In analysis and in particular in the treatment of cohomology problems one is more interested in differential forms with general analytic or smooth coefficients, not polynomial ones. But analytic functions can be seen as limits of polynomials and the operations introduced in this section (exterior derivative, products) carry over while taking the limit. This section was only intended to introduce basic tools.

2. There is the "tensor interface". It is clear that a complete theory of Clifford differential forms including integration, cohomology and monogenicity can be introduced in a fully self-consistent way without any need to define differential forms (and polynomials) as alternating (symmetric) tensors. On the other hand the translation of Clifford differential forms $F \in \Phi$ into tensors can be useful. For instance, the Fischer decomposition carries over from tensors to differential forms and no further proof of it is needed. This translation works as follows. Let $F(u_1, \dots, u_n; du_1, \dots, du_n) \in \Phi$ be homogeneous of degree (k_1, \dots, k_n) in (u_1, \dots, u_n) and of degree (l_1, \dots, l_n) in (du_1, \dots, du_n) . Then one can interpret F as a tensor of degree $k + l$ ($k = \sum k_j, l = \sum l_j$) in the variables (v_1, \dots, v_{k+l}) in a unique way such that $F(v_1, \dots, v_{k+l})$ is symmetric in the sets of variables $(v_1, \dots, v_{k_1}), \dots, (v_{k-k_{n+1}}, \dots, v_k)$ and anti-symmetric in the sets of variables $(v_{k+1}, \dots, v_{k+l_1}), \dots, (v_{k+l-l_{n+1}}, \dots, v_{k+l})$. Moreover, this representation is the inverse of the projection operator which maps a general $k + l$ tensor $F(v_1, \dots, v_{k+l})$ onto a

differential form $F(u_1, \dots, u_n; du_1, \dots, du_n)$ of degree $(k_1, \dots, k_n; l_1, \dots, l_n)$, projection operator which is given by the simple formula

$$F(v_1, \dots, v_{k+l}) \rightarrow F(u_1, \dots, u_k; \dots; u_n, \dots, u_n; du_1, \dots, du_k; \dots; du_n, \dots, du_n).$$

This projection preserves the monogenic decomposition and maps the tensor product to the Φ -product and $\mathcal{T}\mathfrak{b}$ to a proper subclass of $\Phi\mathfrak{b}$.

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