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q-Convexity Properties of the Coverings of a Link Singularity

by

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Abstract

We prove that for a germ of normal isolated singularity (Y, y_0) obtained by contracting a curve, if the fundamental group of the link singularity is infinite then the universal covering of $Y \setminus \{y_0\}$ can be written as the union of $n-1$ Stein open subsets.

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§1. Introduction

Let (Y, y_0) be the germ of a normal 2-dimensional singularity and let K be the associated link singularity. It was shown in [4] that if $\pi_1(K)$ is an infinite group then the universal covering of $Y \setminus \{y_0\}$ is Stein for Y small enough.

In this paper we generalize this result to the case when (Y, y_0) is a normal isolated singularity of dimension $n \geq 2$ obtained by contracting a complex curve. More precisely we prove:

Theorem. *Suppose that (Y, y_0) is a germ of normal isolated singularity obtained by contracting a curve, $\dim(Y) = n \geq 2$, and let K be the corresponding link singularity. If $\pi_1(K)$ is infinite then the universal covering space of $Y \setminus y_0$ for Y small can be written as the union of $n-1$ Stein open subsets. In particular it is $(n-1)$-complete.*

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The theory of q-convexity was introduced by A. Andreotti and H. Grauert in [1] and is one of the basic tools in the study of the geometry of non-compact complex spaces.

§2. Preliminaries

For the following result see [15].

Theorem 1. *Let X be a complex space and $p : Y \to X$ a covering. If X is Stein then Y is Stein as well.*

Theorem 2 was proved by Y. T. Siu in [14].

Theorem 2. *If X is a complex space and Y is a Stein subspace then there exists an open Stein subset U of X such that $Y \subset U$.*

Definition 1. Suppose that X is a Stein space and U is an open subset. We say that U is *Runge in X* (or that the pair (U, X) is Runge) if U is Stein and the restriction map $\mathcal{O}(X) \to \mathcal{O}(U)$ has dense image.

The following lemma is standard.

Lemma 1. *Suppose that X is a complex space and $\{X_n\}_{n\geq 1}$ is an increasing sequence of Stein open subsets of X. If each pair (X_n, X_{n+1}) is Runge then $\bigcup_{n\geq 1} X_n$ is Stein.*

For the next lemma see [10].

Lemma 2. *Suppose that X is a Stein space and $\phi : X \to \mathbb{R}$ is a plurisubharmonic function. Then for any $r \in \mathbb{R}$ the open set $U = \{x \in X : \phi(x) < r\}$ is Runge in X.*

The lemma below follows from the fact that for a connected locally irreducible complex space the complement of a complex subspace of positive codimension is connected.

Lemma 3. *Suppose that $\pi : X \to Y$ is a proper morphism of complex spaces and that there exists a discrete subset A of Y such that $\pi : X \setminus \pi^{-1}(A) \to Y \setminus A$ is a biholomorphism. If X is locally irreducible then Y is locally irreducible as well.*

The following result is Theorem 2 in [13].

Theorem 3. *Let X be a locally irreducible Stein complex space of pure dimension 2 with isolated singularities and $A \subset X$ a closed complex subvariety without isolated points. Then $X \setminus A$ is Stein.*

Remark. In [3] it was proved that if $\dim(X) = n \geq 2$ then $X \setminus A$ is the union of $n - 1$ Stein open subsets.

Using Theorem 3 and Lemma 3 we obtain:

Corollary 1. *Let X be a locally irreducible complex space of dimension 2 and $A \subset X$ a 1-dimensional closed complex subspace. Assume that X is a proper modification of a Stein space at a discrete set of points, A is connected, has at least one non-compact irreducible component, and $X \setminus A$ has no compact subspaces of positive dimension. Then $X \setminus A$ is Stein.*

For the following proposition see [12, Remark a), p. 165].

Proposition 1. *Suppose that X is a 1-convex complex space and its exceptional set A is 1-dimensional. Then A has a neighborhood that can be embedded into a space $\mathbb{C}^n \times \mathbb{P}^m$.*

The following theorem follows immediately from Theorem 2.4 in [12] using desingularization:

Theorem 4. *Let X be a 1-convex manifold which is embeddable into a space $\mathbb{C}^n \times \mathbb{P}^m$. Then there exist an open 1-convex neighborhood V of the exceptional set and a complex projective manifold Z such that V is an open subset of Z.*

The following theorem was proved in [2] and [9].

Theorem 5. *Let $\pi : X \to T$ be a proper holomorphic surjective map of complex spaces, let $t_0 \in T$ be any point, and denote by $X_{t_0} := \pi^{-1}(t_0)$ the fiber of π at t_0. Assume that $\dim X_{t_0} = 1$. Let $\sigma : \tilde{X} \to X$ be a covering space and let $\tilde{X}_{t_0} = \sigma^{-1}(X_{t_0})$. If $\tilde{X}_{t_0}$ is holomorphically convex, then there exists an open neighborhood Ω of t_0 such that $(\pi \circ \sigma)^{-1}(\Omega)$ is holomorphically convex.*

Lemma 4. *Suppose that X is a Stein space and U and V are two Stein open subsets of X. If (U, X) is Runge then $(U \cap V, V)$ is also Runge.*

Proof. Let $K \subset U \cap V$ be a compact set. We have to show that there exists a plurisubharmonic function $\phi : V \to \mathbb{R}$ with $K \subset \{x \in V : \phi(x) < 0\} \Subset U \cap V$. Let $\phi_1 : X \to \mathbb{R}$ be a plurisubharmonic function such that $K \subset \{x \in X : \phi_1(x) < 0\} \Subset U$ and $\phi_2 : V \to \mathbb{R}$ a plurisubharmonic exhaustion function such that $\phi_{2|K} < 0$. Then $\phi = \max\{\phi_1, \phi_2\}$ has the desired property. □

Corollary 2. *Suppose that X is a complex space and Ω_1, Ω_2, U_1 and U_2 are open Stein subspaces of X such that $U_1, U_2 \subset \Omega_1 \cap \Omega_2$. If (U_1, Ω_1) and (U_2, Ω_2) are Runge then $U_1 \cap U_2$ is Runge in both Ω_1 and Ω_2.*

The next lemma was proved in [4].

Lemma 5. *Let X be a Stein space and let Y, U be Stein open subsets such that $X = U \cup Y$. Assume that $(Y \cap U, U)$ is Runge. Then (Y, X) is also Runge.*

Theorem 6 was proved in [5]; for a more general result see [6].

Theorem 6. *Suppose that X and Y are complex analytic subsets of some neighborhood U of the origin in $\mathbb{C}^n$ such that $0 \in Y$, $Y \subset X$ and $X \setminus Y$ is smooth. If the dimension of each component of $X \setminus Y$ is $\geq n$ and if Y is defined in X by k holomorphic equations, then the pair $(X_\epsilon \setminus \{0\}, Y_\epsilon \setminus \{0\})$ is $(n-k-1)$-connected for $\epsilon > 0$ small enough.*

In the above theorem $X_\epsilon = \{x \in X : \|x\| \leq \epsilon\}$ and similarly for Y_ϵ. We also recall the following definition:

Definition 2. A pair (X, A) with $A \overset{i}{\hookrightarrow} X$ is called *k-connected* if $i_* : \pi_j(A, \{a\}) \to \pi_j(X, \{a\})$ is bijective for $j < k$ and surjective for $j = k$, for all $a \in A$.

Corollary 3. *Suppose that X is a locally irreducible complex space such that all of its irreducible components have dimension at least n, and let Y be a subspace of X. If $X \setminus Y$ is smooth and Y is locally defined in X by at most $n-2$ holomorphic equations, then Y is locally irreducible.*

We shall need the following:

Definition 3. Let L be a connected 1-dimensional complex space and $\bigcup L_i$ be its decomposition into irreducible components. Then L is called an *infinite Nori string* if all L_i are compact and L is not compact.

Definition 4. (a) If Ω is an open subset of $\mathbb{C}^n$, and $\psi : \Omega \to \mathbb{R}$ is a smooth function, then ψ is called *strictly q-convex* if its Levi form has at least $n-q+1$ positive eigenvalues at every point.

(b) Suppose that X is a complex space. A function $\phi : X \to \mathbb{R}$ is called *strictly q-convex* if for every $a \in X$ there exists an embedding of a neighborhood U of a as a closed analytic subset of an open subset Ω of $\mathbb{C}^n$, for some n, and a smooth strictly q-convex function $\psi : \Omega \to \mathbb{R}$ such that $\psi_{|U} = \phi$.

(c) A complex space X is called *q-complete* if there exists a strictly q-convex exhaustion function $\phi : X \to \mathbb{R}$ (i.e. $\{x \in X : \phi(x) < c\} \Subset X$ for every $c \in \mathbb{R}$).

§3. The results

Proposition 2. *Suppose that Z is a complex projective variety with* $\dim(Z) = n$ *and Y is a closed subvariety of Z with* $\dim(Y) = k$ *such that* $\mathrm{Sing}(Z) \subset \mathrm{Sing}(Y)$

and $k \leq (n-1)/2$*. Then there exists a principal hypersurface* H *of* Z *such that* $Y \subset H$ *and* $\mathrm{Sing}(H) \subset \mathrm{Sing}(Y)$*.*

Proof. Let L be a positive line bundle on Z and let $\mathcal{I}$ the ideal of Y. It follows (see for example [8]) that there exists $m_0 \in \mathbb{N}$ such that for any $m \geq m_0$ the canonical map $\psi_x : \Gamma(Z, \mathcal{I} \otimes L^m) \to \Gamma(Z, \mathcal{I}/\mathfrak{m}_x \mathcal{I} \otimes L^m)$ is surjective for every $x \in Y$. If x is a regular point of Y then $\Gamma(X, \mathcal{I}/\mathfrak{m}_x \mathcal{I} \otimes L^m) = \Gamma(X, \mathcal{I} \otimes \mathcal{O}/\mathfrak{m}_x \otimes L^m)$ is a vector space of dimension $n-k$. It follows that for such a point $\dim(\mathrm{Ker}(\psi_x)) = N-n+k$ where $N = \dim(\Gamma(Z, \mathcal{I} \otimes L^m))$.

We consider the diagram

$$\begin{array}{ccc} \mathcal{R} \hookrightarrow \mathrm{Reg}(Y) \times \Gamma(Z, \mathcal{I} \otimes L^m) & \xrightarrow{p_2} & \Gamma(Z, \mathcal{I} \otimes L^m) \\ \Big\downarrow{\scriptstyle p_1} & & \\ \mathrm{Reg}(Y) & & \end{array}$$

where $\mathcal{R} = \{(x,s) \in \mathrm{Reg}(Y) \times \Gamma(Z, \mathcal{I} \otimes L^m) : \mathrm{d}s(x) = 0\}$. Then $\dim(\mathcal{R} \cap p_1^{-1}(x)) = N - n + k$ for every $x \in \mathrm{Reg}(Y)$ and hence $\dim(\mathcal{R}) \leq N - n + 2k$. We assumed that $k \leq (n-1)/2$ and therefore $\dim(\mathcal{R}) < N$. We deduce that $p_2(\mathcal{R})$ has measure zero. If s is a section in $\Gamma(Z, \mathcal{I} \otimes L^m) \setminus p_2(\mathcal{R})$ and we set $H = \{x \in X : s(x) = 0\}$ then H is smooth at every point of $\mathrm{Reg}(Y)$. Also it follows from Bertini's theorem (see [7]) that for almost every s the hypersurface H is smooth at every point of $Z \setminus \mathrm{Sing}(Z)$, hence at every point of $Z \setminus Y$. We conclude that $\mathrm{Sing}(H) \subset \mathrm{Sing}(Y)$ for almost every $s \in \Gamma(Z, \mathcal{I} \otimes L^m)$. □

Lemma 6. *Suppose that* C *is a* 1*-dimensional connected compact complex space such that* C *has an irreducible component which is not locally irreducible. Then there exists a connected infinite Nori string* $\tilde{C}$ *and an unbranched covering map* $p : \tilde{C} \to C$*.*

Proof. Let C^1 be an irreducible component of C and $x_0 \in C^1$ a point such that $C^1_{x_0}$, the germ of C^1 at x_0, is not irreducible. Let $\bigcup_{i \in I_1} C_{i,x_0}$ be the decomposition of $C^1_{x_0}$ into irreducible components (according to our assumption, I_1 has at least two elements), and $\bigcup_{i \in I} C_{i,x_0}$ the decomposition of C_{x_0} into irreducible components with $I_1 \subset I$. Let U and V be open neighborhoods of x_0 such that $\overline{U} \subset V$ and there exist closed analytic subsets C_i, $i \in I$, of V, which are representatives for C_{i,x_0}. We pick an index $j \in I_1$ and let $C' = \bigcup_{i \in I \setminus \{j\}} C_i$, which is a closed analytic subset of V. Let $F := ((C \setminus \overline{U}) \sqcup C' \sqcup C_j)/\sim$ where the equivalence relation is defined as follows. Let $x \in (C \setminus \overline{U}) \cap V$. Note that $(C \setminus \overline{U}) \cap V = (C' \setminus \overline{U}) \cup (C_j \setminus \overline{U})$ and that $C' \setminus \overline{U}$ and $C_j \setminus \overline{U}$ are disjoint. Then if $x \in C' \setminus \overline{U}$ we identify it with the corresponding point in C', while if $x \in C_j \setminus \overline{U}$ we identify it with the corresponding point in C_j. We have a projection $\tau : F \to C$ whose fiber above x_0 has exactly two elements, $P \in C'$ and $Q \in C_j$, and $\tau : F \setminus \{P, Q\} \to C \setminus \{x_0\}$ is a biholomorphism. Note that F is connected and compact.

Let now $\{F_k\}_{k\in\mathbb{Z}}$ be 1-dimensional complex spaces, each biholomorphic to F via $\pi_k : F_k \to F$ and $P_k = \pi_k^{-1}(P)$, $Q_k = \pi_k^{-1}(Q)$. We set $\tilde{C} = (\bigsqcup F_k)/\sim$ where Q_k is identified with P_{k-1}. If we put $p : \tilde{C} \to C$, $p(x) = \tau(\pi_k(x))$ for each $x \in F_k \setminus \{P_k, Q_k\}$ and $p(Q_k) = x_0$, we obtain an unramified covering. Obviously, $\tilde{C}$ is a connected infinite Nori string. □

Remark. Let T_k be the equivalence class of F_k in $\tilde{C}$ and z_k be the unique intersection point of T_k and T_{k+1} (i.e. z_k is the equivalence class of Q_k). The identity map $F \to F$ induces a biholomorphism $T_k \to F$ which in turn induces a biholomorphism $g_k : T_k \to T_{k+1}$. Then $g : \tilde{C} \to \tilde{C}$, $g_{|T_k} = g_k$, is a (well-defined) covering transformation map.

Proposition 3. *Let X and Y be two n-dimensional normal complex spaces, $n \geq 3$, $y_0 \in Y$ and $\pi : X \to Y$ a proper holomorphic map such that $C = \pi^{-1}(y_0)$ is a connected 1-dimensional complex space and $\pi : X \setminus C \to Y \setminus \{y_0\}$ is a biholomorphism. Assume that $H_1(C)$ is infinite and that there exists a locally irreducible 2-dimensional complex subspace S of X with isolated singularities such that $C \subset S$. Then there exist an open neighborhood W of C in X and an unbranched covering $p : \tilde{W} \to W$ such that $p^{-1}((S \setminus C) \cap W)$ is Stein.*

Proof. We consider the decomposition $C = \bigcup C_i$ into irreducible components. Because $H_1(C)$ is infinite we distinguish three possible cases:

1. All irreducible components C_i are locally irreducible, their graph is a (connected) tree, and at least one them has genus greater than or equal to 1.
2. There exists an irreducible component C_{i_0} which is not locally irreducible.
3. All irreducible components C_i are locally irreducible, and their graph contains a cycle.

Case 1. In this case let $p : \tilde{C} \to C$ be a connected holomorphically convex covering of C that has at least one non-compact irreducible component. There exists such a covering because at least one irreducible component of C has genus greater than or equal to 1. We also choose an open neighborhood W_1 of C in X such that W_1 has a continuous deformation retraction onto $W_1 \cap S$ and $W_1 \cap S$ has a continuous deformation retraction onto C. We extend the covering $p : \tilde{C} \to C$ to a covering $p : \tilde{W}_1 \to W_1$, which in turn induces a covering $p : \tilde{S} \to S \cap W_1$. We apply Theorem 5 to deduce that we can find a neighborhood W of C in X such that $p^{-1}(W)$ is holomorphically convex and therefore $p^{-1}(S \cap W)$ is holomorphically convex. Note that every compact 1-dimensional subspace of $p^{-1}(S \cap W)$ is included in $\tilde{C}$ and therefore $p^{-1}(S \cap W)$ is a proper modification of a Stein space at a discrete set of points. Corollary 1 then implies that $p^{-1}(S \cap W) \setminus \tilde{C} = p^{-1}((S \setminus C) \cap W)$ is Stein.

Case 2. We apply Lemma 6 to get a covering space $p : \tilde{C} \to C$ such that $\tilde{C}$ is an infinite Nori string. As in Case 1, we choose an open neighborhood W_1 of C in X such that W_1 has a continuous deformation retraction on $W_1 \cap S$ and $W_1 \cap S$ has a continuous deformation retraction in S onto C, and we extend p to a covering $p : \tilde{W}_1 \to W_1$ which induces a covering $p : \tilde{S} \to S \cap W_1$. At the same time the covering transformation map g extends to a covering transformation map $g : \tilde{S} \to \tilde{S}$. We are using here the notation of the proof of Lemma 6 and of the Remark that follows. Let $U_0 \subset \tilde{S}$ be a strictly pseudoconvex, relatively compact neighborhood of T_0. For $k \in \mathbb{Z}$, $k > 0$, we denote by $g^{(k)}$ the k-th iterate $g \circ \cdots \circ g$ and for $k \in \mathbb{Z}$, $k < 0$, we put $g^{(k)} = (g^{-1})^{(k)}$. We set $U_k = g^{(k)}(U_0)$. Then U_k is a strictly pseudoconvex neighborhood of T_k. Shrinking U_0 we can assume that $\overline{U}_0 \cap \bigcup_{|k| \geq 2} \overline{U}_k = \emptyset$ and that $p_{|U_0 \cap U_1}$ and $p_{|U_0 \cap U_{-1}}$ are 1-1. In particular U_0 does not contain any T_k, $k \neq 0$. It follows, obviously, that $U_p \cap U_q = \emptyset$ if $|k - p| > 1$. By Corollary 1, $U_k \setminus T_{k-1}$ and $U_{k+1} \setminus T_{k+2}$ are Stein open subsets of $\tilde{S}$. We now choose an open Stein neighborhood B_0 of z_0 such that $B_0 \subset U_0 \cap U_1$ and B_0 is Runge in both $U_0 \setminus T_{-1}$ and $U_1 \setminus T_2$ (see Corollary 2). Moreover we assume that there exists an open Stein neighborhood V_1 of y_0 in Y such that $V_1 \supset p(\overline{B}_0)$. It follows from Lemma 4 that $B_0 \setminus \tilde{C}$ is Runge in both $U_0 \setminus \tilde{C}$ and $U_1 \setminus \tilde{C}$. We set $B_k = g^{(k)}(B_0)$. Note that $p(B_k) = p(B_0)$ for every $k \in \mathbb{Z}$, and $B_k \setminus \tilde{C}$ is Runge in both $U_k \setminus \tilde{C}$ and $U_{k+1} \setminus \tilde{C}$.

We choose a strictly plurisubharmonic exhaustion function $\phi : V_1 \to \mathbb{R}$ for V_1 such that $\phi(y_0) = 0$ and $\phi(y) > 0$ for $y \in V_1 \setminus \{y_0\}$. Let $\epsilon > 0$ be such that $V = \{y \in V_1 : \phi(y) < \epsilon\} \Subset p(B_0)$. We claim that $(p \circ \pi)^{-1}(V \cap S) \setminus \tilde{C}$ is Stein. To prove this we consider for $k, l \in \mathbb{Z}$, $l < k$, $\Omega_{k,l} = (\bigcup_{j=l}^{k} U_j) \cap ((p \circ \pi)^{-1}(V \cap S))$ and $M_{k,l} = \Omega_{k,l} \setminus \tilde{C}$. Note that since $p(\partial U_{j_1} \cap \partial U_{j_2}) \cap B_0 = \emptyset$ for $j_1 \neq j_2$, each $\Omega_{k,l}$ is a strictly pseudoconvex, relatively compact open subset of $\tilde{S}$. Its maximal compact 1-dimensional subvariety is $T_l \cup \cdots \cup T_k$, which is exceptional. Hence $\Omega_{k,l}$ is 1-convex. On the other hand $\tilde{C} \cap \Omega_{k,l} = (\bigcup_{j=l}^{k} T_j) \cup (T_{k+1} \cap \Omega_{k,l}) \cup (T_{l-1} \cap \Omega_{k,l})$. Because Ω_k does not contain T_{k+1} or T_{l-1} it follows from Corollary 1 that $M_{k,l}$ is Stein. Note now that $M_{k+1,l} = M_{k,l} \cup ((U_{k+1} \cap (p \circ \pi)^{-1}(V \cap S)) \setminus \tilde{C})$ and that $M_{k,l} \cap ((U_{k+1} \cap (p \circ \pi)^{-1}(V \cap S)) \setminus \tilde{C}) = (B_k \cap (p \circ \pi)^{-1}(V \cap S)) \setminus \tilde{C}$, which by Lemma 2 is Runge in $(U_{k+1} \cap (p \circ \pi)^{-1}(V \cap S)) \setminus \tilde{C}$. We deduce from Lemma 5 that $M_{k,l}$ is Runge in $M_{k+1,l}$. Similarly $M_{k,l}$ is Runge in $M_{k,l-1}$. Therefore $M_{k,-k}$ is Runge in $M_{k+1,-k-1}$ for every $k \in \mathbb{Z}$, $k > 0$. As $(p \circ \pi)^{-1}(V \cap S) \setminus \tilde{C} = \bigcup_{k=1}^{\infty} M_{k+1,-k-1}$ it follows from Lemma 1 that $(p \circ \pi)^{-1}(V \cap S) \setminus \tilde{C}$ is Stein as claimed.

Case 3. Let $C_1, \dots, C_k$ be irreducible components of C such that their graph forms a minimal cycle (i.e. no proper subset of $\{C_1, \dots, C_k\}$ forms a cycle). We contract $C_2 \cup \dots \cup C_k$ in X to obtain a normal complex space X'. Let S' and C' be the images of S and C respectively. It follows from Lemma 3 that S' is locally irreducible. Notice at the same time that C' is not locally irreducible anymore and hence we can apply Case 2. We obtain a neighborhood W' of C' and a covering map $p' : \tilde{W}' \to W'$ such that $p'^{-1}((S' \setminus C') \cap W')$ is Stein. We pull back this covering via the contraction map to obtain a covering for a neighborhood of C with the desired property. □

Theorem 7. *Suppose that (Y, y_0) is a germ of normal isolated singularity obtained by contracting a curve, $\dim(Y) = n \geq 2$, and let K be the corresponding link singularity. If $\pi_1(K)$ is infinite, then the universal covering space of $Y \setminus y_0$ for Y small can be written as the union of $n-1$ Stein open subsets. In particular it is $(n-1)$-complete.*

Proof. If $\dim(Y) = 2$ the theorem was proved in [4]. Hence we assume that $\dim(Y) \geq 3$. Let $\pi : X \to Y$ be a local resolution of singularities and C the exceptional curve. As we are assuming that $n \geq 3$ it follows that $H_1(C)$ is infinite since $\pi_1(C)$ is infinite (note that C has real codimension > 2 in X so $\pi_1(X) = \pi_1(X \setminus C)$). On the other hand from Proposition 1 it follows that C has a strictly pseudoconvex neighborhood which can be embedded into a space $\mathbb{C}^n \times \mathbb{P}^m$, and then by Theorem 4, there exist an open 1-convex neighborhood V of the exceptional set and a complex projective manifold Z such that V is an open subset of Z. We will show now that we can find a two-dimensional locally irreducible subvariety S of Z such that $\mathrm{Sing}(S) \subset \mathrm{Sing}(C)$ and $Z \setminus S$ is the union of $n-2$ Stein open subsets. The local irreducibility will follow from Corollary 3 if we can choose S to be a local set-theoretic complete intersection. To obtain S we apply Proposition 2, $n-2$ times, to we obtain a sequence of projective varieties $H_1 \supset \dots \supset H_{n-2} =: S \supset C$ such that H_{j+1} is a principal hypersurface in H_j and $\mathrm{Sing}(H_j) \subset \mathrm{Sing}(C)$. Each $H_j \setminus H_{j+1}$, $j = 1, \dots, n-3$, is Stein and Theorem 2 implies that there exists a Stein open subset Ω_{j+1} of Z such that $\Omega_{j+1} \cap H_j = H_j \setminus H_{j+1}$. If we put $\Omega_1 = Z \setminus H_1$ we get $Z \setminus H_{n-2} = Z \setminus S = \Omega_1 \cup \dots \cup \Omega_{n-2}$. In particular, since V is strictly pseudoconvex, $V \setminus S$ is the union of $n-2$ Stein open subsets.

We now apply Proposition 3 to find a strictly pseudoconvex neighborhood W of C in X such that on one hand $W \setminus S = W_1 \cup \dots \cup W_{n-2}$ where W_j, $j = 1, \dots, n-2$, are Stein open subsets of X and on the other hand there exists an (unbranched) covering space $p : \tilde{W} \to W$ for which $p^{-1}((S \setminus C) \cap W)$ is Stein.

It remains to notice is that $\tilde{W}_j := p^{-1}(W_j)$, $j = 1, \dots, n-2$, are Stein (see Theorem 1) and, at the same time, by Theorem 2 there exists a Stein open subset $\tilde{W}_{n-1}$ of $\tilde{W}$ such that $\tilde{W}_{n-1} \cap p^{-1}(S) = p^{-1}((S \setminus C) \cap W)$. Obviously $\tilde{W} = \tilde{W}_1 \cup \cdots \cup \tilde{W}_{n-1}$ and hence $\tilde{W}$ is the union of $n-1$ Stein open sets. As the universal covering $\hat{W}$ of W is a covering of $\tilde{W}$, Theorem 1 implies that $\hat{W}$ is the union of $n-1$ Stein open sets. The $(n-1)$-completeness of $\hat{W}$ follows from [11, Satz 2.3]. □

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