

The Modified Canonical Proboscis

R. Finn and J. Marek

Abstract: A canonical proboscis domain Ω corresponding to contact angle γ_0 , as introduced by Fischer and Finn and later studied by Finn and Leise, has the property that a solution of the capillary problem exists in Ω for contact angle γ if and only if $|\gamma - \frac{\pi}{2}| < |\gamma_0 - \frac{\pi}{2}|$. We show in this paper that every such domain can be modified so as to yield the existence of a bounded solution also at the angle γ_0 . The modification can be effected in such a way that for prescribed $\varepsilon > 0$ the solution height must physically become infinite when $|\gamma - \frac{\pi}{2}| > |\gamma_0 - \varepsilon - \frac{\pi}{2}|$, over a subdomain that includes as large a portion of Ω as desired.

Keywords: *Capillarity, contact angle, mean curvature, canonical proboscis, subsidiary variational problem*

AMS subject classification: 76 B 45, 53 A 10, 49 Q 10

1. Background material

The underlying idea for this paper can be traced to a discovery of Concus and Finn [1] that solutions of the capillary surface equation in wedge domains depend discontinuously on the data. We restrict ourselves here to configurations in the absence of gravity, in which case the equation for a surface $S: u(x, y)$ over a domain Ω takes the form

$$\operatorname{div} Tu = 2H, \quad Tu = \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}; \quad (1)$$

here the constant H is the *mean curvature* of S . The physical requirement that S meet vertical walls over the boundary $\Sigma = \partial\Omega$ in a prescribed constant angle γ yields the boundary condition

$$\mathbf{v} \cdot Tu = \cos \gamma \quad (2)$$

on Σ ; here \mathbf{v} is outer directed unit normal.

Physically, one seeks a solution of (1), (2) with prescribed volume V , corresponding to that amount of liquid in a capillary tube. It can be shown that whenever such a solution exists

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over a given base section Ω , it is uniquely determined by V and by γ . The dependence on V is evidenced only by addition of constants; if Ω is the physical base of the tube, then in order for the solution to have physical meaning V must be sufficiently large that $u(x,y) > 0$ throughout Ω .

It was shown in [1] that when Σ contains an isolated protruding corner of opening $2\alpha < \pi$ such that $|\gamma - \frac{\pi}{2}| > \alpha$ then regardless of V there is no surface $u(x,y)$ satisfying (1) as a classical solution in Ω and (2) on the smooth part of Σ . On the other hand, explicit examples can be given of surfaces with those properties that are uniformly bounded and continuous on the closed domain, and for which $|\gamma - \frac{\pi}{2}| = \alpha$.

This discontinuous dependence suggests use of the property as a way to measure very precisely the contact angle between various liquids and solids. The effectiveness of the method has already been shown in earth-based experiments (where a less pronounced discontinuity appears) using corners with rectilinear sides, by Coburn [2, p. 220] and by Weislogel [3] in configurations for which the critical angle γ_{cr} was about 80° . These results can presumably be sharpened further in a gravity-free environment, and correspondingly improved accuracy can be expected in general for configurations with γ close to 90° . But when γ is close to zero or to π the procedure can be subject to experimental error, owing to the smallness of the region near the vertex in which the discontinuity would manifest itself. With a view to obtaining more clearly observable behavior, *canonical proboscis domains* were introduced in [4]. For prescribed $\gamma = \gamma_0$ and arbitrary but fixed R_0 , the rectilinear sides

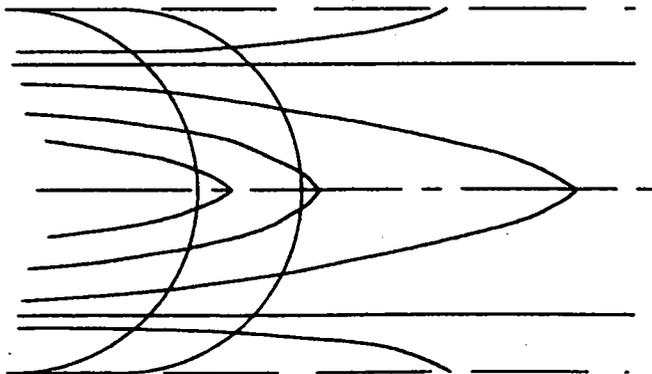


Figure 1. Integral curves of (3). These curves meet all horizontal translates of the indicated circular arcs in the given angle γ_0

of the corner used in the above experiments are replaced, for $\gamma > 0$, by curves satisfying the

equation

$$x + c = \sqrt{R_0^2 - y^2} + R_0 \sin \gamma_0 \ln \frac{\sqrt{R_0^2 - y^2} \cos \gamma_0 - y \sin \gamma_0}{R_0 + y \cos \gamma_0 + \sqrt{R_0^2 - y^2} \sin \gamma_0}, \tag{3}$$

and for $y < 0$ by the reflected images in the x -axis. In the case $0 < \gamma_0 < \frac{\pi}{2}$, these curves are illustrated in Figure 1, for varying values of c . Only those that lie between the asymptotes $y = \pm R_0 \cos \gamma_0$ are of interest for us. All of these are horizontal translates of a single curve, which meets the x -axis in the angle $\alpha_0 = |\frac{\pi}{2} - \gamma_0|$, forming with its reflected image a protruding angle $2\alpha_0$ there, and which extends backward to $x = -\infty$, approaching the asymptotic lines.

Geometrically, the curves are determined by the property that they meet all translates C_{R_0} of the right semicircle of radius R_0 centered on the x -axis in the constant angle γ_0 , see Figures 1 and 2. For simplicity in exposition, we assume in what follows that $0 < \gamma_0 < \frac{\pi}{4}$. The discussion for the complementary case is analogous.

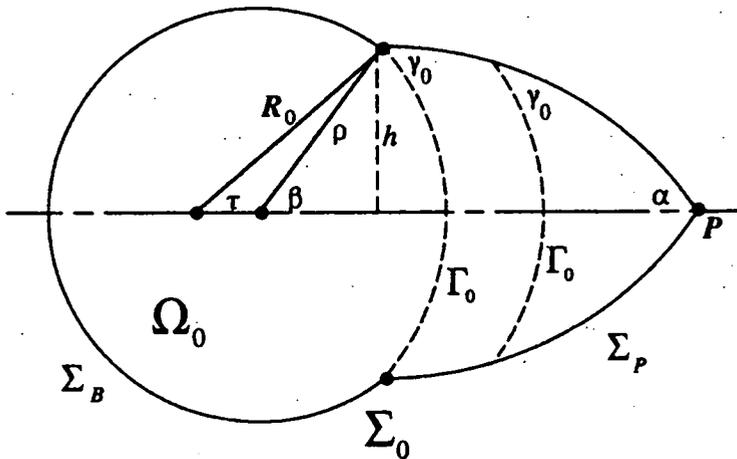


Figure 2. Canonical proboscis configuration

We form a *canonical proboscis domain* by choosing an arbitrary point on one of the curves and its reflection in the x -axis, and joining the points by a circular "bubble" of radius ρ chosen so that the equation

$$R_0 = \frac{|\Omega_0|}{|\Sigma_0| \cos \gamma_0} \tag{4}$$

will hold in the closed domain thus formed; here $|\Omega_0|$ and $|\Sigma_0|$ are the respective area and circumference of the configuration, see Figure 2; the subarcs Γ_0 of the above semicircles that meet the proboscis curves then become *extremals* of a subsidiary variational problem, see, e.g., [5, Chapter 6]. These curves sweep out a subdomain Ω_ρ which we refer to as the proboscis portion of Ω_0 , or simply proboscis. In [6] it is proved that this construction is always possible, and that $R_0 \cos \gamma_0 < \rho < 2R_0$, so that the proboscis can be chosen to occupy as large a portion of Ω_0 as desired. Further, it is proved in [6] that γ_0 is exactly the critical angle γ_{cr} for the configuration, in the senses that for fixed prescribed V :

- (i) A solution of (1), (2) exists in Ω_0 if and only if $|\gamma - \frac{\pi}{2}| < |\gamma_0 - \frac{\pi}{2}|$, and
- (ii) As $|\gamma - \frac{\pi}{2}| \nearrow |\gamma_0 - \frac{\pi}{2}|$ the solution rises to positive infinity throughout the entire portion of the domain that lies between any of the constructed extremals and the vertex P .

From (ii) we see immediately that if we start with a value of γ for which a solution exists over the entire domain, choose V large enough so that the surface will cover the base, and then let $\gamma \searrow \gamma_0$, then the surface must either become very large over a relatively large portion of Ω_0 or else uncover a portion of the base in the "bubble". Assuming that the changes occur slowly until γ is very close to γ_0 , and very rapidly thereafter, we will have found a procedure that could presumably be used for accurate measurements of small contact angles. In fact [7] computer calculations do suggest that this kind of behavior actually occurs, and that the procedure should therefore be feasible in many cases of interest. Nevertheless, the change is not strictly discontinuous as occurs in the case of rectilinear bounding segments at the corner (i.e., the solution no longer remains bounded as γ tends to γ_{cr}) and the question arises as to whether a similar global behavior can be achieved in the context of a configuration for which the solution continues to exist and remain bounded globally even at the critical angle γ_0 . We intend in this paper to answer that question affirmatively.

The key to the matter appears to be the difference in the magnitudes of the curvatures near the vertex P of the two configurations. In the former case the curvature vanishes, in the latter the limiting magnitude is $R_0^{-1} \cos \gamma_0 > 0$. We can prove that any solution is necessarily bounded whenever the limiting magnitude is less than $R_0^{-1} \cos \gamma_0$, but it suffices for our purpose to consider the case in which the corner is formed by two straight segments, in which case it is easier for us to prove the existence of the desired solutions. What we shall do is make a continuous deformation of the boundary, depending on a parameter $\varepsilon > 0$, such that a neighborhood of the vertex goes into straight segments of length $\sigma(\varepsilon)$ tending to zero

with ϵ , the remainder of the proboscis portion of Ω goes into a new proboscis, corresponding to a contact angle $\gamma^\epsilon = \gamma_0 - \epsilon$, and the radius $\rho(\epsilon)$ of the bubble adjusts itself so that (4) continues to hold. We shall then show that for a suitable range of ϵ , γ_0 continues to be the critical angle, and that the solution exists and is bounded throughout the perturbed domain when $\gamma = \gamma_0$. But if γ decreases to $\gamma_0 - \epsilon$, then the solution becomes infinite in the entire proboscis; thus a simple experiment reflecting these facts will suffice to bound the actual contact angle between $\gamma_0 - \epsilon$ and γ_0 .

2. The deformation

Since the equations are formally invariant under similarity transformation, we may normalize the configuration by fixing R_0 . We observe that the given proboscis is one of the family of curves (3) simply covering the strip $0 < y < R_0 \cos \gamma_0$, and for the resulting field of tangential directions, $y'(x)$ is a decreasing function of y . The semicircles C_{R_0} meet all these curves in the angle γ_0 . Given ϵ , $0 < \epsilon < \gamma_0$, we introduce a rectilinear segment T_ϵ tangent to the given proboscis at its vertex, and of length such that it meets the curve of the family (3)

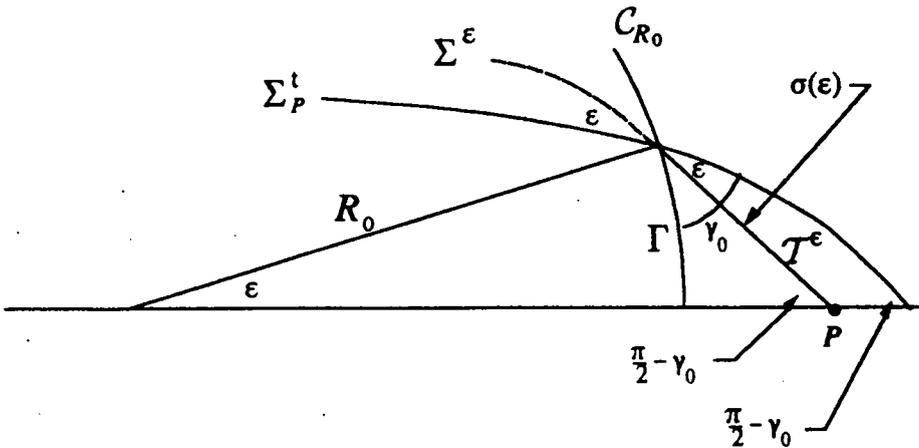


Figure 3. Construction of the modified proboscis; the arc Σ_ϵ^t is a horizontal translate of the original Σ_P through P . It is that arc of the family that passes through the endpoint of T^ϵ

passing through its endpoint in the angle ϵ . This point is easily determined, as the semicircle C_{R_0} through it will then subtend the angle ϵ with the x -axis, see Figure 3. We compute easily

$$\sigma(\epsilon) = R_0 \frac{\sin \epsilon}{\cos \gamma} \tag{5}$$

We now construct a new family (3), corresponding to the angle $\gamma^\epsilon = \gamma_0 - \epsilon$. That curve of the family passing through the endpoint of the segment T^ϵ just constructed will meet the segment tangentially; we use this curve together with T^ϵ to construct a new proboscis boundary Σ^ϵ . We take the points at the same height h as new initial points for "bubble" construction, so that (4) will continue to hold, in the form

$$|\Omega^\epsilon| - R_0 |\Sigma^\epsilon| \cos \gamma_0 = 0 \tag{6}$$

see Figure 4. We proceed to show that this procedure uniquely determines a new configuration.

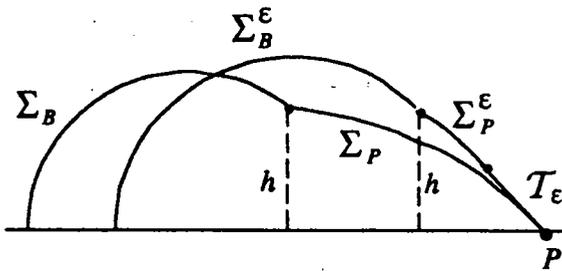


Figure 4. Bubble construction

Lemma 1. *The procedure just described can be continued until a maximum value $\epsilon_M = \sin^{-1}(h / R_0)$ is obtained; throughout the deformation, $\rho(\epsilon)$ is uniquely determined, with $\rho(0) = \rho_0$, and there holds $\rho > R_0 \cos \gamma_0$.*

The value ϵ_M is achieved when the proboscis portion Ω_p of Ω^ϵ consists entirely of rectilinear segments, at which point the domain has the appearance of a section of an ice cream cone, with the ice cream surface tangent to the cone on the intersection circle.

Proof of Lemma 1: We assume that the procedure has been completed up to some given value of ϵ , and we wish to continue it further. Setting

$$\mathcal{F}(\rho; \epsilon) = |\Omega^\epsilon| - R_0 |\Sigma^\epsilon| \cos \gamma_0 \tag{7}$$

it will suffice to show that $\mathcal{F}_\rho(\rho; \epsilon) \neq 0$. We note that Ω_p is unvaried in the differentiation. Referring to Figure 2 for notation, we may thus write

$$|\Omega'_B| = \rho^2 [(\pi - \beta) + \sin \beta \cos \beta], \quad |\Sigma'_B| = 2\rho(\pi - \beta), \tag{8}$$

and compute

$$\begin{aligned} \frac{\partial}{\partial \rho} |\Omega'_B| &= 2\rho [(\pi - \beta) + \tan \beta] \\ \frac{\partial}{\partial \rho} |\Sigma'_B| &= 2 [(\pi - \beta) + \tan \beta] \end{aligned}$$

so that $\mathcal{F}_\rho(\rho; \varepsilon) = 0$ is equivalent to

$$\rho = R_0 \cos \gamma_0$$

which by (6) is equivalent to

$$(\pi - \beta) - \sin \beta \cos \beta + \frac{1}{\rho^2} (\rho |\Sigma'_p| - |\Omega'_p|) = 0.$$

Since $(\pi - \beta) > \pi/2 > 1$, it suffices to prove that $\rho |\Sigma'_p| - |\Omega'_p| > 0$. This is however evident due to the monotonicity of the height of Σ_p , since ρ exceeds the height change and $|\Sigma'_p|$ exceeds the change in abscissa, so that the product of the two exceeds $|\Omega'_p|$.

Since by Corollary 4.1 of [6], $\rho > R_0 \cos \gamma_0$ in the initial configuration, our proof has shown that this inequality persists throughout the deformation. Since $R_0 \cos \gamma_0 > h$, the construction remains physically (and mathematically) possible, with a bubble protruding above and below the proboscis, until the segments extend to the maximum height h of Ω_p . In this configuration, ε achieves its largest possible value $\varepsilon_M = \sin^{-1}(h / R_0)$ that the procedure can yield, see Figure 2. \square

In practice, the chief interest in the procedure will presumably center on small values $\varepsilon \approx 1^\circ$, for the purpose of locating the contact angle between values γ_0 and $\gamma_0 - \varepsilon$.

3. Canonical properties

We must show that throughout the indicated procedure, γ_0 remains the critical angle for the configuration, in the sense that a bounded solution of (1), (2) exists over Ω^* whenever $\gamma_0 \leq \gamma \leq \pi/2$, but no solution exists when $0 \leq \gamma < \gamma_0$. We follow in general outline the corresponding discussion of [6]; there are, however, necessarily differences in detail. As in [6], we begin by observing that $\gamma_{cr} \geq \gamma_0$, in view of the protruding corner of opening $2\alpha = \pi - \gamma_{cr}$. We must prove that $\gamma_{cr} \leq \gamma_0$. In [6], that conclusion was obtained by showing that only a particular continuum of "extremal" subarcs Γ of semicircles of radius R_0 could meet the boundary Σ in angles that equal γ_0 at smooth intersection points or are not less than γ_0 at re-entrant corner points, and minimize the functional

$$\Phi = |\Gamma| - |\Sigma^*| \cos \gamma + \frac{1}{R} |\Omega^*|. \tag{9}$$

among all such arcs. See the "Theorem" in [6]; here Σ^* and Ω^* are the portions of Σ^* and of Ω^* cut off by Γ on the side opposite to that in which the curvature vector points.

In the present instance, we establish $\gamma_{cr} \leq \gamma_0$ by showing that there exist no subarcs at all of semicircles of radius R_0 with the properties just described. Using results of [8] or of [9], we then obtain additionally that a bounded solution does in fact exist for all γ in $\gamma_0 \leq \gamma \leq \pi/2$, yielding the desired discontinuous dependence.

We distinguish cases, as in [6]. We may assume, as in [6], that $0 < \gamma_0 < \pi/2$. We note that no minimizing extremal can include a semicircle, see Theorem 6.11 of [5].

Case 1. Γ meets the circular arc Σ_p^* of Σ in two interior points p and q , see Figure 5. We may assume as just noted that $\delta < \pi/2$. From the relation (6.53) of [5] with $\sigma = 0$, we find that the second variation \mathcal{J} of Φ takes the form, for the particular variation $\eta = a \cos \theta$,

$$\mathcal{J} = -a^2 \sin 2\delta + 2a^2 \frac{\cos^2 \delta}{\sin \gamma} \left(\cos \gamma - \frac{R_0}{\rho} \right). \tag{10}$$

Since $R_0 \sin \delta = \rho \sin(\delta + \gamma)$ we conclude $\mathcal{J} = -a^2 \cot \delta < 0$, so that Γ cannot minimize.

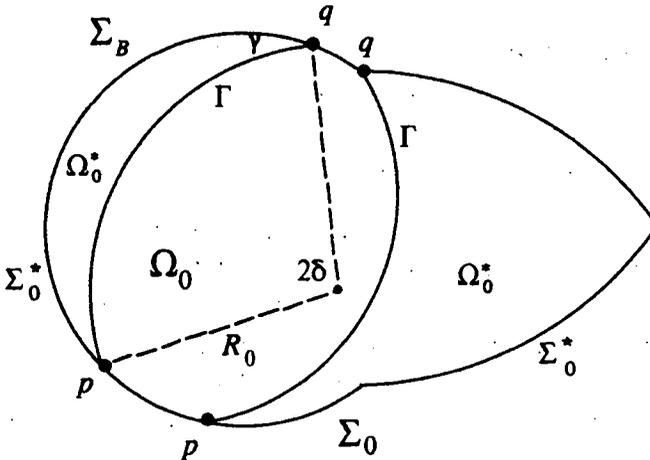


Figure 5. Cases 1 and 2

Case 2. Γ meets Σ_p^s at an interior point p and at a re-entrant corner q , see Figure 5. A small rotation of Γ about the center of Σ_p^s leaves Φ invariant and reduces this case to the previous one.

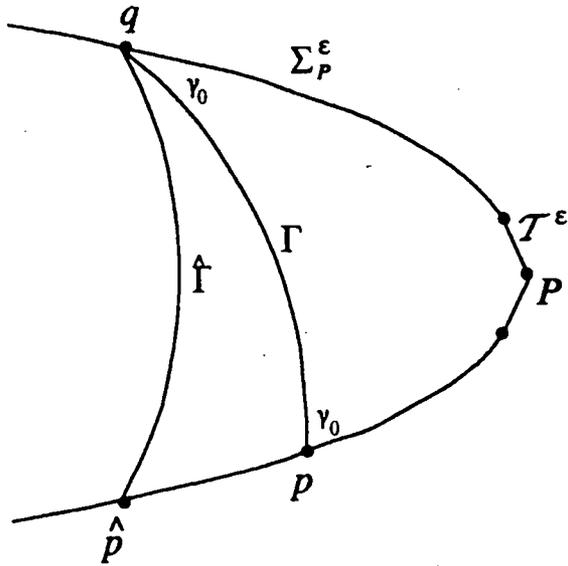


Figure 6. Case 3

Case 3. Γ meets the proboscis Σ_p^s at points p and q above and below, each of which is either an interior point or a re-entrant corner, and Γ is oriented as in Figure 6. The configuration cannot be symmetric, as the construction was effected so that the incident angles in that case would be less than γ_0 . We may suppose the point p to be the one closer to the vertex; we introduce the symmetrical arc $\hat{\Gamma}$ that meets Σ_p^s in the symmetric points q and \hat{p} . $\hat{\Gamma}$ is a rotation of Γ about q , and meets Σ_p^s at q in an angle exceeding γ_0 . But according to the construction, $\hat{\Gamma}$ meets Σ_p^s in equal angles less than γ_0 . So this case cannot occur.

Case 4. Γ meets two interior points p and q of the proboscis, with orientation as in Figure 7. We may assume that either the configuration is symmetric or else that p is the point closer to the vertex. We introduce the center point and line \mathcal{L} of symmetry of Γ , as indicated in the figure. The line through q parallel to \mathcal{L} must enter Ω^s as indicated, as the slope of Σ_p^s is negative at q . We then find $\delta + \tau = \pi/2$ and hence $\delta + \gamma_0 > \pi/2$. It follows from Theorem 6.16 of [5] that Γ cannot minimize, and thus this case is excluded.

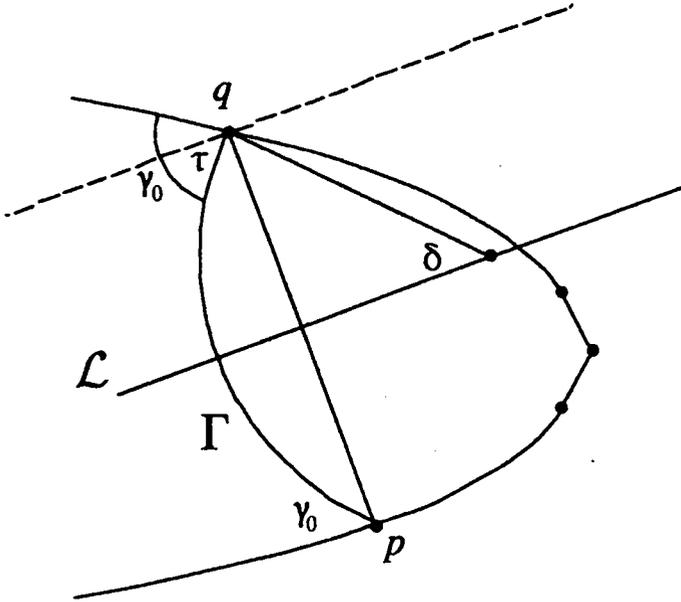


Figure 7. Case 4

This reasoning extends also to the case in which one or both of the contact points is a juncture point with Σ_b^c . That is so because according to Theorem 6.10 of [5] the angle between Γ and Σ_b^c equals or exceeds γ_0 when Γ minimizes, and the proof of Theorem 6.16 of [5] carries over without change when that occurs; the variations need only be restricted to positive ones, and these suffice for the proof.

Case 5. Γ meets two points on a single arc of the proboscis, with orientation as in Figure 8. Since the vertex angle with the vertical is γ_0 and since Σ_p^c is convex, the orthogonal to Γ at p must have positive slope. Since that arc of Σ_p^c has negative slope, Γ would have to include a semicircle and hence could not minimize.

Case 6. Γ meets two points on a single arc of the proboscis, and has orientation opposite to that indicated in Figure 8. From the differential equation

$$\frac{dy}{dx} = \frac{y \sin \hat{\gamma}_0 - \sqrt{R_0^2 - y^2} \cos \hat{\gamma}_0}{y \cos \hat{\gamma}_0 + \sqrt{R_0^2 - y^2} \sin \hat{\gamma}_0} \tag{11}$$

defining the curved portion of Σ_p^c (see [4]), we may calculate its curvature κ . Setting $y = R_0 \sin \tau$, we find

$$\kappa = -\frac{\cos(\tau + \hat{\gamma}_0)}{R_0 \cos \tau} \tag{12}$$

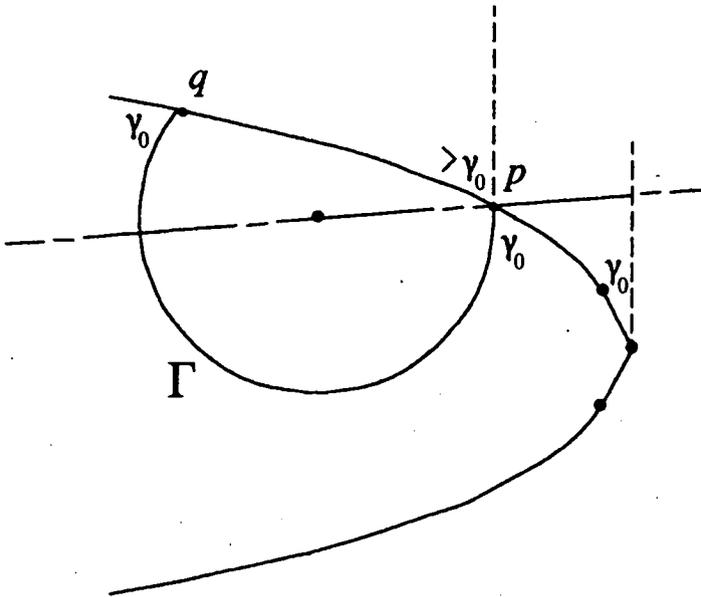


Figure 8. Case 5

and since

$$\tau = \sin^{-1} \frac{y}{R_0} < \sin^{-1} \cos \hat{\gamma}_0 \tag{13}$$

we find $\tau + \hat{\gamma}_0 < \pi/2$, from which follows $|\kappa| < 1/R_0$. On the rectilinear segments we have $\kappa = 0$. Thus, in any event the oriented curvature of Γ exceeds that of the boundary in the indicated orientation, so that this case cannot occur.

Case 7. One end of Γ terminates in the vertex V of the proboscis. This case is excluded by Theorem 6.10 of [5].

8. Γ meets Σ at a point p interior to Σ_p^* and a point q interior to Σ_b^* . We will exclude this case by using that q is exterior to the closed disk formed by completing Σ_b^* . The essential geometrical features that are needed are shown in Figure 10. The configuration has been rotated about the center of Σ_b^* so as to bring p and q to the same height. The completed Σ_b^* is shown and Σ_p^* suppressed for clarity. If $\gamma_0 + \delta \leq \pi/2$ then the segment \overline{pq} cannot lie above the center of Σ_b^* . Since q is exterior to Σ_b , \overline{pq} must have length exceeding $2\rho \sin(\gamma_0 + \delta)$.

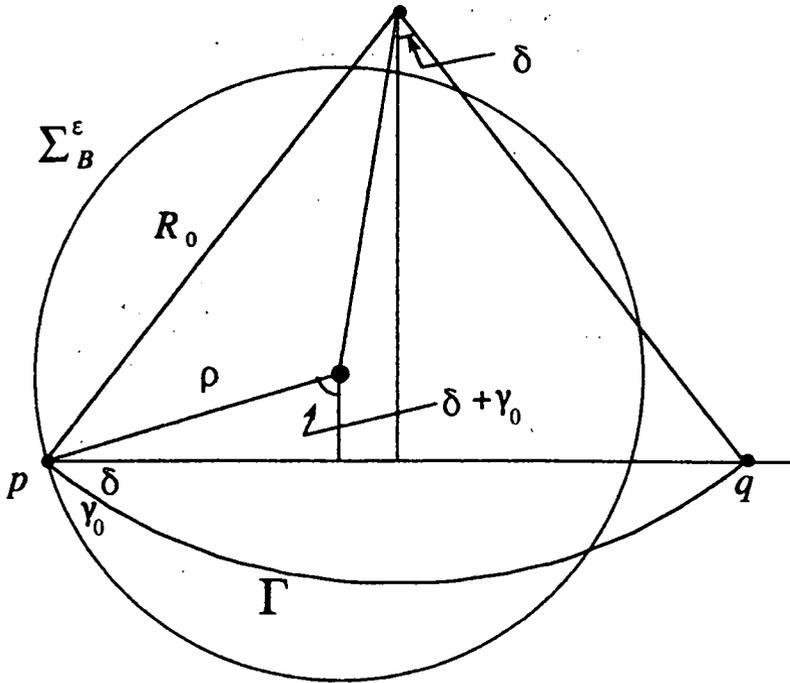


Figure 9. Case 8

There follows

$$\begin{aligned}
 R_0 \sin \delta &> \rho \sin(\gamma_0 + \delta) = \rho(\sin \gamma_0 \cos \delta + \cos \gamma_0 \sin \delta) \\
 R_0 \cos \delta &> \rho(\sin \gamma_0 \cot \delta \cos \gamma_0 + \cos^2 \gamma_0) \\
 &\geq \rho(\sin \gamma_0 \tan \gamma_0 \cos \gamma_0 + \cos^2 \gamma_0) \\
 &= \rho
 \end{aligned}
 \tag{14}$$

which contradicts Lemma 1. Therefore $\gamma_0 + \delta > \pi/2$. By Theorem 6.16 of [5], Γ could not minimize, a contradiction.

This exhausts all cases, and establishes that the functional Φ is positive for all extremals Γ . By the "Theorem" in [6] and also [8] and [9] (see also Lemmas 6.3, 6.4 in [6]) and taking into account the analogous discussion for the range $\gamma_0 > \pi/2$, we obtain

Theorem 1. *A solution of (1), (2) exists in a canonical proboscis domain Ω^* and is uniformly bounded in Ω^* , throughout the closed range $|\gamma - \frac{\pi}{2}| \leq |\gamma_0 - \frac{\pi}{2}|$, for any ϵ in the range $0 < \epsilon \leq \epsilon_M$.*

As γ crosses γ_0 , an extremal near P will meet Σ_p^* in the angle γ , forming an extremal subdomain Ω^* adjacent to P for which $\Phi \leq 0$. There is no longer a classical solution throughout Ω^* , but a solution continues to exist as a *soluzione illimitata* in the sense of Miranda [10], remaining smooth exterior to Ω^* but achieving the value $u = +\infty$ throughout Ω^* . Physically, the solution attempts to rise to infinity throughout Ω^* . If γ_0 is close to 0 or π and γ close to γ_0 , then $|\Omega^*|$ will be very small and the discontinuity difficult to observe. We note, however, that according to the construction, when γ has decreased to $\hat{\gamma}_0 = \gamma_0 - \epsilon$, then the arcs Γ will meet the curved part of Σ_p^* in the angle γ and will therefore become extremal for Ω^* . We assert:

Theorem 2. *In any domain Ω_r^* bounded between one of the arcs Γ and the vertex P , there holds $\Phi(\Omega_r^*; \gamma) < 0$.*

It follows that no smooth solution can exist in Ω_r^* for this angle, and it must be expected that physically the surface will attempt to rise to infinity throughout this subdomain. Thus at the angle $\gamma = \gamma_0 - \epsilon$ the singular behavior occurs throughout a domain that can be chosen to occupy as large a portion of Ω^* as desired, and it should therefore be possible to make it easily observable. This basic property of the modified canonical proboscis should facilitate a very accurate determination of γ_{cr} ; if the discontinuous behavior is not observed at $\gamma = \gamma_0$ but is observed at $\gamma = \gamma_0 - \epsilon$ then $\gamma_0 - \epsilon \leq \gamma_{cr} \leq \gamma_0$.

Proof of Theorem 2. Since the arcs Γ are extremal for the functional Φ on the curved part of Σ_p^* , there follows $\frac{\partial \Phi}{\partial x} = 0$ whenever Γ abuts on that part of Σ_p^* , thus Φ is constant on that set. At the vertex P , $\Phi = 0$ obviously. As one moves through the family of arcs Γ in the direction away from P , the angle γ decreases, since all arcs have the same radius. Thus, it suffices to show that on the rectilinear part of Σ_p^* there holds $\frac{\partial \Phi}{\partial y} > 0$. We calculate easily

$$\frac{\partial \Phi}{\partial y} = 2R \frac{\sin \gamma}{\sin \alpha} \cos(\alpha + \gamma) \tag{15}$$

which is positive since $\alpha + \gamma < \pi/2$ according to the construction. \square

Remark: Our construction has been effected so as to keep the theoretical critical contact angle unchanged from that of a given (initial) proboscis domain. From the point of view of practical construction of a modified proboscis, it would presumably be preferable to start with a proboscis domain corresponding to contact angle $\gamma_0 - \varepsilon$ and draw external tangents to Σ_p from a point P_ε on the x -axis, making angles $\alpha = \frac{\pi}{2} - \gamma_0$ with that axis. The construction is equivalent to what we have done, and the same experiment would again yield $\gamma_0 - \varepsilon \leq \widehat{\gamma}_{cr} \leq \gamma_0$.

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