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Blow-up and Polynomial Decay of Solutions for a Viscoelastic Equation with a Nonlinear Source

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Abstract. In this paper we investigate the blow-up and decay phenomenon of solutions for a viscoelastic equation with a nonlinear source. Even for vanishing initial energy, we show the solution blows up in finite time. We also prove the solution decays under suitable conditions.

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1. Introduction

In this paper, we consider the following viscoelastic equation

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau) d\tau = |u|^\gamma u, & (x,t) \in \mathbb{R}^n \times (0,\infty) \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $\gamma > 0$, u_0, u_1 are two compactly supported functions and g is a positive nonincreasing function defined on $\mathbb{R}^+$. A special case without $|u|^\gamma u$ was considered in [4], where it is shown that the energy of the solution decays exponentially and polynomially. There are many literatures regarding similar equations. For example, Messaoudi [3] and Tatar [6] considered

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau) d\tau + u_t|u_t|^{m-2} = |u|^{p-2}, & u, x \in \Omega, \ t > 0 \\ u(x,t)|_{\partial\Omega} = 0, & t \geq 0 \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), & x \in \Omega, \end{cases}$$

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where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Zhou [1, 7–10] showed the blow-up, global existence and nonexistence of solutions to related equations. Recently, the asymptotic behavior for the wave equation was discussed in [5].

The rest of this paper is organized as follows. In Section 2, we recall some preliminary results. Then, some blow-up criteria will be established in Section 3. In Section 4, we discuss the polynomial decay for this equation.

In this paper, we use $\|\cdot\|_p$ to denote the L^p-norm.

2. Preliminaries

First, we define the corresponding energy to problem (1) as

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(\tau)d\tau\right)\|\nabla u\|_2^2 + \frac{1}{2}(g\circ\nabla u)(t) - \frac{1}{\gamma+2}\|u\|_{\gamma+2}^{\gamma+2}, \quad (2)$$

here

$$(g\circ v)(t) = \int_0^t g(t-\tau)\|v(t)-v(\tau)\|_2^2 d\tau,$$
$$E'(t) = \frac{1}{2}(g'\circ\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u\|_2^2 \le 0. \quad (3)$$

Hence, we can deduce that $E(t) \le E(0)$.

Then, we denote:

(H1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable function such that

$$1 - \int_0^\infty g(\tau)d\tau = l > 0, \quad t \ge 0.$$

(H2) There exists $a > 0$ such that

$$g'(t) \le -ag(t), \quad t \ge 0. \quad (4)$$

Lemma 2.1. *If we assume that $\gamma < \frac{2}{n-2}$, there exists a positive constant $C > 1$ (throughout this paper, C denotes a generic positive constant, it may be different from line to line), such that*

$$\|u\|_{\gamma+2}^s \le C\left(\|\nabla u\|_2^2 + \|u\|_{\gamma+2}^{\gamma+2}\right), \quad (5)$$

with $2 \le s \le \gamma+2$, for any u being a solution to (1) on $[0,T)$. And consequently,

$$\|u\|_{\gamma+2}^s \le C\left(H(t) + \|u_t\|_2^2 + (g\circ\nabla u)(t) + \|\nabla u\|_2^2\right),$$

with $2 \le s \le \gamma+2$ on $[0,T)$ and here $H(t) := -E(t)$.

Proof. Since $\|u\|_{\gamma+2} \leq 1$, we have $\|u\|^s_{\gamma+2} \leq \|u\|^2_{\gamma+2} \leq B^2\|\nabla u\|_2^2$ is true. When $\|u\|_{\gamma+2} > 1$, we get $\|u\|^s_{\gamma+2} \leq \|u\|^{\gamma+2}_{\gamma+2}$. (5) follows from the definition of energy corresponding to the solution. □

The supremum of all T for which the solution exists on $[0,T) \times \mathbb{R}^n$ is called the lifespan of the solution of (1). The lifespan is denoted by T^*. If $T^* = \infty$, we say the solution is global, while it is nonglobal if $T^* < \infty$, and we say that the solution blows up in finite time.

3. Blow-up phenomenon

Before presenting our blow-up criteria, let us recall the lemma first:

Lemma 3.1 ([2]). *Suppose that $\psi(t)$ is a twice continuously differential function satisfying*

$$\begin{cases} \psi''(t) \geq C_0\psi^{1+\alpha(t)}, & t > 0,\ C_0 > 0,\ \alpha > 0, \\ \psi(0) > 0,\ \psi'(0) > 0. \end{cases}$$

Then $\psi(t)$ blows up in finite time. Moreover, the blow-up time can be estimated explicitly.

Just as in [9], the first main theorem in this section reads:

Theorem 3.2. *Assume that both of* (H1) *and* (H2) *hold, $0 < \gamma < \frac{2}{n-2}$, if $n > 2$; $0 < \gamma$, if $n = 1, 2$. Suppose $\int_0^\infty g(\tau)d\tau < \frac{2\gamma}{2\gamma+1}$. Then for any initial data $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ with compact support satisfying $E(0) \leq 0$, $\int_{\mathbb{R}^n} u_0u_1dx > 0$, the corresponding solution blows up in finite time.*

Proof. Defining $\psi(t) = \frac{1}{2}\int_{\mathbb{R}^n} |u(x,t)|^2dx$, choosing suitable $\delta > 0$ and differentiating twice, yields

$$\begin{aligned}
\psi''(t) &= \int_{\mathbb{R}^n} u_{tt}udx + \int_{\mathbb{R}^n} |u_t|^2dx \\
&= -\int_{\mathbb{R}^n} |\nabla u|^2dx + \int_{\mathbb{R}^n} \nabla u(t)\int_0^t g(t-\tau)\nabla u(\tau)d\tau dx \\
&\quad + \int_{\mathbb{R}^n} |u|^{\gamma+2}dx + \int_{\mathbb{R}^n} |u_t|^2dx \\
&\geq \left(-1-\delta+\int_0^t g(\tau)d\tau\right)\|\nabla u\|_2^2 - \frac{1}{4\delta}\left(\int_0^t g(\tau)d\tau\right)(g\circ\nabla u)(t) \\
&\quad + \int_{\mathbb{R}^n} |u|^{\gamma+2}dx + \int_{\mathbb{R}^n} |u_t|^2dx.
\end{aligned} \tag{6}$$

Here we use

$$\begin{aligned}
&\int_{\mathbb{R}^n} \nabla u(t) \int_0^t g(t-\tau)\nabla u(\tau) d\tau dx \\
&= -\int_{\mathbb{R}^n} \nabla u(t) \int_0^t g(t-\tau)(\nabla u(t) - \nabla u(\tau)) d\tau dx + \left(\int_0^t g(\tau)d\tau\right) \|\nabla u\|_2^2 \\
&\geq -\delta \|\nabla u\|_2^2 - \frac{1}{4\delta}\left(\int_0^t g(\tau)d\tau\right)(g \circ \nabla u)(t) + \left(\int_0^t g(\tau)d\tau\right) \|\nabla u\|_2^2.
\end{aligned}$$

Now, we exploit (2) to substitute for $\|\nabla u\|_2^2$, thus (6) takes the form

$$\begin{aligned}
\psi''(t) \geq & -2\frac{(1+\delta-\int_0^t g(\tau)d\tau)}{(1-\int_0^t g(\tau)d\tau)}E(t) + \left[\frac{(1+\delta-\int_0^t g(\tau)d\tau)}{(1-\int_0^t g(\tau)d\tau)} + 1\right]\|u_t\|_2^2 \\
& + \left[1 - \frac{(1+\delta-\int_0^t g(\tau)d\tau)}{(1-\int_0^t g(\tau)d\tau)} \cdot \frac{2}{\gamma+2}\right]\|u\|_{\gamma+2}^{\gamma+2} \\
& + \left[\frac{(1+\delta-\int_0^t g(\tau)d\tau)}{(1-\int_0^t g(\tau)d\tau)} - \frac{1}{4\delta}\left(\int_0^t g(\tau)d\tau\right)\right](g \circ \nabla u)(t).
\end{aligned} \tag{7}$$

If we choose $\delta > 0$ such that

$$\frac{(1+\delta-\int_0^t g(\tau)d\tau)}{(1-\int_0^t g(\tau)d\tau)} - \frac{1}{4\delta}\left(\int_0^t g(\tau)d\tau\right) \geq 0, \quad 1 - \frac{(1+\delta-\int_0^t g(\tau)d\tau)}{(1-\int_0^t g(\tau)d\tau)} \cdot \frac{2}{\gamma+2} > 0,$$

inequality (7) becomes into $\psi''(t) \geq \lambda \|u\|_{\gamma+2}^{\gamma+2}$.

Since $\operatorname{supp}\{u_0(x), u_1(x)\} \subset B(L)$, it follows that

$$\psi''(t) \geq \lambda 2^{\frac{\gamma+2}{2}} \psi^{\frac{\gamma+2}{2}}(t)(W_n)^{\frac{-\gamma}{2}}(t+L)^{\frac{-n\gamma}{2}},$$

where W_n is the volume of the unit ball. Then by Lemma 3.1, we see that the solution blows up in finite time. □

Theorem 3.3. *Assume that both of* (H1) *and* (H2) *hold;* $0 < \gamma < \frac{2}{n-2}$, *if* $n > 2$; $0 < \gamma$, *if* $n = 1, 2$. *Suppose* $\int_0^\infty g(\tau)d\tau < \frac{\frac{\gamma+2}{2}-1}{\frac{\gamma+2}{2}-1+\frac{1}{2(\gamma+2)}}$ *and* $E(0) < 0$, *then the solution blows up in finite time.*

Proof. By the definition

$$H(t) = -E(t) \quad \text{and} \quad H'(t) = -\frac{1}{2}(g' \circ \nabla u)(t) + \frac{1}{2}g(t)\|\nabla u\|_2^2 \geq 0,$$

we have $0 < H(0) \leq H(t) \leq \frac{1}{\gamma+2}\|u\|_{\gamma+2}^{\gamma+2}$. Moreover, we also define

$$L(t) = H^{1-\alpha}(t) + \epsilon \int_{\mathbb{R}^n} u u_t dx,$$

for ϵ small to be choose later and $0 < \alpha \leq \frac{\gamma}{2(\gamma+2)}$.

By differentiating the above equality and applying Young and Schwarz inequalities, we have

$$\begin{aligned}
L'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \epsilon\int_{\mathbb{R}^n}|u_t|^2dx + \epsilon\int_{\mathbb{R}^n}uu_{tt}dx,\\
&= (1-\alpha)H^{-\alpha}(t)\left(-\frac{1}{2}(g'\circ\nabla u)(t) + \frac{1}{2}g(t)\|\nabla u\|_2^2\right) + \epsilon\int_{\mathbb{R}^n}|u_t|^2dx\\
&\quad + \epsilon\left(-\int_{\mathbb{R}^n}|\nabla u|^2dx + \int_{\mathbb{R}^n}\nabla u(t)\int_0^t g(t-\tau)\nabla u(\tau)d\tau dx + \int_{\mathbb{R}^n}|u|^{\gamma+2}dx\right)\\
&\geq \epsilon\|u_t\|_2^2 - \epsilon\left(1-\int_0^t g(\tau)d\tau\right)\|\nabla u\|_2^2\\
&\quad + \epsilon\left[(\gamma+2)H(t) + \frac{\gamma+2}{2}\|u_t\|_2^2 + \frac{\gamma+2}{2}\left(1-\int_0^t g(\tau)d\tau\right)\|\nabla u\|_2^2\right.\\
&\quad \left.+\frac{\gamma+2}{2}(g\circ\nabla u)(t)\right] - \epsilon\delta(go\nabla u)(t) - \frac{\epsilon}{4\delta}\left(\int_0^t g(\tau)d\tau\right)\|\nabla u\|_2^2\\
&= \epsilon\left(1+\frac{\gamma+2}{2}\right)\|u_t\|_2^2 + \epsilon(\gamma+2)H(t) + \epsilon\left(\frac{\gamma+2}{2}-\delta\right)(g\circ\nabla u)(t)\\
&\quad + \epsilon\left[\left(\frac{\gamma+2}{2}-1\right) - \left(\frac{\gamma+2}{2}-1+\frac{1}{4\delta}\right)\left(\int_0^t g(\tau)d\tau\right)\right]\|\nabla u\|_2^2.
\end{aligned}$$

According to the hypothesis in Theorem 3.3 and choosing $0 < \delta < \frac{\gamma+2}{2}$, such that

$$\frac{\gamma+2}{2} - \delta > 0 \quad \text{and} \quad \left(\frac{\gamma+2}{2}-1\right) - \left(\frac{\gamma+2}{2}-1+\frac{1}{4\delta}\right)\int_0^t g(\tau)d\tau > 0,$$

we can deduce that

$$L'(t) \geq C[H(t) + \|u_t\|_2^2 + \|\nabla u\|_2^2 + (g\circ\nabla u)(t)].$$

Thanks to Hölder and Young inequalities, we obtain

$$\begin{aligned}
\left|\int_{\mathbb{R}^n}uu_tdx\right|^{\frac{1}{1-\alpha}} &\leq \|u\|_2^{\frac{1}{1-\alpha}}\|u_t\|_2^{\frac{1}{1-\alpha}}\\
&\leq C\|u\|_{\gamma+2}^{\frac{1}{1-\alpha}}\|u_t\|_2^{\frac{1}{1-\alpha}}\\
&\leq C\left(\|u\|_{\gamma+2}^s + \|u_t\|_2^2\right)\\
&\leq C\left(\|\nabla u\|_2^2 + \|u\|_{\gamma+2}^{\gamma+2} + \|u_t\|_2^2\right)\\
&\leq C\left(H(t) + \|u_t\|_2^2 + (g\circ\nabla u)(t) + \|\nabla u\|_2^2\right),
\end{aligned} \tag{8}$$

where $2 \leq s = \frac{2}{1-2\alpha} \leq \gamma+2$. Hence,

$$\begin{aligned} L^{\frac{1}{1-\alpha}}(t) &= \left(H^{1-\alpha}(t) + \epsilon \int_{\mathbb{R}^n} u u_t dx \right)^{\frac{1}{1-\alpha}} \\ &\leq 2^{\frac{1}{1-\alpha}} \left(H(t) + \left| \int_{\mathbb{R}^n} u u_t dx \right|^{\frac{1}{1-\alpha}} \right) \\ &\leq C \left(H(t) + \|u_t\|_2^2 + (g \circ \nabla u)(t) + \|\nabla u\|_2^2 \right), \end{aligned}$$

which implies that $L'(t) \geq \lambda L^{\frac{1}{1-\alpha}}(t)$, where λ is a constant depending on C and ϵ. Therefore $L(t) = \left(L^{\frac{-\alpha}{1-\alpha}}(0) + \frac{-\alpha}{1-\alpha}\lambda t \right)^{-\frac{1-\alpha}{\alpha}}$. So $L(t)$ goes to infinite as t tends to $\frac{1-\alpha}{\alpha\lambda L^{\frac{\alpha}{1-\alpha}}(0)}$. This completes the proof. □

Lemma 3.4. *Assume that both of* (H1) *and* (H2) *hold, additionally,*

$$\|u_0\|_{\gamma+2} > \lambda_0 \equiv B_0^{\frac{-2}{\gamma}} \quad \textit{and} \quad E(0) < E_0 = \left(\frac{1}{2} - \frac{1}{\gamma+2} \right) B_0^{\frac{-2(\gamma+2)}{\gamma}}.$$

Then

$$\|u\|_{\gamma+2} > \lambda_0 \quad \textit{and} \quad \|\nabla u\|_2 > B_0^{\frac{-(\gamma+2)}{\gamma}}, \quad \textit{for all } t \geq 0,$$

where $B_0 = \frac{B}{l^{\frac{1}{2}}}$ *for* $\|u\|_{\gamma+2} \leq B\|\nabla u\|_2$.

Proof. From (2) and the hypothesis, we know that

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(\tau)d\tau\right)\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{\gamma+2}\|u\|_{\gamma+2}^{\gamma+2} \\ &\geq \frac{1}{2}\left(1 - \int_0^t g(\tau)d\tau\right)\|\nabla u\|_2^2 - \frac{1}{\gamma+2}\|u\|_{\gamma+2}^{\gamma+2} \\ &\geq \frac{l}{2}\|\nabla u\|_2^2 - \frac{1}{\gamma+2}\|u\|_{\gamma+2}^{\gamma+2} \geq \frac{1}{2B_0^2}\|u\|_{\gamma+2}^2 - \frac{1}{\gamma+2}\|u\|_{\gamma+2}^{\gamma+2}. \end{aligned}$$

Set $h(\xi) = \frac{1}{2B_0^2}\xi^2 - \frac{1}{\gamma+2}\xi^{\gamma+2}$, $\xi \geq 0$ and $h(\xi)$ satisfies

$$\begin{cases} h(\xi) \text{ is strictly increasing on } [0, \lambda_0) \\ h(\xi) \text{ takes its maximum value } \left(\frac{1}{2} - \frac{1}{\gamma+2}\right) B_0^{\frac{-2(\gamma+2)}{\gamma}} \text{ at } \lambda_0 \\ h(\xi) \text{ is strictly decreasing on } (\lambda_0, \infty). \end{cases} \tag{9}$$

Since $E_0 > E(0) \geq E(t) \geq h(\|u\|_{\gamma+2})$ for all $t \geq 0$, there is no time t^* such that $\|u(\cdot, t^*)\|_{\gamma+2} = \lambda_0$. By the continuity of the $\|u(\cdot,t)\|_{\gamma+2}$-norm with respect to the time variable, one has $\|u(\cdot,t)\|_{\gamma+2} > \lambda_0 = B_0^{\frac{-2}{\gamma}}$ for all $t \geq 0$, and consequently,

$$\|\nabla u(\cdot,t)\|_2 \geq \frac{1}{l^{\frac{1}{2}}B_0}\|u(\cdot,t)\|_{\gamma+2} > \frac{1}{l^{\frac{1}{2}}}B_0^{\frac{-(\gamma+2)}{\gamma}} > B_0^{\frac{-(\gamma+2)}{\gamma}}.$$

This finishes the proof of Lemma 3.4 □

Theorem 3.5. *Assume that both of* (H1) *and* (H2) *hold,* $0<\gamma<\frac{2}{n-2}$*, if* $n>2$*;* $0<\gamma$*, if* $n=1,2$*. Suppose that* $\int_0^\infty g(\tau)d\tau<\frac{\frac{\gamma+2}{2}-1}{\frac{\gamma+2}{2}-1+\frac{1}{2(\gamma+2)}}$*,* $\|u_0\|_{\gamma+2}>\lambda_0$ *and* $E(0)\le E_0$*. Then the solution of* (1) *blows up in finite time.*

Proof. We set $G(t)=E_0+H(t)$, then $G'(t)=-\frac{1}{2}(g'\circ\nabla u)(t)+\frac{1}{2}g(t)\|\nabla u\|_2^2\ge 0$, from which we have

$$\begin{aligned}0<G(t)&=\left(\frac{1}{2}-\frac{1}{\gamma+2}\right)B_0^{\frac{-2(\gamma+2)}{\gamma}}+H(t)\\&<\left(\frac{1}{2}-\frac{1}{\gamma+2}\right)\|\nabla u\|_2^2+H(t)\\&<C(\|\nabla u\|_2^2+H(t)).\end{aligned}$$

Let

$$F(t)=G^{1-\alpha}(t)+\epsilon\int_{\mathbb{R}^n}uu_tdx,$$

then by direct computing, one can get

$$\begin{aligned}F'(t)&=(1-\alpha)G^{-\alpha}(t)G'(t)+\epsilon\int_{\mathbb{R}^n}|u_t|^2dx+\epsilon\int_{\mathbb{R}^n}uu_{tt}dx,\\&=(1-\alpha)G^{-\alpha}(t)\left(-\frac{1}{2}(g'\circ\nabla u)(t)+\frac{1}{2}g(t)\|\nabla u\|_2^2\right)+\epsilon\int_{\mathbb{R}^n}|u_t|^2dx\\&\quad+\epsilon\left(-\int_{\mathbb{R}^n}|\nabla u|^2dx+\int_{\mathbb{R}^n}\nabla u(t)\int_0^t g(t-\tau)\nabla u(\tau)d\tau+\int_{\mathbb{R}^n}|u|^{\gamma+2}dx\right),\\&\ge\epsilon\|u_t\|_2^2-\epsilon\left(1-\int_0^t g(\tau)d\tau\right)\|\nabla u\|_2^2\\&\quad+\epsilon\left[(\gamma+2)H(t)+\frac{\gamma+2}{2}\|u_t\|_2^2+\frac{\gamma+2}{2}\left(1-\int_0^t g(\tau)d\tau\right)\|\nabla u\|_2^2\right.\\&\quad\left.+\frac{\gamma+2}{2}(g\circ\nabla u)(t)\right]-\epsilon\delta(g\circ\nabla u)(t)-\frac{\epsilon}{4\delta}\left(\int_0^t g(\tau)d\tau\right)\|\nabla u\|_2^2\\&=\epsilon\left(1+\frac{\gamma+2}{2}\right)\|u_t\|_2^2+\epsilon(\gamma+2)H(t)+\epsilon\left(\frac{\gamma+2}{2}-\delta\right)(g\circ\nabla u)(t)\\&\quad+\epsilon\left[\left(\frac{\gamma+2}{2}-1\right)-\left(\frac{\gamma+2}{2}-1+\frac{1}{4\delta}\right)\left(\int_0^t g(\tau)d\tau\right)\right]\|\nabla u\|_2^2.\end{aligned}$$

Using the hypothesis in this theorem and choosing $0<\delta<\frac{\gamma+2}{2}$, such that

$$\frac{\gamma+2}{2}-\delta>0,\quad\left(\frac{\gamma+2}{2}-1\right)-\left(\frac{\gamma+2}{2}-1+\frac{1}{4\delta}\right)\int_0^t g(\tau)d\tau>0,$$

it follows that

$$F'(t)\ge C[H(t)+\|u_t\|_2^2+\|\nabla u\|_2^2+(g\circ\nabla u)(t)].$$

In view of (8) we obtain

$$F^{\frac{1}{1-\alpha}}(t) = \left(G^{1-\alpha}(t) + \epsilon \int_{\mathbb{R}^n} uu_t dx\right)^{\frac{1}{1-\alpha}} \\ \leq 2^{\frac{1}{1-\alpha}} \left(G(t) + \left|\int_{\mathbb{R}^n} uu_t dx\right|^{\frac{1}{1-\alpha}}\right) \\ \leq C\left(H(t) + \|u_t\|_2^2 + (g \circ \nabla u)(t) + \|\nabla u\|_2^2\right).$$

Therefore,

$$F'(t) \geq \lambda F^{\frac{1}{1-\alpha}}(t), \tag{10}$$

where λ is a constant depending on C and ϵ. A simple integration of (10) over $(0,t)$ yields $F(t) = \left(F^{\frac{-\alpha}{1-\alpha}}(0) + \frac{-\alpha}{1-\alpha}\lambda t\right)^{-\frac{1-\alpha}{\alpha}}$, which shows that $F(t)$ blows up in time $T^* \leq \frac{1-\alpha}{\alpha\lambda F^{\frac{\alpha}{1-\alpha}}(0)}$. □

4. Polynomial decay

For extensive studies on decay rate for the wave equations given by Zhou [8], we establish the decay rate for a solution with positive initial energy, let us consider the following four lemmas first.

Lemma 4.1. *Assume that both of* (H1) *and* (H2) *hold. Suppose* $u(x,t)$ *is the solution of* (1) *and let*

$$\Phi_1(t) := (1+t)^{-1} \int_{\mathbb{R}^n} \int_0^t G(t-\tau)|u(t) - u(\tau)|^2 d\tau dx,$$

where $G(t) := e^{-\alpha t} \int_t^\infty e^{\alpha\tau}(-g'(\tau))d\tau$, *then for any* $\delta_1 > 0$, *the following inequality is true:*

$$\frac{d}{dt}[\Phi_1(t)] \leq -(1+t)^{-1}\left\{\left[1+(1+t)\left(\alpha - \frac{\overline{G}}{\delta_1}\right)\right]\Phi_1(t) - (g' \circ u)(t) - \delta_1\|u_t\|_2^2\right\}, \tag{11}$$

here $\overline{G} = \int_0^\infty G(t)dt$.

Proof. Thanks to (H2), we know that for any $\alpha < a$,

$$0 \leq \overline{G} = \int_0^\infty G(t)dt \leq \left(\frac{a}{a-\alpha}\right) \int_0^\infty g(t)dt < \infty.$$

By differentiating $\Phi_1(t)$, we have

$$\begin{aligned}\frac{d}{dt}[\Phi_1(t)] &= -(1+t)^{-2}\int_{\mathbb{R}^n}\int_0^t G(t-\tau)|u(t)-u(\tau)|^2 d\tau dx\\ &\quad + (1+t)^{-1}\int_{\mathbb{R}^n}\int_0^t G'(t-\tau)|u(t)-u(\tau)|^2 d\tau dx\\ &\quad + 2(1+t)^{-1}\int_{\mathbb{R}^n}\int_0^t G(t-\tau)(u(t)-u(\tau))u_t(t) d\tau dx\\ &= (1+t)^{-1}\left[-\Phi_1(t) + 2\int_{\mathbb{R}^n}\int_0^t G(t-\tau)(u(t)-u(\tau))u_t(t) d\tau dx\right]\\ &\quad + (1+t)^{-1}\int_{\mathbb{R}^n}\int_0^t \left[-\alpha G(t-\tau) + g'(t-\tau)\right]|u(t)-u(\tau)|^2 d\tau dx\\ &= -(1+t)^{-1}\Phi_1(t) - \alpha\Phi_1(t)\\ &\quad + (1+t)^{-1}\left[(g'\circ u)(t) + 2\int_{\mathbb{R}^n} u_t \int_0^t G(t-\tau)(u(t)-u(\tau)) d\tau dx\right].\end{aligned}$$

In view of Young and Schwarz inequalities, it follows that

$$\begin{aligned}&\int_{\mathbb{R}^n} u_t \int_0^t G(t-\tau)(u(t)-u(\tau)) d\tau dx\\ &\le \frac{\delta_1}{2}\|u_t\|_2^2 + \frac{1}{2\delta_1}\int_{\mathbb{R}^n}\left|\int_0^t G(t-\tau)(u(t)-u(\tau)) d\tau\right|^2 dx\\ &\le \frac{\delta_1}{2}\|u_t\|_2^2 + \frac{1}{2\delta_1}\int_0^t G(\tau) d\tau \int_{\mathbb{R}^n}\int_0^t G(t-\tau)(u(t)-u(\tau))^2 d\tau dx,\end{aligned}$$

which implies that

$$\frac{d}{dt}[\Phi_1(t)] \le -(1+t)^{-1}\left\{\left[1 + (1+t)\left(\alpha - \frac{\overline{G}}{\delta_1}\right)\right]\Phi_1(t) - (g'\circ u)(t) - \delta_1\|u_t\|_2^2\right\}.$$

Therefore, this completes the proof of this lemma. □

Lemma 4.2. *Assume that both of* (H1) *and* (H2) *hold. Suppose* $u(x,t)$ *is the solution of* (1) *and let*

$$\Phi_2(t) := (1+t)^{-1}\int_{\mathbb{R}^n} u u_t dx,$$

then for any $\delta_2 > 0$,

$$\begin{aligned}\frac{d}{dt}[\Phi_2(t)] &\le (1+t)^{-1}\left(1 + \frac{C}{4\delta_2}\right)\|u_t\|_2^2 - (1+t)^{-1}(l - C\delta_2)\|\nabla u\|_2^2\\ &\quad + \frac{\overline{g}}{4\delta_2}(g\circ\nabla u)(t) + \|u\|^{\gamma+2}\end{aligned} \tag{12}$$

is true.

Proof. Since $\text{supp}\{u_0(x), u_1(x)\} \subset B(L)$, we can get $\|u_t\|_2 \leq C(L+t)\|\nabla u\|_2$, which tells us that

$$\int_{\mathbb{R}^n} uu_t dx \leq C\|u\|_2\|u_t\|_2 \leq C(1+t)\|\nabla u\|_2\|u_t\|_2 \leq C(1+t)\left(\delta_2\|\nabla u\|_2^2 + \frac{1}{4\delta_2}\|u_t\|_2^2\right).$$

By direct computation, we have

$$\begin{aligned}
\frac{d}{dt}[\Phi_2(t)] &= (1+t)^{-2}\int_{\mathbb{R}^n} uu_t dx + (1+t)^{-1}\left[\int_{\mathbb{R}^n} |u_t|^2 dx + \int_{\mathbb{R}^n} uu_{tt} dx\right] \\
&= (1+t)^{-2}\int_{\mathbb{R}^n} uu_t dx + (1+t)^{-1}\int_{\mathbb{R}^n} |u_t|^2 dx \\
&\quad + (1+t)^{-1}\left(-\int_{\mathbb{R}^n} |\nabla u|^2 dx + \int_{\mathbb{R}^n} \nabla u(t)\int_0^t g(t-\tau)\nabla u(\tau) d\tau dx \right.\\
&\quad \left. + \int_{\mathbb{R}^n} |u|^{\gamma+2} dx\right) \\
&\leq (1+t)^{-1}\left(1 + \frac{C}{4\delta_2}\right)\|u_t\|_2^2 - (1+t)^{-1}(l - C\delta_2)\|\nabla u\|_2^2 \\
&\quad + \frac{\overline{g}}{4\delta_2}(g \circ \nabla u)(t) + \|u\|_{\gamma+2}^{\gamma+2}.
\end{aligned}$$

Here we use the fact that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \nabla u(t)\int_0^t g(t-\tau)\nabla u(\tau) d\tau dx \\
&\leq \delta_2\|\nabla u\|_2^2 + \frac{1}{4\delta_2}\int_{\mathbb{R}^n}\left|\int_0^t g(t-\tau)(\nabla u(\tau) - \nabla u(t) d\tau\right|^2 dx + \overline{g}\|\nabla u\|_2^2 \\
&\leq \delta_2\|\nabla u\|_2^2 + \frac{\overline{g}}{4\delta_2}(g \circ \nabla u)(t) + \overline{g}\|\nabla u\|_2^2.
\end{aligned}$$

Thus (12) is established and this lemma holds. □

Lemma 4.3. *Assume that both of* (H1) *and* (H2) *hold, and* $u(x,t)$ *is the solution of* (1)*. If we define*

$$\Phi_3(t) := -(1+t)^{-1}\int_{\mathbb{R}^n} u_t \int_0^t g(t-\tau)(u(t) - u(\tau))^2 d\tau dx,$$

then for any $\delta_2, \delta_3, \delta_4 > 0$*, there exists that*

$$\begin{aligned}
\frac{d}{dt}[\Phi_3(t)] \leq &-(1+t)^{-1}\left(-C\delta_2 + \int_0^t g(\tau)d\tau\right)\|u_t\|_2^2 + 2(1+t)^{-1}\delta_3\|\nabla u\|_2^2 \\
&-(1+t)^{-1}\frac{1}{4\delta_2}(g' \circ u)(t) + (1+t)^{-1}\left(\frac{C}{4\delta_2} + \frac{1}{2\delta_3} + 1\right)\overline{g}(g \circ \nabla u)(t) \\
&+(1+t)^{-1}\delta_4\|u\|_{2(\gamma+1)}^{2(\gamma+1)} + (1+t)^{-1}\frac{1}{4\delta_4}(g \circ u)(t).
\end{aligned} \tag{13}$$

Proof. Similarly, differentiating $\Phi_3(t)$ as before can lead to

$$\begin{aligned}\frac{d}{dt}[\Phi_3(t)] = &(1+t)^{-2}\int_{\mathbb{R}^n} u_t \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx \\ &-(1+t)^{-1}\left[\int_{\mathbb{R}^n} u_{tt}\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx\right. \\ &\left.+\int_{\mathbb{R}^n} u_t \int_0^t g'(t-\tau)(u(t)-u(\tau))d\tau dx + \left(\int_0^t g(\tau)d\tau\right)\|u_t\|_2^2\right].\end{aligned} \tag{14}$$

Combining with (1), we have

$$\begin{aligned}&\int_{\mathbb{R}^n} u_{tt}\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx \\ &=-\int_{\mathbb{R}^n} \nabla u \int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))d\tau dx \\ &\quad+\int_{\mathbb{R}^n}\left[\int_0^t g(t-\tau)\nabla u(\tau)d\tau \int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))d\tau\right]dx \\ &\quad+\int_{\mathbb{R}^n} |u|^\gamma u \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx.\end{aligned} \tag{15}$$

Using Young and Schwarz inequalities again, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n} u_t \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx \\ &\le \delta_2\|u_t\|_2^2 + \frac{1}{4\delta_2}\int_{\mathbb{R}^n}\left|\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau\right|^2 dx \\ &\le C(1+t)\left(\delta_2\|u_t\|_2^2+\frac{\overline{g}}{4\delta_2}(g\circ\nabla u)(t)\right).\end{aligned} \tag{16}$$

Applying the same method, it follows that

$$\begin{cases}\displaystyle\int_{\mathbb{R}^n}\nabla u\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))d\tau dx \le \delta_3\|\nabla u\|_2^2+\frac{\overline{g}}{4\delta_3}(g\circ\nabla u)(t); \\ \displaystyle\int_{\mathbb{R}^n} u_t\int_0^t g'(t-\tau)(u(t)-u(\tau))d\tau dx \le \delta_2\|u_t\|_2^2-\frac{1}{4\delta_2}(g'\circ u)(t); \\ \displaystyle-\int_{\mathbb{R}^n}\left[\int_0^t g(t-\tau)\nabla u(\tau)d\tau\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))d\tau\right]dx \\ \displaystyle\le \overline{g}\delta_3\|\nabla u\|_2^2+\left(1+\frac{1}{4\delta_3}\right)\overline{g}(g\circ\nabla u)(t); \\ \displaystyle\int_{\mathbb{R}^n}|u|^\gamma u\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx \le \delta_4\|u\|_{2(\gamma+1)}^{2(\gamma+1)}+\frac{\overline{g}}{4\delta_4}(g\circ u)(t).\end{cases} \tag{17}$$

Combining (14)–(17) and using $\overline{g} < 1$, we get

$$\begin{aligned}\frac{d}{dt}[\Phi_3(t)] \leq& -(1+t)^{-1}\left(-C\delta_2 + \int_0^t g(\tau)d\tau\right)\|u_t\|_2^2 + 2(1+t)^{-1}\delta_3\|\nabla u\|_2^2\\ &- (1+t)^{-1}\frac{1}{4\delta_2}(g'\circ u)(t) + (1+t)^{-1}\left(\frac{C}{4\delta_2}+\frac{1}{2\delta_3}+1\right)\overline{g}(g\circ\nabla u)(t)\\ &+ (1+t)^{-1}\delta_4\|u\|_{2(\gamma+1)}^{2(\gamma+1)} + (1+t)^{-1}\frac{1}{4\delta_4}(g\circ u)(t).\end{aligned}$$

Thus Lemma 4.3 is proved. □

Now, the last lemma is presented as

Lemma 4.4. *Assume that both* (H1) *and* (H2) *hold. Let*

$$F(t) := E(t) + \sum_{i=1}^{3}\alpha_i\Phi_i(t), \quad t \geq 0, \tag{18}$$

then

$$\xi_1 E(t) \leq F(t) \leq \xi_2[E(t) + \Phi_1(t)], \tag{19}$$

provided any positive constants ξ_1, ξ_2 *are small enough.*

Proof. We can compute directly that

$$\begin{aligned}F(t) =& \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1-\int_0^t g(\tau)d\tau\right)\|\nabla u\|_2^2 + \frac{1}{2}(g\circ\nabla u)(t)\\ &- \frac{1}{\gamma+2}\|u\|_{\gamma+2}^{\gamma+2} + \alpha_1\Phi_1(t) + \alpha_2(1+t)^{-1}\int_{\mathbb{R}^n} uu_t dx\\ &- \alpha_3(1+t)^{-1}\int_{\mathbb{R}^n} u_t\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx.\end{aligned} \tag{20}$$

According to Schwartz and Young inequalities, we get

$$\int_{\mathbb{R}^n} uu_t dx \leq \frac{1}{4\delta_5}\|u\|_2^2 + \delta_5\|u_t\|_2^2 \leq C(1+t)\left(\frac{1}{4}\|\nabla u\|_2^2 + \|u_t\|_2^2\right),$$

and

$$\begin{aligned}\int_{\mathbb{R}^n} u_t\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx &\leq \delta_2\|u_t\|_2^2 + \frac{\overline{g}}{4\delta_2}(g\circ u)(t)\\ &\leq C(1+t)\left(\frac{\overline{g}}{4}(g\circ\nabla u)(t) + \|u_t\|_2^2\right).\end{aligned}$$

Therefore, (20) becomes

$$\begin{aligned} F(t) &\le \alpha_1\Phi_1(t) + \left(\frac{1}{2} + C(\alpha_2+\alpha_3)\right)\|u_t\|_2^2 + \frac{1}{2}\left(1 + \frac{\alpha_3 C\overline{g}}{2}\right)(g\circ\nabla u)(t) \\ &\quad + \frac{1}{2}\left[\left(1 - \int_0^t g(\tau)d\tau\right) + \frac{\alpha_2 C}{2}\right]\|\nabla u\|_2^2 - \frac{1}{\gamma+2}\|u\|_{\gamma+2}^{\gamma+2} \\ &\le \xi_2[E(t) + \Phi_1(t)]. \end{aligned} \tag{21}$$

On the other hand, we have

$$\begin{aligned} F(t) &\ge \left(\frac{1}{2} + C(\alpha_2+\alpha_3)\right)\|u_t\|_2^2 + \frac{1}{2}\left(l - \frac{\alpha_2 C}{2}\right)\|\nabla u\|_2^2 \\ &\quad + \frac{1}{2}\left(1 - \frac{\alpha_3 C\overline{g}}{2}\right)(g\circ\nabla u)(t) - \frac{1}{\gamma+2}\|u\|_{\gamma+2}^{\gamma+2} \\ &\ge \xi_1 E(t). \end{aligned} \tag{22}$$

Combining (21) and (22), this completes the proof. □

Now, our result is

Theorem 4.5. *Let both of* (H1) *and* (H2) *hold; and* $0 < \gamma < \frac{2}{n-2}$, *if* $n > 2$; $0 < \gamma$, *if* $n = 1, 2$. *If the initial data satisfy* $\|u_0\|_{\gamma+2} < \lambda_0 \equiv B_0^{\frac{-2}{\gamma}}$ *and* $E(0) < E_0 = \left(\frac{1}{2} - \frac{1}{\gamma+2}\right)B_0^{\frac{-2(\gamma+2)}{\gamma}}$ *there exist two positive constants* K *and* k *such that*

$$E(t) \le K(1+t)^{-k}.$$

Proof. Direct differentiation of (18), yields

$$\begin{aligned} F'(t) &= E'(t) + \sum_{i=1}^{3}\alpha_i\Phi_i'(t) \\ &\le \frac{1}{2}(g'\circ\nabla u)(t) + \sum_{i=1}^{3}\alpha_i\Phi_i'(t) \\ &\le -\frac{a}{2}(g\circ\nabla u)(t) + \sum_{i=1}^{3}\alpha_i\Phi_i'(t). \end{aligned} \tag{23}$$

Because of (H1), for any $t \ge t_0 > 0$, we have

$$\int_0^t g(\tau)d\tau \ge \int_0^{t_0} g(\tau)d\tau = g_0 > 0.$$

Inserting (11)–(13) into (23), one gets

$$
\begin{aligned}
F'(t) \le & -(1+t)^{-1}\left[1+(1+t)\left(\alpha-\frac{\overline{G}}{\delta_1}\right)\right]\alpha_1\Phi_1(t) \\
& -(1+t)^{-1}\left(\frac{\alpha_3}{4\delta_2}-\alpha_1\right)(g'\circ u)(t) \\
& -(1+t)^{-1}\left\{\frac{a}{2}-\overline{g}\left[\frac{\alpha_2}{4\delta_2}+\alpha_3\left(\frac{C}{4\delta_2}+\frac{1}{2\delta_3}+1\right)\right]\right\}(g\circ\nabla u)(t) \\
& -(1+t)^{-1}\left[\alpha_2(l-C\delta_2)-2\delta_3\alpha_3\right]\|\nabla u\|_2^2 \\
& -(1+t)^{-1}\left[\alpha_3(g_0-C\delta_2)-\alpha_2\left(1+\frac{C}{4\delta_2}\right)-\alpha_1\delta_1\right]\|u_t\|_2^2 \\
& +(1+t)^{-1}\alpha_2\|u\|_{\gamma+2}^{\gamma+2}+(1+t)^{-1}\alpha_3\delta_4\|u\|_{2(\gamma+1)}^{2(\gamma+1)}+(1+t)^{-1}\frac{\alpha_3}{4\delta_4}(g\circ u)(t).
\end{aligned}
$$

From (3), $E'(t)\le 0$, it follows that

$$
E(t)\le E(0)<E_0=\left(\frac{1}{2}-\frac{1}{\gamma+2}\right)B_0^{\frac{-2(\gamma+2)}{\gamma}}. \tag{24}
$$

We claim that $\|u\|_{\gamma+2}<\lambda_0$, for all $t\ge 0$.

Suppose not, thanks to the continuity of $\|u(\cdot,t)\|_{\gamma+2}$-norm, then there exists a t_0 such that $\|u(\cdot,t_0)\|_{\gamma+2}=\lambda_0$. But from (9), we have

$$
\begin{aligned}
E(t_0) &= \frac{1}{2}\|u_t(\cdot,t_0)\|_2^2+\frac{1}{2}\left(1-\int_0^t g(\tau)d\tau\right)\|\nabla u(\cdot,t_0)\|_2^2 \\
&\quad +\frac{1}{2}(g\circ\nabla u)(t_0)-\frac{1}{\gamma+2}\|u(\cdot,t_0)\|_{\gamma+2}^{\gamma+2} \\
&\ge \frac{1}{2}\left(1-\int_0^t g(\tau)d\tau\right)\|\nabla u(\cdot,t_0)\|_2^2-\frac{1}{\gamma+2}\|u(\cdot,t_0)\|_{\gamma+2}^{\gamma+2} \\
&\ge \left(\frac{1}{2}-\frac{1}{\gamma+2}\right)B_0^{\frac{-2(\gamma+2)}{\gamma}}=E_0,
\end{aligned}
$$

which contradicts (24). On the other hand, for all $t\ge 0$,

$$
\begin{aligned}
\|\nabla u\|_2^2 &= \frac{1}{1-\int_0^t g(\tau)d\tau}\left(2E(t)-(g\circ\nabla u)(t)+\frac{2}{\gamma+2}\|u\|_{\gamma+2}^{\gamma+2}-\|u_t\|_2^2\right) \\
&< \frac{1}{l}\left(2E(t)+\frac{2}{\gamma+2}\|u\|_{\gamma+2}^{\gamma+2}\right) \\
&< \frac{1}{l}\left[\left(1-\frac{2}{\gamma+2}\right)B_0^{\frac{-2(\gamma+2)}{\gamma}}+\frac{2}{\gamma+2}B_0^{\frac{-2(\gamma+2)}{\gamma}}\right]=\frac{1}{l}B_0^{\frac{-2(\gamma+2)}{\gamma}},
\end{aligned}
$$

and

$$\|u\|_{2(\gamma+1)}^{2(\gamma+1)} \le C\|\nabla u\|_2^{2\gamma}\|\nabla u\|_2^2 < B_0^{2(\gamma+1)}\frac{B_0^{-2(\gamma+2)}}{l^\gamma}\|\nabla u\|_2^2 = \frac{C}{l^\gamma}B_0^{-2(\gamma+2)}\|\nabla u\|_2^2.$$

Next, we get

$$\begin{aligned} F'(t) \le &-(1+t)^{-1}\left[1+(1+t)\left(\alpha-\frac{\overline{G}}{\delta_1}\right)\right]\alpha_1\Phi_1(t) \\ &-(1+t)^{-1}\left[a\left(\alpha_1-\frac{\alpha_3}{4\delta_2}\right)-\frac{\alpha_3}{4\delta_4}\right](g\circ u)(t) \\ &-(1+t)^{-1}\left\{\frac{a}{2}-\overline{g}\left[\frac{\alpha_2}{4\delta_2}+\alpha_3\left(\frac{C}{4\delta_2}+\frac{1}{2\delta_3}+1\right)\right]\right\}(g\circ\nabla u)(t) \qquad (25)\\ &-(1+t)^{-1}\left[\alpha_2(l-C\delta_2)-2\delta_3\alpha_3-\frac{C}{l^\gamma}B_0^{-2(\gamma+2)}\alpha_3\delta_4\right]\|\nabla u\|_2^2 \\ &-(1+t)^{-1}\left[\alpha_3(g_0-C\delta_2)-\alpha_2\left(1+\frac{C}{4\delta_2}\right)-\alpha_1\delta_1\right]\|u_t\|_2^2. \end{aligned}$$

Now we choose suitable δ_2, δ_3, δ_4, such that

$$\delta_2 < \min\left\{\frac{g_0}{C},\frac{l}{C}\right\} \quad \text{and} \quad \frac{\alpha_3\left(2\delta_3+\frac{CB_0^{-2(\gamma+2)}\delta_4}{l^\gamma}\right)}{l-C\delta_2} < \alpha_2 < \frac{\alpha_3(g_0-C\delta_2)}{1+\frac{C}{4\delta_2}},$$

which implies that

$$\alpha_2(l-C\delta_2)-2\delta_3\alpha_3-\frac{C}{l^\gamma}B_0^{-2(\gamma+2)}\alpha_3\delta_4 > 0, \quad \alpha_3(g_0-C\delta_2)-\alpha_2\left(1+\frac{C}{4\delta_2}\right) = k_1 > 0.$$

Moreover, let α_2 and α_3 be small enough,

$$\frac{a}{2}-\overline{g}\left[\frac{\alpha_2}{4\delta_2}+\alpha_3\left(\frac{C}{4\delta_2}+\frac{1}{2\delta_3}+1\right)\right] > 0.$$

Then, we choose α_1 large enough, i.e., $\alpha_1-\frac{\alpha_3}{4\delta_2}>0$ and $a\left(\alpha_1-\frac{\alpha_3}{4\delta_2}\right)-\frac{\alpha_3}{4\delta_4}>0$, and δ_1 small enough so that $k_1-\alpha_1\delta_1>0$.

Therefore, if a in (4) is large enough so that $a>\alpha>\frac{1}{\delta_1}\overline{G}$, consequently (25) becomes

$$F'(t) \le -C(1+t)^{-1}[E(t)+\Phi_1(t)] \le \frac{-C}{\xi_2}(1+t)^{-1}F(t), \qquad (26)$$

for all $t \ge t_0$. Integrating (26) over (t_0,t), yields

$$F(t) \le \frac{F(t_0)(1+t_0)^{\frac{C}{\xi_2}}}{(1+t)^{\frac{C}{\xi_2}}}.$$

Due to (19), we finish the proof. □

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