Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 31 (2012), 267–282
DOI: 10.4171/ZAA/1459

Diffusion Phenomenon for Linear Dissipative Wave Equations

Belkacem Said-Houari

Abstract. In this paper we prove the diffusion phenomenon for the linear wave equation. To derive the diffusion phenomenon, a new method is used. In fact, for initial data in some weighted spaces, we prove that for $2 \le p \le \infty$, $\|u-v\|_{L^p(\mathbb{R}^N)}$ decays with the rate $t^{-\frac{N}{2}(1-\frac{1}{p})-1-\frac{\gamma}{2}}$, $\gamma \in [0,1]$ faster than that of either u or v, where u is the solution of the linear wave equation with initial data $(u_0,u_1) \in \big(H^1(\mathbb{R}^N)\cap L^{1,\gamma}(\mathbb{R}^N)\big) \times \big(L^2(\mathbb{R}^N)\cap L^{1,\gamma}(\mathbb{R}^N)\big)$ with $\gamma \in [0,1]$, and v is the solution of the related heat equation with initial data $v_0 = u_0 + u_1$. This result improves the result in H. Yang and A. Milani [Bull. Sci. Math. 124 (2000), 415 − 433] in the sense that, under the above restriction on the initial data, the decay rate given in that paper can be improved by $t^{-\frac{\gamma}{2}}$.

Keywords. Diffusion phenomenon, Cauchy problem, damped wave equation, heat equation, asymptotic behavior
Mathematics Subject Classification (2010). Primary 35L60, 37L15, secondary 35L70, 35A05, 35B40

1. Introduction

In this paper we study the diffusion phenomenon of the following Cauchy problem with linear damping

$$\begin{cases} u_{tt}(t,x) - \Delta u(t,x) + u_t(t,x) = 0, & x \in \mathbb{R}^N,\ t \ge 0 \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^N. \end{cases} \tag{1}$$

Our main goal is to show that the asymptotic profile of the solution $u(t,x)$ of (1) is given by the solution $v(t,x)$ of the following parabolic problem

$$\begin{cases} v_t(t,x) - \Delta v(t,x) = 0, & x \in \mathbb{R}^N,\ t \ge 0 \\ v(0,x) = u_0(x) + u_1(x), & x \in \mathbb{R}^N. \end{cases} \tag{2}$$

B. Said-Houari: Division of Mathematical and Computer Sciences and Engineering, King Abdullah University of Science and Technology (KAUST), Thuwal, Saudi Arabia; belkacem.saidhouari@kaust.edu.sa

Let us define the rescaling function $u_\lambda(x,t) = \lambda u(\lambda^2 t, \lambda x)$, then from (1), the function $u_\lambda(x,t)$ satisfies

$$\lambda^{-2} u_{\lambda,tt}(t,x) - \Delta u_\lambda(t,x) + u_{\lambda,t}(t,x) = 0, \quad x \in \mathbb{R}^N,\ t \geq 0.$$

The above formula indicates that, the term involving second order in time derivatives becomes negligible. According to this formal argument it is natural to expect the asymptotic behavior of the solutions of problem (1) to be the same as that of the problem (2).

Up to our knowledge, the early result dealing with the diffusion phenomenon is the one given by Hsiao and Liu [2] where the authors studied the p-system with linear damping (which can be seen as a damped wave equation). Namely, they considered the problem

$$\begin{cases} v_t - u_x = 0 \\ u_t + p(v)_x = -\alpha u, \end{cases} \tag{3}$$

with initial data

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x)$$

satisfying $(u,v)(0,x) = (u_0(x), v_0(x)) \to (u_\pm, v_\pm)$ as $x \to \pm\infty$. We recall that in system (3) $v = v(t,x)$ and $u = u(t,x)$ represent the specific volume and velocity, respectively, whereas the function p represents the pressure where is assumed to be a smooth function of v such that $p(v) > 0$, $p'(v) < 0$ and α is a positive constant. System (3) can be viewed as the isentropic Euler equations in lagrangian coordinates with the frictional term $-\alpha u$ in the momentum equation. Thus it models the compressible flow through porous media. Also system (3) can be used to model the motion of one-dimensional elastic continua interacting with media exerting frictional forces with deformation gradient strain v and stress $-p$. The damping term $-\alpha u$ may prevent shock formation, at least for smooth and small initial data.

The authors in [2] showed that for $v_+ \neq v_-$, the solutions of (3) time asymptotically behave like those governed by Darcy's law

$$\begin{cases} \overline{v}_t - \overline{u}_x = 0 \\ p(\overline{v})_x = -\alpha \overline{u}, \end{cases}$$

and the convergence rates

$$\|(v - \overline{v}, u - \overline{u})\| = O(t^{-\frac{1}{2}}, t^{-\frac{1}{2}})$$

have been shown. System (3) was also considered by Nishihara in [10] and he succeeded in proving the following decay rates of convergence

$$\|(v - \overline{v}, u - \overline{u})\|_{L^\infty} = O(t^{-1}, t^{-\frac{3}{2}}),$$

when $v_+ = v_-$. The results in [2,10] have been extended and improved in several directions. The interested reader is referred to [6,7,12,14] and references therein.

Nishihara [11] proved that the solutions of the nonlinear one-dimensional wave equation

$$\begin{cases} u_{tt} - (a(u_x))_x + \alpha u_t = 0 \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \end{cases} \tag{4}$$

behave asymptotically as those of the linear parabolic problem

$$\begin{cases} \alpha v_t - a'(0) v_{xx} = 0 \\ v(0,x) = u_0(x) + \dfrac{1}{\alpha} u_1(x). \end{cases} \tag{5}$$

This holds because the term u_{tt} decays faster than the other terms in (4) and can be neglected as t tends to infinity. More precisely, he proved the decay estimates

$$(u - v, u_x - v_x, u_t - v_t) = O(t^{-1}, t^{-\frac{3}{2}}, t^{-2}).$$

Subsequently, the result in [11] has been extended to an abstract setting by Ikehata and Nishihara [5], where they investigated the diffusion phenomenon for a more general second order linear evolution equation of the form

$$u_{tt}(t) + u_t(t) + Au(t) = 0, \quad u(0) = u_0,\ u_t(0) = u_1, \tag{6}$$

where $A : D(A) \subset H \to H$ is a nonnegative self-adjoint operator in a Hilbert space H with dense domain. They proved the estimate

$$\|u(t) - v(t)\|_H \le C(1+t)^{-1} (\log(2+t))^{\frac{1+\varepsilon}{2}},$$

for any $\varepsilon > 0$, where v is the solution of the problem

$$v_t(t) + Av(t) = 0, \quad v(0) = u_0 + u_1.$$

They also conjecture that the optimal estimate which implies the diffusion phenomenon for (6) is

$$\|u(t) - v(t)\|_H \le C(1+t)^{-1}. \tag{7}$$

Chill and Haruax [1] proved the validity of Ikehata and Nihihara's conjecture (i.e., estimate (7)) by applying spectral analysis for unbounded self-adjoint operators. A related result on the diffusion phenomenon of the wave equation in exterior domains was also given by Ikehata in [3].

Yang and Milani [15] considered the N-dimensional version of (4) and showed that there holds an improvement decay estimate of the form

$$\|u(t) - v(t)\|_{L^\infty(\mathbb{R}^N)} = O(t^{-\frac{N}{2}-1}), \tag{8}$$

where v is the solution of the N-dimensional problem

$$\begin{cases} v_t - \operatorname{div}\left(a\left(\nabla v\right)\nabla v\right) = 0 \\ \qquad v\left(0,x\right) = u_0\left(x\right) + \dfrac{1}{\alpha}u_1\left(x\right). \end{cases} \tag{9}$$

It is by now well known (see [8]) that the solution u of the wave equation with a linear damping decays like

$$\left\|u\left(t\right)\right\|_{L^\infty(\mathbb{R}^N)} \le C\left(1+t\right)^{-\frac{N}{2}} \tag{10}$$

and the solution v of (9) decays as

$$\left\|v\left(t\right)\right\|_{L^\infty(\mathbb{R}^N)} \le C\left(1+t\right)^{-\frac{N}{2}}. \tag{11}$$

Consequently, the estimate (8) implies that the norm $\left\|u\left(t\right)-v\left(t\right)\right\|_{L^\infty(\mathbb{R}^N)}$ of the difference $u-v$ decays faster than that of either u or v.

Under the additional hypothesis $u_0, u_1 \in L^{1,\gamma}(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} u_i(x)dx = 0$, $i = 0,1$, Ikehata [4] proved that the estimate (10) can be improved to be

$$\left\|u\left(t\right)\right\|_{L^\infty(\mathbb{R}^N)} \le C\left(1+t\right)^{-\frac{N}{2}-\frac{\gamma}{2}}.$$

On the other hand, we prove (see Theorem 3.2) that under the condition $v_0 \in L^{1,\gamma}(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} v_0(x)dx = 0$, then the decay rate (11) of the solution of problem (15) can be also improved to

$$\left\|v\left(t\right)\right\|_{L^\infty(\mathbb{R}^N)} \le C\left(1+t\right)^{-\frac{N}{2}-\frac{\gamma}{2}}.$$

Now, a natural question to be asked is *whether or not the decay rate given in* (8) *can be further improved?* The main result of this paper (Section 3) is to give a positive answer to the above question (see Theorem 3.3). In fact, we show that, for initial data restricted in some weighted spaces, we prove that $\left\|u-v\right\|_{L^\infty(\mathbb{R}^N)}$ decays with the rate $t^{-\frac{N}{2}-1-\frac{\gamma}{2}}$, $\gamma \in [0,1]$, faster than the one in (8) given by Yang and Milani in [15]. To obtain our result, we use some estimates from [15], which are still valid in our case, but we use also new ideas to get better decay rates (see for example the proof of Lemma 3.6). In addition Theorem 3.2 is completely new. Moreover, the method that we use in the proof (especially to get the L^1-estimate) has not been introduced in [4].

The remaining part of this paper is organized as follows: in Section 2, we fix notations and for the convenience of the reader, we recall without proofs some useful results. In Section 3, we state and prove our main results.

2. Preliminaries

In this section, we introduce some notations and some technical lemmas to be used throughout this paper.

Throughout this paper, $\|.\|_q$ and $\|.\|_{H^l}$ stand for the $L^q(\mathbb{R}^N)$-norm $(1 \leq q \leq \infty)$ and the $H^l(\mathbb{R}^N)$-norm. Also, for $\gamma \in [0, +\infty)$, we define the weighted function space $L^{p,\gamma}(\mathbb{R}^N)$, $1 \leq p < \infty$, $N \geq 1$, as follows: $u \in L^{p,\gamma}(\mathbb{R}^N)$ iff

$$\|u\|_{p,\gamma}^p = \int_{\mathbb{R}^N} (1 + |x|)^\gamma |u(x)|^p dx < +\infty.$$

Next, we introduce some inequalities which will be used later in this paper.

Lemma 2.1 ([9]). *Let $N \geq 1$. Let $1 \leq p, q, r \leq \infty$, and let k be a positive integer. Then for any integer j with $0 \leq j \leq k$, we have*

$$\left\|\partial_x^j u\right\|_{L^p} \leq C \left\|\partial_x^k u\right\|_{L^q}^a \|u\|_{L^r}^{1-a} \tag{12}$$

where $\frac{1}{p} = \frac{j}{N} + a\left(\frac{1}{q} - \frac{k}{N}\right) + (1-a)\frac{1}{r}$ for a satisfying $\frac{j}{k} \leq a \leq 1$ and C is a positive constant. There are the following exceptional cases:

1. *If $j = 0$, $qk < N$ and $r = \infty$, then we made the additional assumption that either $u(x) \to 0$ as $|x| \to \infty$ or $u \in L^{q'}$ for some $0 < q' < \infty$.*
2. *If $1 < r < \infty$ and $k - j - N/r$ is nonnegative integer, then* (12) *holds only for $\frac{j}{k} \leq a < 1$.*

Now, we introduce the Hausdorff-Young inequality.

Lemma 2.2. *For $p, q, r\,(1 \leq p, q, r \leq \infty)$ satisfying $\frac{1}{q} - \frac{1}{p} = 1 - \frac{1}{r}$, the inequality*

$$\|f * g\|_p \leq C \|f\|_r \|g\|_q$$

holds, where $*$ denotes the convolution.

3. The diffusion phenomenon

In this section, we consider problem (1) and (2) and show that if we restrict the initial data $(u_0, u_1) \in \left(H^1(\mathbb{R}^N) \cap L^{1,\gamma}(\mathbb{R}^N)\right) \times \left(L^2(\mathbb{R}^N) \cap L^{1,\gamma}(\mathbb{R}^N)\right)$ with $\gamma \in [0, 1]$, then we can derive faster decay estimates of the difference $w = u - v$ than those given in [15]. First consider the wave equation (1) and exploiting the method in [8], then the solution of (1) can be represented as follows

$$u(x,t) = K_1 * u_1 + K_2 * u_0,$$

where the asterisk $*$ denotes the convolution, K_1 and K_2 can be represented as

$$K_j(x,t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^N} e^{ix\xi} R_j(\xi,t) d\xi, \quad j = 1,2.$$

Here $R_j(t,\xi)$ is the Fourier transform of $K_j(t,x)$. Thus, R_j satisfies the following ODE systems

$$\begin{aligned} &\frac{d^2}{dt^2} R_j + \alpha \frac{d}{dt} R_j + \beta |\xi|^2 R_j = 0, \quad j = 1,2 \\ &\qquad\qquad R_1(0,\xi) = 0, \quad \frac{d}{dt} R_1(0,\xi) = 1 \\ &\qquad\qquad R_2(0,\xi) = 1, \quad \frac{d}{dt} R_2(0,\xi) = 0. \end{aligned} \tag{13}$$

Based on the ODE form (13) and on some estimates in [8], Ikehata [4] has proved the following result:

Theorem 3.1 ([4]). *Let $N \geq 1$ and $\gamma \in [0,1]$. Assume that $u_0 \in (H^1(\mathbb{R}^N) \cap L^{1,\gamma}(\mathbb{R}^N))$ and $u_1 \in \left(L^2(\mathbb{R}^N) \cap L^{1,\gamma}(\mathbb{R}^N)\right)$. Then the solution $u(t,x)$ of problem* (1) *satisfies*

$$\begin{cases} \|u(t,\cdot)\|_2 \leq CI_0 (1+t)^{-\frac{N}{4}-\frac{\gamma}{2}} + CI_1 (1+t)^{-\frac{N}{4}} \\ \|\nabla u(t,\cdot)\|_2 \leq CI_0 (1+t)^{-\frac{N}{4}-\frac{\gamma}{2}-\frac{1}{2}} + CI_1 (1+t)^{-\frac{N}{4}-\frac{1}{2}} \\ \|u_t(t,\cdot)\|_2 \leq CI_0 (1+t)^{-\frac{N}{4}-\frac{\gamma}{2}-1} + CI_1 (1+t)^{-\frac{N}{4}-1}, \end{cases} \tag{14}$$

where C is a positive constant and

$$I_0 = \|u_0\|_{H^1} + \|u_0\|_{1,\gamma} + \|u_1\|_2 + \|u_1\|_{1,\gamma}, \quad I_1 = \left| \int_{\mathbb{R}^N} (u_0(x) + u_1(x))\, dx \right|.$$

Theorem 3.1 shows that for a particular class of initial data the classical Matsumura's decay estimates can be improved by using a device to treat the low frequency part in the Fourier integral formula of the solution.

Now, we want to extend Ikehata's result to the heat equation. In order to achieve this goal, let us consider the problem

$$\begin{cases} v_t(t,x) - \Delta v(t,x) = 0, & x \in \mathbb{R}^N,\ t \geq 0 \\ v(0,x) = v_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{15}$$

Our first new result reads as follows.

Theorem 3.2. *Let $N \geq 1$, $\gamma \in [0,1]$ and k be nonnegative integer. Then, we have*

$$\left\|\partial_x^k v(t,x)\right\|_p \leq Ct^{-\alpha-\frac{k}{2}-\frac{\gamma}{2}} \|v_0\|_{1,\gamma} + C \left| \int_{\mathbb{R}} v_0(x)\, dx \right| t^{-\alpha-\frac{k}{2}}, \tag{16}$$

for $1 \leq p \leq \infty$, where $\alpha = \frac{N}{2}\left(1 - \frac{1}{p}\right)$ and C is a positive constant.

Proof. Let us prove (16) for the L^∞ and L^2-norms.

The solution of (15) can be given explicitly. Indeed applying the Fourier transform in the x variable, problem (15) can be rewritten as

$$\begin{cases} \hat{v}_t(t,\xi) + |\xi|^2 \hat{v}(t,\xi) = 0, & \xi \in \mathbb{R}^N,\ t \geq 0 \\ \hat{v}(0,\xi) = \hat{v}_0(\xi), & \xi \in \mathbb{R}^N. \end{cases} \tag{17}$$

Solving the first order ordinary differential equation (17), we get $\hat{v}(t,\xi) = e^{-|\xi|^2 t}\hat{v}_0(\xi)$. Consequently, we have from the above formula

$$\begin{aligned} \left\|\widehat{\partial_x^k v}\right\|_{L^1(\mathbb{R}^N)} = \left\|(i\xi)^k \hat{v}\right\|_{L^1(\mathbb{R}^N)} &\leq C \left\||\xi|^k e^{-|\xi|^2 t}\hat{v}_0\right\|_{L^1(\mathbb{R}^N)} \\ &= C \int_{\mathbb{R}^N} |\xi|^k e^{-|\xi|^2 t} |\hat{v}_0(\xi)|\, d\xi \end{aligned} \tag{18}$$

Our goal now is to estimate $|\hat{v}_0|$. Indeed, we have (see [4])

$$\begin{aligned} |\hat{v}_0(\xi)| &= \left|\int_{\mathbb{R}^N} e^{-ix\xi} v_0(x)\, dx\right| \\ &\leq \int_{\mathbb{R}^N} |\cos(x\cdot\xi) - 1|\, |v_0(x)|\, dx + \int_{\mathbb{R}^N} |\sin(x\cdot\xi)|\, |v_0(x)|\, dx + \left|\int_{\mathbb{R}^N} v_0(x)\, dx\right|. \end{aligned}$$

Since $K_\gamma = \sup_{\theta\neq 0} \frac{|1-\cos\theta|}{|\theta|^\gamma} < +\infty$, $M_\gamma = \sup_{\theta\neq 0} \frac{\sin\theta}{|\theta|^\gamma} < +\infty$ for $0 \leq \gamma \leq 1$ we deduce

$$|\hat{v}_0(\xi)| \leq C_\gamma |\xi|^\gamma \|v_0\|_{1,\gamma} + \left|\int_{\mathbb{R}^N} v_0(x)\, dx\right| \tag{19}$$

with $C_\gamma = K_\gamma + M_\gamma$. Consequently, inserting (19) in (18) yields

$$\left\|\widehat{\partial_x^k v}\right\|_{L^1(\mathbb{R}^N)} \leq C \|v_0\|_{1,\gamma} \int_{\mathbb{R}^N} |\xi|^{k+\gamma} e^{-|\xi|^2 t} d\xi + C \left|\int_{\mathbb{R}^N} v_0(x) dx\right| \int_{\mathbb{R}^N} |\xi|^k e^{-|\xi|^2 t} d\xi.$$

Hence, passing to polar coordinates, we get

$$\left\|\widehat{\partial_x^k v}\right\|_{L^1(\mathbb{R}^N)} \leq C \|v_0\|_{1,\gamma} \int_0^{+\infty} r^{N-1+k+\gamma} e^{-r^2 t} dr + C \left|\int_{\mathbb{R}^N} v_0(x) dx\right| \int_0^{+\infty} r^{N-1+k} e^{-r^2 t} dr.$$

Thus, by the well known inequality $\int_0^{+\infty} r^\sigma e^{-r^2 t} dr \leq C t^{-\frac{\sigma+1}{2}}$, we get

$$\left\|\widehat{\partial_x^k v}\right\|_{L^1(\mathbb{R})} \leq C \|v_0\|_{1,\gamma} t^{-\frac{N+k+\gamma}{2}} + C \left|\int_{\mathbb{R}^{\mathbb{N}}} v_0(x)\, dx\right| t^{-\frac{N+k}{2}}. \tag{20}$$

By using the inequality

$$\|f\|_{L^p(\mathbb{R}^N)} \leq \|\hat{f}\|_{L^q(\mathbb{R}^N)}, \quad \frac{1}{p} + \frac{1}{q} = 1,\ 1 \leq q \leq 2,$$

we have $\left\|\partial_x^k v\right\|_{L^\infty(\mathbb{R}^N)} \leq \left\|\widehat{\partial_x^k v}\right\|_{L^1(\mathbb{R}^N)}$, and therefore, this last inequality together with the estimate (20) imply

$$\left\|\partial_x^k v\right\|_{L^\infty(\mathbb{R}^N)} \leq C t^{-\frac{N}{2}-\frac{k+\gamma}{2}} \left\|v_0\right\|_{1,\gamma} + C \left|\int_{\mathbb{R}^N} v_0(x)\, dx\right| t^{-\frac{N}{2}-\frac{k}{2}},$$

which is equivalent to (16), for $p = \infty$.

By the same method, and using the Plancherel theorem, we can easily show the L^2-decay estimate. Once (16) is true for $p = 2$ and $p = \infty$, then (16) for $2 < p < \infty$ follows from the interpolation inequality (12) by choosing $k = \frac{p}{2}j$, $q = 2$, $r = \infty$ and $a = \frac{2}{p}$.

Now, to complete the proof of (16), we have only to prove (16) for $p = 1$. Let us first prove the estimate (16) for $k = 0$. Indeed, the solution of (15), can be given as

$$v(t,x) = G(t,x) * v_0(x) = \int_{\mathbb{R}^N} G(t, x-y)\, v_0(y)\, dy \tag{21}$$

where

$$G(t,x) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}, \tag{22}$$

is the heat kernel (the fundamental solution of the heat equation). Consequently, (21) takes the form

$$v(t,x) = (4\pi t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} v_0(y)\, dy. \tag{23}$$

Then (23) easily takes the form

$$\begin{aligned} v(t,x) &= (4\pi t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} \left(e^{-\frac{|x-y|^2}{4t}} - e^{-\frac{|x|^2}{4t}}\right) v_0(y) dy + (4\pi t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|x|^2}{4t}} v_0(y) dy \\ &= v_1(t,x) + v_2(t,x). \end{aligned}$$

It is clear that

$$\int_{\mathbb{R}^N} |v_2(t,x)|\, dx \leq (4\pi t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} e^{-\frac{|x|^2}{4t}} dx \right| |v_0(y)|\, dy \leq C \left| \int_{\mathbb{R}^N} v_0(x) dx \right|.$$

On the other hand,

$$\begin{aligned} |v_1(t,x)| &\leq (4\pi t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} \left| e^{-\frac{|x-y|^2}{4t}} - e^{-\frac{|x|^2}{4t}} \right| |v_0(y)|\, dy \\ &= (4\pi t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} \left(\left| e^{-\frac{|x-y|^2}{4t}} - e^{-\frac{|x|^2}{4t}} \right|^\gamma \left| e^{-\frac{|x-y|^2}{4t}} - e^{-\frac{|x|^2}{4t}} \right|^{1-\gamma} |v_0(y)| \right) dy \\ &\leq C_\gamma (4\pi t)^{-\frac{N}{2}} \int_0^1 \int_{\mathbb{R}^N} \left| \frac{y(x-\theta y)}{2t} e^{-\frac{|x-\theta y|^2}{4t}} \right|^\gamma |v_0(y)|\, dy d\theta \\ &\leq C_\gamma \frac{(4\pi t)^{-\frac{N}{2}}}{t^{\frac{\gamma}{2}}} \int_0^1 \int_{\mathbb{R}^N} \left| \frac{y(x-\theta y)}{2\sqrt{t}} e^{-\frac{|x-\theta y|^2}{4t}} \right|^\gamma |v_0(y)|\, dy d\theta. \end{aligned} \tag{24}$$

By putting $z = \frac{x-\theta y}{2\sqrt{t}}$, and since $\int_{\mathbb{R}^N} |z|^\gamma e^{-|z|^{2\gamma}}$ is bounded, then (24) implies

$$\|v_1(t,x)\|_1 \le Ct^{-\frac{\gamma}{2}} \int_{\mathbb{R}^N} |y|^\gamma |v_0(y)|\, dy \le Ct^{-\frac{\gamma}{2}} \|v_0\|_{1,\gamma}. \tag{25}$$

Therefore, (3) together with (25) imply the inequality (16) for $k = 0$.

It is sufficient to use the induction on k to obtain higher order estimates for v_1 and v_2. (This higher order estimates of v_1 are sharp for k even). In order to let the reader understand the core of the argument, we prove here the estimate of v_1 (the most difficult term) for $k = 1$. We have

$$\partial_x v_1 = (4\pi t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} \left(-\frac{2(x-y)}{4t} e^{-\frac{(x-y)^2}{4t}} + \frac{x}{2t} e^{-\frac{x^2}{4t}} \right) v_0(y) dy,$$

this implies that

$$|\partial_x v_1| \le Ct^{-\frac{N}{2}} \int_{\mathbb{R}^N} \left| \frac{x-y}{2t} e^{-\frac{(x-y)^2}{4t}} - \frac{x}{2t} e^{-\frac{x^2}{4t}} \right|^{1-\gamma} \left| \frac{x-y}{2t} e^{-\frac{(x-y)^2}{4t}} - \frac{x}{2t} e^{-\frac{x^2}{4t}} \right|^{\gamma} |v_0(y)|\, dy. \tag{26}$$

However, for the right-hand side in (26), we set first $z = \frac{|x-y|}{2\sqrt{t}}$ for the first term and $z = \frac{|x|}{2\sqrt{t}}$ for the second term. Then, using the facts that the function ze^{-z^2} is bounded in $z \ge 0$ and that $\left| \frac{x}{2t} e^{-\frac{x^2}{4t}} \right| \le \frac{1}{\sqrt{t}} z e^{-z^2}$, we get

$$\begin{aligned}
|\partial_x v_1| &\le Ct^{\frac{\gamma-1}{2}} t^{-\frac{N}{2}} \int_{\mathbb{R}^N} \left| \frac{(x-y)}{2t} e^{-\frac{(x-y)^2}{4t}} - \frac{x}{2t} e^{-\frac{x^2}{4t}} \right|^\gamma |v_0(y)|\, dy \\
&\le Ct^{\frac{\gamma-1}{2}} t^{-\frac{N}{2}} \int_{\mathbb{R}^N} \left| \int_0^1 \frac{d}{d\theta} \left\{ \frac{(x-\theta y)}{2t} e^{-\frac{(x-\theta y)^2}{4t}} \right\} d\theta \right|^\gamma |v_0(y)|\, dy \\
&= Ct^{\frac{\gamma-1}{2}} t^{-\frac{N}{2}} \int_{\mathbb{R}^N} \left| \int_0^1 \frac{-y}{2t} e^{-\frac{(x-\theta y)^2}{4t}} + \frac{(x-\theta y)^2}{4t^2} e^{-\frac{(x-\theta y)^2}{4t}} d\theta \right|^\gamma |v_0(y)|\, dy.
\end{aligned}$$

Now, putting $z = \frac{x-\theta y}{2\sqrt{t}}$, and since $\int_{\mathbb{R}^N} e^{-|z|^{2\gamma}} dz$ and $\int_{\mathbb{R}} |z|^{2\gamma} e^{-|z|^{2\gamma}} dz$ are bounded, we get

$$\int_{\mathbb{R}^N} |\partial_x v_1|\, dx \le Ct^{\frac{\gamma-1}{2}} t^{-\frac{N}{2}} t^{\gamma \frac{N}{2}} t^{-\gamma} \int_{\mathbb{R}^N} (1+y)^\gamma |v_0(y)|\, dy \le Ct^{-\frac{\gamma+1}{2}} \|v_0\|_{1,\gamma}.$$

Thus (16) is fulfilled for $k = 1$; the rest follows by induction. □

Now, we will prove the diffusion phenomenon of the problem (1). The next theorem improves the result in [15]. Let $u(t,x)$ be the solution of (1) and $v(t,x)$ be the solution of (2). Then, we have the following result.

Theorem 3.3. *Let* $\gamma \in [0,1]$ *and* $N < 4\left(1-\frac{\gamma}{2}\right)$. *Assume that* $(u_0,u_1) \in \left(H^1(\mathbb{R}^N)\cap L^{1,\gamma}(\mathbb{R}^N)\right)\times\left(L^2(\mathbb{R}^N)\cap L^{1,\gamma}(\mathbb{R}^N)\right)$ *satisfying* $\int_{\mathbb{R}^N} u_i(x)dx=0$, $i=0,1$. *Then, we have*

$$\|u(t,\cdot)-v(t,\cdot)\|_\infty \le Ct^{-\frac{N}{2}-1-\frac{\gamma}{2}}. \tag{27}$$

Remark 3.4. If $\gamma = 0$, the result in Theorem 3.3 holds for all $N \ge 1$ (without the restriction $N < 4\left(1-\frac{\gamma}{2}\right)$). In this case a slight modification in the proof of the estimate (43) is needed (see [15]). Moreover, the condition $N < 4\left(1-\frac{\gamma}{2}\right)$ is technical and we conjecture that the estimate (27) holds without it.

Proof of Theorem 3.3. Let us first define $w(t,x) = u(t,x) - v(t,x)$. Then w becomes a solution to the problem:

$$\begin{cases} w_t(t,x) - \Delta w(t,x) = -u_{tt}(t,x), & x\in\mathbb{R}^N,\ t\ge 0 \\ w(0,x) = -u_1(x), & x\in\mathbb{R}^N. \end{cases} \tag{28}$$

Problem (28) can be solved exactly, with

$$w(t,x)=\int_{\mathbb{R}^N} G(x-y,t)(-u_1(y))dy + \int_0^t\int_{\mathbb{R}^N} G(x-y,t-\tau)(-u_{tt}(\tau,y))dyd\tau, \tag{29}$$

where G is the heat kernel defined in (22). As it was shown in [13] the heat kernel satisfying the following decay estimates

$$\begin{aligned} \|G(t,\cdot)\|_1 &\le C \\ \left\|\partial_t^k G(t,\cdot)\right\|_2 &\le Ct^{-\frac{N}{4}-k} \\ \left\|\partial_t^k G(t,\cdot)\right\|_\infty &\le Ct^{-\frac{N}{2}-k} \\ \|\nabla G(t,\cdot)\|_\infty &\le Ct^{-\frac{N}{2}-\frac{1}{2}}. \end{aligned} \tag{30}$$

Now, following [15], we split the second integral in (29) into two parts over the intervals $[0,\frac{t}{2}]$ and $[\frac{t}{2},t]$, and obtain (see [15] for details)

$$\begin{aligned} w(t,x) = &-\int_{\mathbb{R}^N} G\left(\frac{t}{2},x-y\right)u_t\left(\frac{t}{2},y\right)dy \\ &-\int_0^{\frac{t}{2}}\int_{\mathbb{R}^N} G_t(t-\tau,x-y)\,u_t(\tau,y)\,dyd\tau \\ &-\int_{\frac{t}{2}}^t\int_{\mathbb{R}^N} G(t-\tau,x-y)\,u_{tt}(\tau,y)\,dyd\tau \\ = &\ J_1(t,x)+J_2(t,x)+J_3(t,x). \end{aligned}$$

Our goal now, is to estimate the terms J_i, $i=1,2,3$. To this end, we apply the same method as in [11, 15].

First, applying Hölder's inequality, using the last inequality in (14) and the second estimate in (30), we get

$$|J_1(t,x)| \le \left\| G\left(\frac{t}{2},\cdot\right)\right\|_2 \left\| u_t\left(\frac{t}{2},\cdot\right)\right\|_2 \le Ct^{-\frac{N}{4}}(1+t)^{-\frac{N}{4}-\frac{\gamma}{2}-1} = Ct^{-\frac{N}{2}-1-\frac{\gamma}{2}}. \tag{31}$$

To estimate the term $J_3(t,x)$, we have by [14, Lemma 3.1] (of course the estimates in that lemma can be easily generalized to the high dimensional case and to the L^∞-norm)

$$\|u_{tt}(t,x)\|_\infty \le C(1+t)^{-\frac{N}{2}-2-\frac{\gamma}{2}}. \tag{32}$$

Consequently, Hölder's inequality together with the first estimate in (30) and (32) give

$$\begin{aligned} |J_3(t,x)| &\le \int_{\frac{t}{2}}^{t} \|G(t-\tau,\cdot)\|_1 \|u_{tt}(\tau,\cdot)\|_\infty \, d\tau \\ &\le C\int_{\frac{t}{2}}^{t} (1+\tau)^{-\frac{N}{2}-2-\frac{\gamma}{2}} \, d\tau \\ &= Ct^{-\frac{N}{2}-1-\frac{\gamma}{2}}. \end{aligned} \tag{33}$$

Now, going back to J_2, and similarly to [15], we have

$$\begin{aligned} J_2(t,x) = &-\int_{\mathbb{R}^N} G_t\left(\frac{t}{2},x-y\right) u\left(\frac{t}{2},y\right) dy + \int_{\mathbb{R}^N} G_t(t,x-y)\, u_0(y)\, dy \\ &- \int_0^{\frac{t}{2}} \int_{\mathbb{R}^N} G_{tt}(t-\tau,x-y)\, u(\tau,y)\, dy d\tau \\ = &\ J_{21}(t,x) + J_{22}(t,x) + J_{23}(t,x). \end{aligned} \tag{34}$$

Concerning J_{21}, we have by Hölder's inequality, the first estimate in Theorem 3.1 and the second inequality in (30)

$$|J_{21}(t,x)| \le \left\| G_t\left(\frac{t}{2},\cdot\right)\right\|_2 \left\| u\left(\frac{t}{2},\cdot\right)\right\|_2 \le Ct^{-\frac{N}{4}-1}(1+t)^{-\frac{N}{4}-\frac{\gamma}{2}} \le Ct^{-\frac{N}{2}-1-\frac{\gamma}{2}}. \tag{35}$$

The estimate of J_{22} is also crucial in deriving the diffusion phenomenon with the optimal decay rate $t^{-\frac{N}{2}-1-\frac{\gamma}{2}}$. In [15], the authors showed the following estimate

$$|J_{22}(t,x)| \le \|G_t(t,\cdot)\|_\infty \|u_0\|_1 \le Ct^{-\frac{N}{2}-1}. \tag{36}$$

However, we can not use the same estimate here, otherwise we will lose $t^{-\frac{\gamma}{2}}$. Instead, of the L^1-norm of the initial data u_0, we will use the $L^{1,\gamma}$-norm of u_0 in order to gain $t^{-\frac{\gamma}{2}}$ in the estimate (36).

Lemma 3.5. *It follows that, for all $\gamma \in [0,1]$,*

$$|J_{22}(t,x)| \leq Ct^{-\frac{N}{2}-1-\frac{\gamma}{2}}. \tag{37}$$

Proof. For simplicity, we take $\gamma = 1$. Using (22), we get

$$G_t(t,x) = -\frac{4\pi N}{2}(4\pi t)^{-\frac{N}{2}-1} e^{-\frac{|x|^2}{4t}} + \frac{|x|^2}{16t^2}(4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}} \tag{38}$$

Consequently,

$$\begin{aligned} J_{22}(t,x) &= \int_{\mathbb{R}^N} G_t(t,x-y)\, u_0(y)\, dy \\ &= -\frac{4\pi N}{2}(4\pi t)^{-\frac{N}{2}-1}\int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} u_0(y)\, dy \\ &\quad + c\,(4\pi t)^{-\frac{N}{2}-1}\int_{\mathbb{R}^N} \frac{|x-y|^2}{4t} e^{-\frac{|x-y|^2}{4t}} u_0(y)\, dy \\ &= K_1(t,x) + cK_2(t,x). \end{aligned}$$

Now, let us estimate $K_1(t,x)$. Indeed, since $\int_{\mathbb{R}^N} u_0(y)\, dy = 0$, we have

$$\begin{aligned} K_1(t,x) &= -2\pi N(4\pi t)^{-\frac{N}{2}-1}\int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} u_0(y)\, dy \\ &= -2\pi N(4\pi t)^{-\frac{N}{2}-1}\int_{\mathbb{R}^N}\left(e^{-\frac{|x-y|^2}{4t}} - e^{-\frac{|x|^2}{4t}}\right) u_0(y)\, dy \\ &= -\frac{2\pi N(4\pi t)^{-\frac{N}{2}-1}}{\sqrt{t}}\int_0^1\int_{\mathbb{R}^N}\frac{y\cdot(x-\theta y)}{2\sqrt{t}} e^{-\frac{|x-\theta y|^2}{4t}} u_0(y)\, dy d\theta \end{aligned} \tag{39}$$

The estimate (39) gives

$$|K_1(t,x)| \leq C_1 t^{-\frac{N}{2}-\frac{3}{2}} \|u_0\|_{1,1}, \tag{40}$$

where $C_1 = 2\pi N(4\pi)^{-\frac{N}{2}-1}\sup_{z\in\mathbb{R}^N}\{|z|\, e^{-|z|^2}\}$.

By the same method, and since $\int_{\mathbb{R}^N} u_0(y)\, dy = 0$, we estimate $K_2(t,x)$ as follows

$$\begin{aligned} K_2(t,x) &= (4\pi t)^{-\frac{N}{2}-1}\int_{\mathbb{R}^N}\frac{|x-y|^2}{4t} e^{-\frac{|x-y|^2}{4t}} u_0(y)\, dy \\ &= (4\pi t)^{-\frac{N}{2}-1}\int_{\mathbb{R}^N}\left(\frac{|x-y|^2}{4t} e^{-\frac{|x-y|^2}{4t}} - \frac{|x|^2}{4t} e^{-\frac{|x|^2}{4t}}\right) u_0(y)\, dy \\ &= \frac{(4\pi t)^{-\frac{N}{2}-1}}{\sqrt{t}}\int_0^1\int_{\mathbb{R}^N}\frac{y\cdot(x-\theta y)}{2\sqrt{t}} e^{-\frac{|x-\theta y|^2}{4t}} u_0(y)\, dy d\theta \\ &\quad + \frac{(4\pi t)^{-\frac{N}{2}-1}}{\sqrt{t}}\int_0^1\int_{\mathbb{R}^N}\frac{|x-\theta y|^2}{4t}\cdot\frac{y\cdot(x-\theta y)}{2\sqrt{t}} e^{-\frac{|x-\theta y|^2}{4t}} u_0(y)\, dy d\theta. \end{aligned}$$

This implies

$$|K_2(t,x)| \leq C_2 t^{-\frac{N}{2}-\frac{3}{2}} \|u_0\|_{1,1}, \tag{41}$$

where $C_2 = (4\pi)^{-\frac{N}{2}-1}\left\{\sup_{z\in\mathbb{R}^N}\{|z|\,e^{-|z|^2}\} + \sup_{z\in\mathbb{R}^N}\{|z|^3\,e^{-|z|^2}\}\right\}$. Taking into account (40) and (41), then (37) is fulfilled. This completes the proof of Lemma 3.5. □

Next, we are going to estimate the term J_{23} in (34). Proceeding as in [15], then J_{23} can be written as

$$\begin{aligned} -J_{23} &= \sum_{i=0}^{2} a_i \int_0^{\frac{t}{2}} \int_{\mathbb{R}^N} \left[\partial_t^{2-i} G\left(\frac{t}{4},\cdot\right) * \partial_t^i G\left(\frac{3t}{4}-\tau,\cdot\right)\right](x-y)\,u(y,\tau)\,dyd\tau \\ &= a_0 L_0 + a_1 L_1 + a_2 L_3. \end{aligned}$$

where a_i, $0 \leq i \leq 2$ are suitable constants. Then, we have the following Lemma.

Lemma 3.6. *It holds that, for all* $\gamma \in [0,1]$,

$$|L_i| \leq C t^{-\frac{N}{2}-1-\frac{\gamma}{2}}, \quad i = 1,2,3. \tag{42}$$

Proof. The term L_0 can be written as (see the term J_0 in [15])

$$L_0 = \int_0^{\frac{t}{2}} \int_{\mathbb{R}^N} G_{tt}\left(\frac{t}{4}, z\right)\left[G\left(\frac{3t}{4}-\tau,\cdot\right) * u(\tau,\cdot)\right](x-z)\,dzd\tau.$$

Then, Hölder's inequality implies

$$|L_0| \leq \int_0^{\frac{t}{2}} \left\|G_{tt}\left(\frac{t}{4},\cdot\right)\right\|_2 \left\|G\left(\frac{3t}{4}-\tau,\cdot\right) * u(\tau,\cdot)\right\|_2 d\tau.$$

Exploiting the second estimate in (30), and using the Hausdorff-Young inequality (Lemma 2.2), with $p = q = 2$ and $r = 1$, we find

$$|L_0| \leq C t^{-\frac{N}{4}-2} \int_0^{\frac{t}{2}} \left\|G\left(\frac{3t}{4}-\tau,\cdot\right)\right\|_1 \|u(\tau,\cdot)\|_2\,d\tau$$

Using the first inequality in (30), and the first estimate in Theorem 3.1 (recall that $I_1 = 0$, since $\int_{\mathbb{R}^N} u_i = 0$, $i = 1,2$), and assume that $\frac{N}{4} + \frac{\gamma}{2} < 1$, then we get $\int_0^{\frac{t}{2}} (1+\tau)^{-\frac{N}{4}-\frac{\gamma}{2}} = O(t^{1-\frac{N}{4}-\frac{\gamma}{2}})$, which gives

$$|L_0| \leq C t^{-\frac{N}{4}-2} \int_0^{\frac{t}{2}} (1+\tau)^{-\frac{N}{4}-\frac{\gamma}{2}} \leq C t^{-\frac{N}{2}-1-\frac{\gamma}{2}}. \tag{43}$$

Consequently, the desired estimate (42) holds for L_0.

Concerning the estimate of L_1, we have by using the second estimate in (30)

$$\begin{aligned} |L_1| &= \int_0^{\frac{t}{2}} \int_{\mathbb{R}^N} G_t\left(\frac{t}{4}, y\right) * G_t\left(\frac{3t}{4} - \tau, y\right)(x-y)\, u(\tau, y) \\ &\leq \int_0^{\frac{t}{2}} \left\| G_t\left(\frac{t}{4}, \cdot\right)\right\|_2 \left\| G_t\left(\frac{3t}{4} - \tau, \cdot\right) * u(\tau, \cdot)\right\|_2 \\ &\leq Ct^{-\frac{N}{4}-1} \int_0^{\frac{t}{2}} \left\| G_t\left(\frac{3t}{4} - \tau, \cdot\right) * u(\tau, \cdot)\right\|_2 . \end{aligned} \tag{44}$$

As above, applying Lemma 2.2, with $p = q = 2$ and $r = 1$, then (44) takes the form

$$|L_1| \leq Ct^{-\frac{N}{4}-1} \int_0^{\frac{t}{2}} \left\| G_t\left(\frac{3t}{4} - \tau, \cdot\right)\right\|_1 \|u(\tau, \cdot)\|_2 \, d\tau. \tag{45}$$

In view of the structure of $G_t(t, \cdot)$ defined in (38), we deduce that $\left\|G_t(\frac{3t}{4} - \tau, \cdot)\right\|_1 \leq Ct^{-1}$. Consequently, the above estimate together with the first estimate in Theorem 3.1 imply

$$\begin{aligned} |L_1| &\leq Ct^{-\frac{N}{4}-1} \int_0^{\frac{t}{2}} \left(\frac{3t}{4} - \tau\right)^{-1} (1+\tau)^{-\frac{N}{4}-\frac{\gamma}{2}} \, d\tau \\ &\leq Ct^{-\frac{N}{4}-1} \int_0^{\frac{t}{2}} \left(\frac{3t}{4} - \tau\right)^{-1} (1+\tau)^{-\frac{N}{4}-\frac{\gamma}{2}} \, d\tau. \end{aligned}$$

Since $\tau \leq \frac{t}{2}$, then we have $\frac{3t}{4} - \tau \geq \frac{t}{4}$. Thus,

$$|L_1| \leq Ct^{-\frac{N}{4}-2} \int_0^{\frac{t}{2}} (1+\tau)^{-\frac{N}{4}-\frac{\gamma}{2}} \, d\tau = Ct^{-\frac{N}{2}-1-\frac{\gamma}{2}},$$

which shows the inequality (42) for L_1. The estimate of L_2 holds exactly like the one of L_0. We omit the details. This completes the proof of Lemma 3.6. □

Completion of the proof of Theorem 3.3. Now, going back to the proof of Theorem 3.3, we deduce that (27) holds from (31), (33), (35), (37) and (42). This completes the proof of Theorem 3.3.

Remark 3.7. In Theorem 3.3 we proved the decay of the L^∞-norm of $(u(t,\cdot) - v(t,\cdot))$. The same method used in the proof of Theorem 3.3 can be adapted to prove that the L^2-norm decays as

$$\|u(t, \cdot) - v(t, \cdot)\|_2 \, Ct^{-\frac{N}{4}-1-\frac{\gamma}{2}}. \tag{46}$$

Corollary 3.8. *Under the assumption of Theorem* 3.3, *and for* $2 \leq p \leq \infty$, *we have the following decay estimate*

$$\|u(t, \cdot) - v(t, \cdot)\|_p \leq Ct^{-\alpha-1-\frac{\gamma}{2}}, \tag{47}$$

where $\alpha = \frac{N}{2}\left(1 - \frac{1}{p}\right)$.

Proof. It is clear that from Theorem 3.3 and Remark 3.7 that (47) holds for $p = 2$ and $p = \infty$. Now, to prove (47) for $2 < p < \infty$, we apply the interpolation inequality (12) by choosing $j = 0$, $q = 2$, $r = \infty$, and $a = \frac{2}{p}$, we get get our desired estimate (47). □

Acknowledgement. The author thanks Prof. Reinhard Racke of the Department of Mathematics and Statistics, University of Konstanz (Germany), for many stimulating conversations and helpful comments.

References

[1] Chill, R. and Haraux, A., An optimal estimate for the difference of solutions of two abstract evolution equations. *J. Diff. Equs.* 193 (2003)(2), 385 – 395.

[2] Hsiao, L. and Liu, T.-P., Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping. *Comm. Math. Phys.* 143 (1992)(3), 599 – 605.

[3] Ikehata, R., Diffusion phenomenon for linear dissipative wave equations in an exterior domain. *J. Diff. Equs.* 186 (2002)(2), 633 – 651.

[4] Ikehata, R., New decay estimates for linear damped wave equations and its application to nonlinear problem. *Math. Meth. Appl. Sci.* 27 (2004)(8), 865 – 889.

[5] Ikehata, R. and Nishihara, K., Diffusion phenomenon for second order linear evolution equations. *Studia Math.* 158 (2003)(2), 153 – 161.

[6] Marcati, P. and Mei, M., Convergence to nonlinear diffusion waves for solutions of the initial boundary problem to the hyperbolic conservation laws with damping. *Quart. Appl. Math.* 58 (2000)(4), 763 – 784.

[7] Marcati, P., Mei, M. and Rubino, B., Optimal convergence rates to diffusion waves for solutions of the hyperbolic conservation laws with damping. *J. Math. Fluid Mech.* 7 (2005)(2), 224 – 240.

[8] Matsumura, A., On the asymptotic behavior of solutions of semi-linear wave equations. *Publ. Res. Inst. Math. Sci.* 12 (1976), 169 – 189.

[9] Nirenberg, L. On elliptic partial differential equations. *Ann. Scuola Norm. Sup. Pisa* 13 (1959)(3), 115 – 162.

[10] Nishihara, K., Convergence rates to nonlinear diffusion waves for solutions of system of hyperbolic conservation laws with damping. *J. Diff. Equs.* 131 (1996)(2), 171 – 188.

[11] Nishihara, K., Asymptotic behavior of solutions of quasilinear hyperbolic equations with linear damping. *J. Diff. Equs.* 137 (1997), 384 – 395.

[12] Nishihara, K., Wang, W. and Yang, T., L_p-convergence rate to nonlinear diffusion waves for p-system with damping. *J. Diff. Equs.* 161 (2000)(1), 191 – 218.

[13] Racke, R., *Lectures on Nonlinear Evolution Equations. Initial Value Problems.* Aspects Math. E19. Braunschweig: Vieweg 1992.

[14] Said-Houari, B., Convergence to strong nonlinear diffusion waves for solutions to p-system with damping. *J. Diff. Equs.* 247 (2009), 917 – 930.

[15] Yang, H. and Milani, A., On the diffusion phenomenon of quasilinear hyperbolic waves. *Bull. Sci. Math.* 124 (2000)(5), 415 – 433.

Received April 30, 2011