Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 15 (1996), No. 2, 279–282

Monogenic Functions of Higher Spin

F. Sommen

Abstract. We present a definition for monogenic functions of higher spin and establish the Fischer decomposition with respect to this notion.

Keywords: *Clifford analysis, Dirac operators, spin representations*

AMS subject classification: 30 G 35

0. Introduction

Let R_m be the Clifford algebra over the Euclidean space $\mathbb{R}^m$ with basis $\{e_j\}_{1\le j\le m}$ and defining relations $e_j e_k + e_k e_j = -2\delta_{jk}$ $(1 \le j,k \le m)$ where δ_{jk} is the Kronecker symbol. Then the Dirac operator ∂ is given by $\partial_{\underline{x}} = \sum_{j=1}^m e_j \partial_{x_j}$ $(\underline{x} = \sum_{j=1}^m x_j e_j)$ and monogenic functions are R_m-valued solutions of the equation $\partial_{\underline{x}} f(\underline{x}) = 0$. Monogenic functions of this type may transform under the spin group Spin(m) in two different ways. First note that Spin(m) is a subgroup of R_m consisting of elements of the form $s = \underline{w}_1 \cdots \underline{w}_{2k}$ whereby $\underline{w}_j \in \mathbb{R}^m$ $(j = 1,\dots,2k)$ are unit vectors (i.e. $\underline{w}_j^2 = -1$). Next let $a \to \bar{a}$ be the main anti-involution on R_m determined by $\overline{ab} = \bar{b}\bar{a}$ and $\bar{e}_j = -e_j$. Then we may consider the two representations

$$L(s)\,f(\underline{x}) = s f(\bar{s}\underline{x}s) \qquad \text{and} \qquad H(s)\,f(\underline{x}) = s f(\bar{s}\underline{x}s)\,\bar{s}$$

transforming monogenic functions into monogenic functions.

The first representation corresponds in fact to fields with spin $\frac{1}{2}$. Usually this representation is defined for spinor-valued functions; but spinor spaces may be seen as minimal left ideals of the real Clifford algebra R_m (or in fact the complexified Clifford algebra C_m which may be represented by spaces of the form $C_m I$ with I being a primitive idempotent). The above definitions carry over to the complex or hyperbolic situation and in particular to the Minkowski space, where fields with spin $\frac{1}{2}$ correspond to the free electron field (see also [1]).

The second representation corresponds to fields with spin 1. Note hereby that special examples of monogenic functions transforming in this way are functions with values in the space $R_{m,k}$ of real k-vectors. Monogenic functions like this may be interpreted as solutions to the Hodge system for harmonic forms. In particular, for $k = 2$ and $m = 4$ (Minkowski space) these functions correspond to the electromagnetic field (see, e.g., [3]).

F. Sommen: Univ. Gent, Dept. Math. Anal., Galglaan 2, 9000 Gent, Belgium

ISSN 0232-2064 / \$ 2.50

But the above definition of monogenicity does not include functions with higher order spin. This is due to the fact that the Clifford algebra R_m only contains the basic representations of the spin group Spin(m) which are the representation $l(s) : a \to s\,a$ on spinor spaces and the representation $h(s) : a \to sa\bar{s}$ on the spaces $R_{m,k}$ of k-vectors. To construct models for irreducible representations of the spin group Spin(m) with higher order weights one may use multilinear functions on $\mathbb{R}^m$ (or even on R_m) with values in R_m, called *Clifford* tensors. In our paper [4] we studied the algebra of Spin(m)-invariant operators on Clifford tensors while in [5] we introduced so called *monogenic* tensors thus leading to explicit models for all irreducible representations of the spin group Spin(m) (see also [2]). But due to the existence of an inner product on the Clifford algebra R_m, spaces of multilinear functions on R_m may also be mapped isomorphically on k-fold tensor products $R_m \otimes \cdots \otimes R_m$ of R_m. Hence monogenic functions with higher order spin can be defined as function on $\mathbb{R}^m$ with values in $R_m \otimes \cdots \otimes R_m$. But there is even a better choice. The tensor product $R_m \otimes \cdots \otimes R_m$ itself, as a vector space, is isomorphic to a Clifford algebra $R_{m\cdot k}$ over $\mathbb{R}^{m\cdot k}$. This is the idea we use in this paper.

In Section 1 we give the definition of monogenic functions with values in $R_{m\cdot k}$ and we discuss the action of the spin group on them. In Section 2 we prove the Fischer decomposition for $R_{m\cdot k}$-valued homogeneous polynomials corresponding to this notion of monogenicity (further examples of Fischer decompositions related to this one may be found in [6, 9]).

Different approaches to Dirac operators of higher spin were presented in [2, 7, 8]. In our approach no specific choice of an irreducible representation space is needed.

1. Definition of monogenic functions of higher spin

Let $\{e_{j,\ell}\}_{\substack{1\le j\le m\\ 1\le \ell\le k}}$ be an orthonormal basis of the space $\mathbb{R}^{m\cdot k}$ generating the Clifford algebra $R_{m\cdot k}$. Then for $\ell = 1,\dots,k$ we put $\partial_{\underline{x}_\ell} = \sum_{j=1}^m e_{j,\ell}\partial_{x_j}$.

Definition 1. A function $f : \mathbb{R}^m \to R_{m\cdot k}$ is called *monogenic of higher spin* if it satisfies the system of equations $\partial_{\underline{x}_\ell} f(\underline{x}) = 0 \ \ (\ell = 1,\dots,k)$.

Next, using the already abvailable Clifford algebra R_m we may introduce embedding maps $(\cdot)_\ell : R_m \to R_{m\cdot k}$ as follows. For $j = 1,\dots,m$ put $(e_j)_\ell = e_{j,\ell}$. This together with the property $(a\,b)_\ell = (a)_\ell\,(b)_\ell$ determines the map $(\cdot)_\ell$. In particular, for each $s \in$ Spin(m) we may consider the element $s_\ell = (s)_\ell$, thus leading to k different realizations of the spin group Spin(m) inside $R_{m\cdot k}$.

On functions $f : \mathbb{R}^m \to R_{m\cdot k}$ we may now consider the so called spin $\frac{k}{2}$-representation

$$L_k(s)\,f : \ f(\underline{x}) \to s_1 \cdots s_k\, f(\bar{s}\,\underline{x}\,s)$$

and we have the following

Theorem 1. *For any function $f : \mathbb{R}^m \to R_{m\cdot k}$ which is monogenic of higher spin, the function $L_k(s)\,f$ is still monogenic of higher order spin.*

Proof. It is sufficient to note that for $\ell \neq n$ and $s_\ell \in$ Spin(m), the element s_ℓ commutes with the n-th Dirac operator $\partial_{\underline{x}_n}$ and that the elements s_ℓ $(\ell = 1,\dots,m)$ are mutually commutative ∎

To make the link with functions with values in tensor products of R_m, note that the operators $\partial_{\underline{x}_\ell}$ are anti-commutative. Hence if we introduce new Clifford algebra elements $E_1, \ldots, E_k$, the operators $D_{\underline{x}_\ell} = \partial_{\underline{x}_\ell} E_\ell$ are mutually commutative and any function $f : \mathbb{R}^m \to R_{m \cdot k}$ satisfying the equations $\partial_{\underline{x}_\ell} f = 0$ $(l = 1, \ldots, k)$ still satisfies the equations $D_{\underline{x}_\ell} f = 0$ $(\ell = 1, \ldots, k)$.

Note also that the algebra generated by the basis elements $e_{j,\ell} E_\ell$ $(j = 1 \ldots m,\ \ell = 1, \ldots, k)$ is isomorphic to the k-fold tensor product $R_m \otimes \cdots \otimes R_m$ of the Clifford algebra R_m. Hence by considering the functions f with values in a somewhat larger Clifford algebra $R_{m \cdot k + k}$ one can incorporate tensor-valued as well as $R_{m \cdot k}$-valued functions.

2. The Fischer decomposition

We first introduce the Spin(m)-invariant Fischer inner product for homogeneous polynomials R_n with values in $R_{m \cdot k}$. On $R_{m \cdot k}$ we consider the main anti-involution $a \to \bar{a}$ determined by $\bar{e}_{j,\ell} = -e_{j,\ell}$ $(\ell = 1, \ldots, k)$ and $\overline{a\, b} = \bar{b}\, \bar{a}$. Then the *Fischer inner product* is given by

$$\big(R_n(\underline{x}),\, S_n(\underline{x})\big) = \bar{R}_n(\partial_{\underline{x}})\, S_n(\underline{x}) \qquad (\underline{x} \in \mathbb{R}^m).$$

It is readily seen that this inner product is invariant under L_k, i.e.

$$\big(L_k(s) R_n,\, L_k(s) S_n\big) = (R_n,\, S_n)$$

for all $s \in \mathrm{Spin}(m)$.

Next consider for $\underline{x} \in \mathbb{R}^m = R_{m,1}$ the corresponding vector variables $(\underline{x})_j = \underline{x}_j = \sum x_\ell\, e_{\ell,j}$ which are anti-commuting $R_{m \cdot k}$-valued functions satisfying $(\underline{x})_j^2 = \underline{x}^2 = -\sum x_\ell^2$ $(j = 1, \ldots, k;\ \ell = 1, \ldots, m)$. Then we may consider the space of polynomials R_n of the form

$$R_n(x) = \sum \underline{x}_j\, R_{j,n-1}(x) \qquad (\underline{x} \in \mathbb{R}^m),$$

$R_{j,n-1}$ being homogeneous of degree $n - 1$, and we have the following

Theorem 2 (Simple Fischer decomposition). *Any homogeneous polynomial R_n admits a unique orthogonal decomposition of the form*

$$R_n(x) = P_n(x) + \sum \underline{x}_j\, R_{j,n-1}(x)$$

whereby P_n is a homogeneous monogenic polynomial of higher spin, i.e. $\partial_{\underline{x}_\ell} P_n = 0$.

Proof. The theorem follows from the fact that the Fischer inner product is positive definite so that R_n may always be decomposed as an orthogonal sum $R_n(x) = P_n(x) + \sum \underline{x}_j\, R_{j,n-1}(x)$, and from the orthogonality and the definition of the Fischer inner product it follows that P_n is monogenic ∎

To arrive at a complete Fischer decomposition we consider the spaces $\mathcal{P}_{(n,j)}$ of homogeneous polynomials of degree n and type j to be defined recursively as follows : $\mathcal{P}_{(n,0)}$ is the space of all homogeneous polynomials of degree n while $\mathcal{P}_{(n,j)}$ is the subspace of $\mathcal{P}_{(n,j-1)}$ of polynomials of the form $\sum \underline{x}_\ell\, R_{\ell,n-1}(x)$ with $R_{\ell,n-1} \in \mathcal{P}_{(n-1,j-1)}$. We now come to

Theorem 3 (Complete Fischer decomposition). *Any polynomial* $R_{(n,j)} \in \mathcal{P}_{(n,j)}$ *admits a unique orthogonal decomposition of the form* $R_{(n,j)} = P_{(n,j)} + R_{(n,j+1)}$ *with* $R_{(n,j+1)} \in \mathcal{P}_{(n,j+1)}$ *and whereby* $P_{(n,j)} \in \mathcal{P}_{(n,j)}$ *is* $(j+1)$*-monogenic of higher spin, i.e.* $\partial_{\underline{x}_{\ell_1}} \cdots \partial_{\underline{x}_{\ell_{j+1}}} P_{(n,j)} = 0$.

Proof. The proof is similar to that of the previous theorem taking into account that the j-monogenicity condition is satisfied by any homogeneous polynomial of degree k which is Fischer orthogonal to the space $\mathcal{P}_{(n,j+1)}$ ∎

By recursive application of this theorem it follows that any homogeneous polynomial R_n of degree n admits a unique orthogonal decomposition of the form

$$R_n = \sum_{j=0}^{n} P_{(n,j)}$$

whereby $P_{(n,j)} \in \mathcal{P}_{(n,j)}$ is left $(j+1)$-monogenic of higher spin. This in fact establishes the canonical form of the Fischer decomposition. One can now look for characterizations of polynomials of the form $P_{(n,j)} \in \mathcal{P}_{(n,j)}$ which are left $(j+1)$-monogenic. They can be characterized in terms of solutions of special systems of equations similar to the monogenicity condition.

References

[1] Delanghe, R., Sommen, F. and V. Souček: *Clifford Algebra and Spinor-Valued Functions: a Function Theory for the Dirac Operator* (Mathematics and its Applications: Vol. 53). Dordrecht: Kluwer Acad. Publ. 1992.

[2] Gilbert, J. and M. Murray: *Clifford Algebras and Dirac Operators in Harmonic Analysis.* Cambridge: Cambridge Univ. Press 1990.

[3] Jancewicz, B.: *Multivectors and Clifford Algebras in Electrodynamics.* Singapore: World Sci. 1988.

[4] Sommen, F.: *SO(M)-invariant operators on Clifford tensors.* Fund. Theories Phys. 55 (1993), 193 - 202.

[5] Sommen, F.: *Clifford tensor calculus.* In: Proc. XXIIth Conf. Diff. Geom. Meth. Theor. Phys. (DGM), Ixtapa 1993 (eds.: J. Keller and Z. Oziewicz). Adv. Appl. Clifford Alg. (Proc. Suppl.), 4 S 1 (1994), 423 - 436.

[6] Sommen, F. and N. Van Acker: *Functions of two vector variables.* Adv. Appl. Clifford Alg. 4 (1994), 65 - 72.

[7] Souček, V.: *Clifford analysis for higher spin.* Fund. Theories Phys. 55 (1993), 223 - 232.

[8] Stein, E. and G. Weiss: *Generalization of the Cauchy-Riemann equations and representations of the rotation group.* Amer. J. Math. 90 (1968), 163 - 196.

[9] Bernades, G., Sommen, F. and R. Delanghe: *Fischer decomposition for two-sided monogenic functions.* In preparation.

Received 13.10.1995