Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 15 (1996), No. 2, 329–343

On the Dirichlet Problem for the Ekman Equation

K. Frischmuth and J. Rossmann

Abstract. The Ekman partial differential equation for the stream function of turbulent mass flow in shallow and small-sized surface waters are discussed. The Dirichlet problem for the Ekman equation is shown to be well-posed in a weighted Sobolev space. Conditions for the existence of classical solutions are given. The dependence of regularity and asymptotics of the solution on the properties of the depth profile are studied.

Keywords: *Second order elliptic equations, asymptotics, hydrodynamics, Ekman equation*

AMS subject classification: 35 J 20, 35 J 25

1. Introduction

In the present paper we are concerned with the Dirichlet problem for the partial differential equation

$$\Delta u - \frac{2}{h}(\nabla h, \nabla u) = k_A(\nabla h, QW) \qquad \text{in } \Omega \tag{1}$$

which models the wind-induced hydrodynamic flow in small and shallow surface waters. Here the bounded domain $\Omega \subset \mathbb{R}^2$ describes the projection of the considered surface water on the plane $\mathbb{R}^2$, $(\cdot,\cdot)$ denotes the Euclidean scalar product in $\mathbb{R}^2$, $u : \Omega \longrightarrow \mathbb{R}$ the *stream function*, $h : \Omega \longrightarrow [0, +\infty)$ the *depth profile*, $W \in \mathbb{R}^2$ the *wind vector*,

$$Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

a clockwise rotation by an angle of $\frac{\pi}{2}$ and, finally, $k_A \geq 0$ an empirical constant.

In [17] equation (1) was derived in the given form and discussed under the Dirichlet condition

$$u = u_0 \qquad \text{on } \partial\Omega \tag{2}$$

and a finite difference scheme for the numerical solution was presented. However, typical depth profiles h vanish at least on parts of $\partial\Omega$, and up to now there has been no proof that problem (1) - (2) is well-posed for that case. The aim of the present paper is to prove existence and uniqueness of solutions to problem (1) - (2) and, furthermore,

K. Frischmuth: FB Mathematik der Univ., Universitätspl. 1, D – 18051 Rostock
J. Rossmann: FB Mathematik der Univ., Universitätspl. 1, D – 18051 Rostock

ISSN 0232-2064 / $ 2.50

to discuss regularity of the solution under possibly weak assumptions. To this end we transform equation (1) into divergence form and consider an associated variational problem

$$a(u,v) = b(v) \qquad \text{for all } v \tag{3}$$

in a weighted Sobolev space $H(\Omega)$. Using the Lax-Milgram lemma we obtain unique solvability of the Dirichlet problem (1) - (2). If, in addition, ∇h is bounded this allows us to infer the existence of an element $f \in L_2(\Omega)$ such that the solution u of problem (3) satisfies

$$\Delta u = f. \tag{4}$$

Hence, we conclude that $u \in H^2_{loc}(\Omega)$. We are able to strengthen this regularity result, if Ω is a polygonal domain and the depth profile h fulfils $h(x) = g(x)\,\mathrm{dist}(x, \partial\Omega)^\beta$ with a function g satisfying $g(x) \geq c > 0$ for all x. In this case we obtain $u \in H^{1+\beta}(\Omega)$. Further, the asymptotic behaviour of the solution in a neighbourhood of smooth parts of the boundary will be derived.

2. Weak formulations, existence and uniqueness

If we derived a weak formulation of problem (1) - (2) by multiplying equation (1) by a test function v and integrating by parts we would obtain a non-symmetric bilinear form. Moreover, without assuming rather undesirable conditions on the depth profile h it proves to be difficult to obtain existence and uniqueness of a weak solution that way. Weak coercivity (cf. [1]) can be obtained for $h \geq h_0 > 0$ by using a maximum principle (cf. [4]). The assumptions of the Lax-Milgram Lemma can be fitted under certain conditions for Ω and h expressed in terms of embedding constants (cf. [13]). Both results hold in $H^1(\Omega)$, and both require rather unrealistic restrictions on h.

A proper variational formulation of problem (1) - (2) is obtained after multiplying equation (1) by h^{-2}. In fact, observing that

$$h^{-2}\Delta u - 2h^{-3}(\nabla h, \nabla u) = \mathrm{div}(h^{-2}\nabla u)$$

we can rewrite equation (1) in the form

$$\mathrm{div}(h^{-2}\nabla u) = k_A h^{-2}(\nabla h, QW). \tag{5}$$

We introduce the weighted Sobolev space

$$H = \mathring{H}^1_{h^{-2}}(\Omega) = \overline{C_0^\infty(\Omega)} \tag{6}$$

with completion taken with respect to the norm

$$\|v\| = \|v\|_{h^{-2}} = \left(\int_\Omega h^{-2}|\nabla v|^2 dx\right)^{1/2}. \tag{7}$$

Observe that $\|\cdot\|$ makes H a Hilbert space with scalar product

$$\langle u,v\rangle = \int_\Omega h^{-2}(\nabla u, \nabla v)\,dx. \tag{8}$$

For the boundary condition u_0 we assume that there exists an element $\hat{u}_0 \in H^1(\Omega)$ such that

$$\operatorname{tr}(\hat{u}_0, \partial\Omega) = u_0 \qquad \text{and} \qquad \int_\Omega h^{-2}|\nabla \hat{u}_0|^2 dx < +\infty . \tag{9}$$

Now we introduce the difference $w = u - \hat{u}_0$ and obtain for it the equation

$$\operatorname{div}(h^{-2}\nabla w) = k_A h^{-2}(\nabla h, QW) - \operatorname{div}(h^{-2}\nabla \hat{u}_0) \qquad (w \in H) \tag{10}$$

with homogeneous boundary conditions. Multiplication by $v \in H$ and integration by parts yields

$$\int_\Omega h^{-2}(\nabla w, \nabla v)\, dx = -\int_\Omega h^{-2}(\ell, \nabla h) v\, dx - \int_\Omega h^{-2}(\nabla \hat{u}_0, \nabla v)\, dx \tag{11}$$

where we denote $\ell = k_A QW$. We define

$$a(w,v) = \int_\Omega h^{-2}(\nabla w, \nabla v)\, dx$$

$$b(v) = -\int_\Omega h^{-2}(\ell, \nabla h) v\, dx - \int_\Omega h^{-2}(\nabla \hat{u}_0, \nabla v)\, dx ..$$

Hence, the variational problem

Find a function $w \in H$ such that for all $v \in H$ there holds

$$a(w,v) = b(v) . \tag{12}$$

is set up. As a matter of course we can state the following

Lemma 1. *The bilinear form $a(\cdot,\cdot)$ is symmetric, bounded and coercive on $H \times H$, both constants being equal to 1, i.e.*

$$a(w,v) \le \|w\|\|v\| \qquad \text{and} \qquad a(w,w) \ge \|w\|^2 . \tag{13}$$

In fact, the second relation holds as an equality. Further, we have the following

Lemma 2. *The terms*

$$\int_\Omega h^{-2}(\ell, \nabla h) v\, dx \qquad \text{and} \qquad \int_\Omega h^{-2}(\nabla \hat{u}_0, \nabla v)\, dx$$

both define linear continuous functionals on H.

Proof. Let $v \in C_o^\infty(\Omega) \subset H$. Then

$$\int_\Omega h^{-2} v \nabla h\, dx = -\int_\Omega v \nabla(h^{-1})\, dx = \int_\Omega h^{-1}\nabla v\, dx .$$

Hence, by the Hölder inequality

$$\left|\int_\Omega h^{-2} v \nabla h\, dx\right|^2 \le \left(\int_\Omega h^{-2}|\nabla v|^2\, dx\right)\left(\int_\Omega 1\, dx\right) = \|v\|^2 \operatorname{mes}(\Omega) .$$

Consequently, the first term in the statement of Lemma 2 is a well defined linear operator for v in the dense subspace $C_0^\infty(\Omega)$ of H and its norm is bounded by $|\ell|\sqrt{\operatorname{mes}(\Omega)} = k_A|W|\sqrt{\operatorname{mes}(\Omega)}$. Hence, there exists a unique extension of this operator onto the whole space H which satisfies the same bound. To $\int_\Omega (\nabla \hat{u}_0, \nabla v) h^{-2}\, dx$ we apply once more the Hölder inequality which yields

$$\int_\Omega h^{-2}|(\nabla \hat{u}_0, \nabla v)|\, dx \le \|v\|\, \|\hat{u}_0\|$$

and the lemma is proved ∎

Remark 1. In the last inequality the norm $\|\hat{u}_0\|$ is finite due to (9), although the inclusion $\hat{u}_0 \in H$ does not necessarily hold.

Lemmas 1 and 2 together with the Lax-Milgram Lemma provide a proof of

Theorem 1. *The variational problem* (12) *possesses a unique solution* $w \in H$. *An upper bound on the norm of the solution is defined by the given wind force* $k_A|W|$, *the measure of the domain and the boundary condition, namely*

$$\|w\| \leq k_A |W| \sqrt{\mathrm{mes}(\Omega)} + \|\hat{u}_0\| \tag{14}$$

with $\hat{u}_0$ *from* (9) *holds.*

For $h \leq h_0 < +\infty$ we have $H \subseteq H^1(\Omega)$, and we are able to deduce the existence of a solution $u = \hat{u}_0 + w$ to equation (1) in the Sobolev space $H^1(\Omega)$. Moreover, under reasonable restrictions on $|\nabla h|$ and for a suitable class of domains Ω, the finiteness of the norm $\|u\|$ implies stronger regularity.

3. Regularity of the solution

We denote by $r = r(x)$ the distance of the point x to the boundary $\partial\Omega$ and by ρ a real function on $C^\infty(\Omega)$ satisfying the conditions

(i) $c_1\, r(x) \leq \rho(x) \leq c_2\, r(x)$

(ii) $|D_x^\alpha \rho(x)| \leq c_\alpha\, r(x)^{-|\alpha|+1}$

for $x \in \Omega$, where c_1, c_2 and c_α are positive constants independent of x. Such function ρ always exists (see, e.g., [18: Chapter 6/Section 2]) and is called *regularized distance* of x to $\partial\Omega$. If $\partial\Omega$ is smooth, we can set $\rho = r$.

The goal of this section is to prove regularity assertions for the solution of problem (1) - (2) in the case when the depth profile h has the representation

$$h(x) = g(x)\, \rho(x)^\beta \tag{15}$$

where β is an arbitrary non-negative real number and g is a real function on $C^\infty(\overline{\Omega})$ satisfying the inequality

$$g(x) \geq g_0 > 0 \qquad \text{in } \Omega.$$

We restrict ourselves to the case when Ω is a domain of polygonal type, i.e. the boundary $\partial\Omega$ of Ω is piecewise smooth and in a neighbourhood of each corner the domain Ω is diffeomorphic to a plane wedge

$$K = \Big\{ x = (x_1, x_2) \in \mathbb{R}^2 : -\phi_0 < \arg(x_1 + i x_2) < +\phi_0 \Big\} \tag{16}$$

where $0 < \phi_0 < \pi$. From (15) and the conditions on ρ it follows that

$$|D^\alpha (h^{-1} \nabla h)| \leq c_\alpha\, r^{-1-\alpha}$$

for every multi-index α, where c_α are constants independent of $x \in \Omega$. We define the space $V^l_\beta(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{V^l_\beta(\Omega)} = \left(\int_\Omega \sum_{|\alpha|\le l} r^{2(\beta-l+|\alpha|)}\, |D^\alpha u|^2\, dx\right)^{1/2}.$$

It is evident that the norm in H is equivalent to the norm

$$\|u\| = \left(\int_\Omega r^{-2\beta}|\nabla u|^2 dx\right)^{1/2}. \tag{17}$$

Moreover, from the Hardy inequality it follows that the last norm is equivalent to the norm

$$\|u\| = \left(\int_\Omega (r^{-2\beta-2}\,|u|^2 + r^{-2\beta}\,|\nabla u|^2) dx\right)^{1/2}$$

in $C_0^\infty(\Omega)$. Consequently, H coincides with the space $V^1_{-\beta}(\Omega)$.

In the following we will prove that the variational solution of the boundary value problem (1) - (2) belongs to the space $H^{1+\beta}(\Omega)$. We consider at first the boundary value problem

$$\left.\begin{array}{ll} \Delta u - 2\,h^{-1}(\nabla h, \nabla u) = (\ell, \nabla h) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{array}\right\} \tag{18}$$

with homogeneous Dirichlet condition which leads to the variational problem

$$a(u,v) = \int_\Omega h^{-2}\cdot(\ell,\nabla h)\cdot v\,dx \qquad \text{for all } v \in H. \tag{19}$$

Obviously, every solution $u \in H$ of this variational problem is a weak solution of the differential equation (18). Since $h^{-1}\nabla h$ and $(\ell, \nabla h)$ are smooth in Ω, we have the inclusion $u \in H^2_{loc}(\Omega)$.

We observe now what happens, if we apply a diffeomorphism $x' = \kappa(x)$ to equation (18). First note that the space V^l_β is invariant under diffeomorhisms. Indeed, let

$$r_\kappa(x') = \inf_{y'\in\partial\Omega'} |x'-y'| = \inf_{y\in\partial\Omega} |\kappa(x)-\kappa(y)|$$

be the distance of the point $x' = \kappa(x)$ to the boundary $\partial\Omega'$ of the domain $\Omega' = \kappa(\Omega)$. Then it follows from the differentiability of κ that $r_\kappa(x') \le c\,r(x)$ and, analogously, $r(x) \le c\,r_\kappa(x')$, where c is a positive constant. Hence κ induces an isomorphism $V^l_\beta(\Omega) \to V^l_\beta(\Omega')$. The operator which arises from the Laplace operator Δ via coordinate change $x \to x'$ is a second order differential operator with smooth coefficients in $\overline{\Omega}$, while the κ-image of the operator $u \to 2\,h^{-1}\,(\nabla h, \nabla u)$ is a differential operator of first order with coefficients of the form $a(x')\,h^{-1}\,\frac{\partial h}{\partial x'_j}$ $(a \in C^\infty(\overline{\Omega}))$.

For the investigation of the smoothness of u up to the boundary we need the following lemma.

Lemma 3. *Let K be the plane wedge (16) and let*

$$L = \sum_{i,j=1}^{2} a_{i,j}(x) \frac{\partial^2}{\partial x_i \, \partial x_j} + \sum_{j=1}^{2} a_j(x) \frac{\partial}{\partial x_j} \tag{20}$$

be an elliptic differential operator in K with coefficients $a_{i,j} \in C^\infty(\overline{K})$ and $a_j \in C^\infty(K)$ such that in $\overline{K}$

$$\left|a_{i,j}(x) - a_{i,j}(0)\right| < \varepsilon \qquad \text{and} \qquad \left|D^\alpha a_j\right| \le c_\alpha \, r^{-1-|\alpha|}$$

for every multi-index $\alpha = (\alpha_1, \alpha_2)$ with $|\alpha| \le l-2$ $(l \in \{2,3,\dots\})$ and ε a sufficiently small positive real number. Furthermore, let $u \in V^{l-1}_{-\beta+l-2}(K) \cap H^2_{loc}(K)$ be a solution of the boundary value problem

$$\left.\begin{aligned} Lu &= f \qquad \text{in} \;\; K \\ u &= 0 \qquad \text{on} \;\; \partial K \end{aligned}\right\}$$

where $f \in V^{l-2}_{-\beta+l-1}(K)$. Then $u \in V^{l}_{-\beta+l-1}(K)$ and

$$\|u\|_{V^l_{-\beta+l-1}(K)} \le c \left(\|f\|_{V^{l-2}_{-\beta+l-1}(K)} + \|u\|_{V^{l-1}_{-\beta+l-2}(K)} \right) \tag{21}$$

with a constant c independent of u.

Proof. We denote by $L_0(x, D)$ the principal part of the operator L. Since $u \in V^{l-1}_{-\beta+l-2}(K)$, we have $Lu - L_0(0,D)u \in V^{l-2}_{-\beta+l-1}(K)$. Consequently, u is a solution of the problem

$$\left.\begin{aligned} L_0(x,D)u &= F \qquad \text{in} \;\; K \\ u &= 0 \qquad \text{on} \;\; \partial K \end{aligned}\right\}$$

where $F = f - (L - L_0(x,D))\,u \in V^{l-2}_{-\beta+l-1}(K)$. Let $\{U_\nu\}_{\nu \ge 1}$ be a countable covering of K such that $\operatorname{dist}(U, \partial K) = \operatorname{diam} U_\nu =: d_\nu$. Furthermore, let ψ_ν $(\nu \ge 1)$ be smooth functions satisfying the conditions

$$\operatorname{supp} \psi_\nu \subset U_\nu \,, \qquad \sum_{\nu=0}^{+\infty} \psi_\nu = 1 \;\; \text{in} \; K \,, \qquad |D^\alpha \psi_\nu| \le c\, d_\nu^{-|\alpha|} . \tag{22}$$

If $x \in U_\nu$, then $x' = \frac{x}{d_\nu} \in d_\nu^{-1} U_\nu =: U'_\nu$, where $\operatorname{diam} U'_\nu = 1$. We define the function u_ν by the equation

$$u_\nu(x') = \psi_\nu(d_\nu x')\, u(d_\nu x') = \psi_\nu(x)\, u(x).$$

Obviously, u_ν is equal to zero on the boundary of U'_ν and the support of u_ν is contained in U'_ν. Using classical estimates for solutions of elliptic equations in the domain U'_ν, we get

$$\begin{aligned} \|\psi_\nu u\|^2_{V^l_{-\beta+l-1}(K)} &\le c\, d_\nu^{-2\beta} \, \|u_\nu\|^2_{H^l(U'_\nu)} \\ &\le c\, d_\nu^{-2\beta} \left(\|L_0(0,D)\, u_\nu\|^2_{H^{l-2}(K)} + \|u_\nu\|^2_{H^{l-1}(K)} \right) \end{aligned} \tag{23}$$

with a constant c independent of u and ν. Since for $|\alpha| \geq 1$

$$|a_{i,j}(d_\nu x') - a_{i,j}(0)| < \varepsilon \quad \text{and} \quad |D^\alpha_{x'}(a_{i,j}(d_\nu x') - a_{i,j}(0)| \leq c\, d_\nu^{|\alpha|}$$

from (23) it follows that

$$\begin{aligned}\|\psi_\nu u\|^2_{V^l_{-\beta+l-1}(K)} &\leq c\, d_\nu^{-2\beta}\Big(\|L_0(d_\nu x', D_{x'})u_\nu\|^2_{H^{l-2}(U'_\nu)} + \|u_\nu\|^2_{H^{l-1}(U'_\nu)}\Big)\\ &\leq c\Big(\|L_0(x,D)(\psi_\nu u)\|^2_{V^{l-2}_{-\beta+l-1}(K)} + \|\psi_\nu u\|^2_{V^{l-1}_{-\beta+l-2}(K)}\Big)\\ &\leq c\Big(\|\psi_\nu F\|^2_{V^{l-2}_{-\beta+l-1}(K)} + \textstyle\sum_{\mu\in I_\nu}\|\psi_\mu u\|^2_{V^{l-1}_{-\beta+l-2}(K)}\Big)\end{aligned}$$

where I_ν denotes a set of integer numbers such that $\sum_{\mu\in I_\nu}\psi_\mu = 1$ on U_ν. We can assume that there exists a number N such that every of the sets I_ν consists of not more than N elements and every μ is contained in not more than N of the sets I_ν. Using equivalence of the norm in $V^l_\beta(K)$ with the norm

$$\|u\| = \left(\sum_{\nu=0}^{+\infty}\|\psi_\nu u\|^2_{V^l_\beta(K)}\right)^{1/2}$$

we get the estimate (21). This proves the lemma ∎

Since the differential operator in (18) has the form given in the foregoing lemma, as a consequence of this lemma the following theorem holds.

Theorem 2. *Let $u \in H$ be a solution of the variational problem* (19) *and function h has the representation* (15) *with a function $g \in C^\infty(\overline{\Omega})$ satisfying the inequality $|g(x)| \geq g_0 > 0$. Then $u \in V^l_{-\beta+l-1}(\Omega)$ for $l \in \mathbb{N}$. In particular, $u \in H^{1+\beta}(\Omega)$.*

Proof. If $\operatorname{supp} u \cap \partial\Omega = \emptyset$, then the assertion is trivial. Suppose that $\operatorname{supp} u$ is contained in a neighbourhood of any boundary point. (Otherwise, we apply a suitable diffeomorphism. As we have seen before Lemma 3, any diffeomorphism transforms the differential equation (18) into the equation $Lu = f$, where $f \in V^{l-2}_{-\beta+l-1}$ and L has the form (20). For sufficiently small $\operatorname{supp} u$ it can be assumed that the conditions on the coefficients of L in Lemma 3 are satisfied.) Then with no loss of generality, we may assume that the domain Ω coincides with a half-plane or a plane wedge in this neighbourhood. In both cases we can apply Lemma 3 and obtain $u \in V^l_{-\beta+l-1}(\Omega)$. For functions with arbitrary support this assertion holds by means of a suitable partition of unity on Ω. Using the fact that the space $V^l_{-\beta+l-1}(\Omega)$ is continuously imbedded into $H^{1+\beta}(\Omega)$ for $l \geq 1+\beta$ (cf. [16: Theorem 1]), we get $u \in H^{1+\beta}(\Omega)$. The proof is complete ∎

We consider now the boundary value problem (1) - (2) with inhomogeneous Dirichlet condition. Under condition (15) assumption (9) can be characterized as follows.

Lemma 4. *Let u_0 be an arbitrary function on $\partial\Omega$. Then there exists a function $\hat{u}_0 \in H^1(\Omega)$ with properties* (9) *if and only if*

a) *u_0 is a constant in the case $\beta \geq \frac{1}{2}$*

b) *$u_0 \in H^{\frac{1}{2}+\beta}(\partial\Omega)$ if $0 \leq \beta < \frac{1}{2}$.*

Proof. Let $W^l_\beta(\Omega)$ be the closure of the set of all functions $u \in C^\infty(\overline{\Omega})$ such that

$$\|u\|_{W^l_\beta(\Omega)} = \left(\|u\|^2_{L_2(\Omega)} + \int_\Omega \sum_{|\alpha|=l} r^{2\beta}\,|D^\alpha u|^2 dx\right)^{1/2} < +\infty \tag{24}$$

with respect to norm (24). Then the function $\hat{u}_0$ in (9) belongs to $W^1_{-\beta}(\Omega)$. In case b) the trace of each function from $W^1_{-\beta}(\Omega)$ belongs to the space $H^{\frac{1}{2}+\beta}(\partial\Omega)$ (cf. [15: Lemmas 1.1 and 1.2]).

We consider the case $\beta \geq \frac{1}{2}$. Let $\hat{u}_0$ be a function from $W^1_{-\beta}(\Omega)$ and $\{\hat{u}_n\}_{n\geq 1} \subset C^\infty(\overline{\Omega}) \cap W^1_{-\beta}(\Omega)$ a sequence converging to $\hat{u}_0$ in $W^1_{-\beta}(\Omega)$. Since $\int_\Omega r^{-2\beta}\,|\nabla \hat{u}_n|^2 dx < +\infty$, we have $\nabla \hat{u}_n|_{\partial\Omega} = 0$ and therefore $\hat{u}_n|_{\partial\Omega} = c_n =$ const. From the Hardy inequality it follows that

$$\int_\Omega r^{-2\beta-2}\,|\hat{u}_n - c_n|^2\, dw \leq c \int_\Omega r^{-2\beta}\,|\nabla \hat{u}_n|^2\, dx$$

with a constant c independent of $\hat{u}_n$ and c_n. This implies $|c_n| \leq c\,\|\hat{u}_n\|_{W^1_{-\beta}(\Omega)}$. Analogously we obtain $|c_n - c_m| \leq c\|\hat{u}_n - \hat{u}_m\|_{W^1_{-\beta}(\Omega)}$. Hence the limit of the sequence $\{c_n\}_{n\geq 1}$ exists and coincides with the trace of $\hat{u}_0$ on $\partial\Omega$ ∎

Remark 2. In both cases $\beta \geq \frac{1}{2}$ and $\beta < \frac{1}{2}$ of Lemma 4 the function u_0 can be extended to a function $\hat{u}_0 \in W^l_{-\beta+l-1}(\Omega)$, where $l \geq 1$ is an arbitrary integer (cf. [15: Lemma 1.2]). Since $W^l_{-\beta+l-1}(\Omega) \subset H^{1+\beta}(\Omega)$ for $l \geq \beta + 1$ (see [16: Theorem 3]), we have $\hat{u}_0 \in H^{1+\beta}(\Omega)$.

Let u be a solution of problem (1) - (2) and let $\hat{u}_0 \in W^l_{-\beta+l-1}(\Omega)$ for $l \in \mathbb{N}$ be an extension of u_0. Then $\Delta\hat{u}_0 - 2h^{-1}(\nabla h, \nabla \hat{u}_0) \in V^{l-2}_{-\beta+l-1}(\Omega)$ for $l \in \mathbb{N}$ and $w = u - \hat{u}_0$ is a solution of the equation

$$\Delta w - 2\,h^{-1}\,(\nabla h, \nabla w) = (\ell, \nabla h) - \Delta \hat{u}_0 + 2\,h^{-1}\,(\nabla h, \nabla \hat{u}_0) \tag{25}$$

with homogeneous Dirichlet condition. Since the right side of this equation belongs to the space $V^{l-2}_{-\beta+l-1}(\Omega)$ for $l \in \mathbb{N}$ we can apply Lemma 3 and obtain $u \in H^{1+\beta}$.

4. Asymptotics of the solution in a neighbourhood of smooth parts of the boundary

Let $u \in H$ be a solution of the boundary value problem (18), and let χ and ψ be smooth cut-off functions which are equal to one in a neighbourhood U of any point $x^\circ \in \partial\Omega$ and equal to zero outside of a neighbourhood $U' \supset U$ such that $\chi\psi = \chi$. We suppose again that the function h has the representation (15), where $g \in C^\infty(\Omega)$ with $|g(x)| \geq g_0 > 0$. For simplicity, we further assume that $\partial\Omega$ coincides with the x_2-axis in the neighbourhood U'.

The function ψu can be considered as a solution of the problem

$$\left.\begin{aligned} \Delta(\psi u) - 2h^{-1}(\nabla h, \nabla(\psi u)) = F \qquad & \text{in } \Omega \\ \psi u = 0 \qquad & \text{on } \partial\Omega \end{aligned}\right\} \tag{26}$$

where

$$F = \psi(\ell, \nabla h) + 2(\nabla\psi, \nabla u) + u\,\Delta\psi - 2h^{-1}(\nabla h, \nabla\psi)\,u.$$

We prove at first a regularity assertion for the derivative of ψu in x_2-direction.

Lemma 5. *If $u \in H = V^1_{-\beta}(\Omega)$ is a solution of problem* (18)*, then the inclusion*

$$\frac{\partial(\psi u)}{\partial x_2} \in V^1_{-\beta}(\Omega)$$

holds.

Proof. Let v be an arbitrary function in Ω and $\delta \neq 0$ an arbitrary real number. Then we define the function v_δ by the equality

$$v_\delta(x_1, x_2) = \delta^{-1}\big((v(x_1, x_2 + \delta) - v(x_1, x_2)\big).$$

For sufficiently small δ, the function $(\psi u)_\delta$ satisfies the equations

$$h^{-2}\Delta(\psi u)_\delta - 2h^{-3}(\nabla h, \nabla(\psi u)_\delta) = h^{-2}F_\delta + 2h^{-2}((h^{-1}\nabla h)_\delta, \nabla(\psi u)) \quad \text{in } \Omega \tag{27}$$

$$(\psi u)_\delta = 0 \qquad \text{on } \Omega. \tag{28}$$

Hence Lemma 1 yields

$$\|(\psi u)_\delta\|_H \leq c\left\|h^{-2}F_\delta + 2h^{-2}\left((h^{-1}\nabla h)_\delta\,,\ \nabla(\psi u)\right)\right\|_{H^*} \tag{29}$$

where H^* denotes the dual space of H. We show that the right side of this inequality can be majorized by the sum of the norm of F in $V^0_{-\delta}(\Omega)$ and the norm of ψu in $V^1_{-\beta}(\Omega)$. For every smooth function v with $\operatorname{supp} v \subset U'$ we have

$$\begin{aligned} \|v_\delta\|_{V^0_{-\beta}(\Omega)} &= \int_\Omega r^{-2\beta}\,|v_\delta|^2\,dx \\ &= \int_\Omega x_1^{-2\beta}\left|\int_0^1 \frac{\partial v}{\partial x_2}(x_1, x_2 + t\delta)\,dt\right| dx \\ &\leq \|v\|^2_{V^1_{-\beta}(\Omega)}\,. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} \left|\int_\Omega h^{-2}\,F_\delta\,v\,dx\right| &= \left|\int_\Omega F\cdot(h^{-2}\,v)_{-\delta}\,dx\right| \\ &\leq c\,\|F\|_{V^0_{-\beta}(\Omega)}\left(\|v\|_{V^0_{-\beta}(\Omega)} + \|v_{-\delta}\|_{V^0_{-\beta}(\Omega)}\right) \\ &\leq c\,\|F\|_{V^0_{-\beta}(\Omega)}\,\|v\|_{V^1_{-\beta}(\Omega)} \end{aligned}$$

and, therefore, $\|h^{-2} F_\delta\|_{H^\bullet} \le c\,\|F\|_{V^0_{-\beta}(\Omega)} < +\infty$. Furthermore,

$$\left\|2h^{-2}\left((h^{-1}\nabla h)_\delta\,,\,\nabla(\psi u)\right)\right\|_{H^\bullet} \le \left\|2h^{-2}\left((h^{-1}\nabla h)_\delta\,,\,\nabla(\psi u)\right)\right\|_{V^0_{\beta+1}(\Omega)}$$
$$\le c\left(\int_\Omega r^{2\beta+2}\left|r^{-2\beta-1}\,\nabla(\psi u)\right|^2 dx\right)^{1/2}$$
$$\le c\,\|\psi u\|_{V^1_{-\beta}(\Omega)}\,.$$

Thus, we get the desired estimate for the right side of (29). The constant in this estimate is independent of δ. By the Fatou lemma we further have

$$\left\|\frac{\partial(\psi u)}{\partial x_2}\right\|_{V^1_{-\beta}(\Omega)} = \left\|\lim_{\delta\to 0}(\psi u)_\delta\right\|_{V^1_{-\delta}(\Omega)} \le \varliminf_{\delta\to 0}\|(\psi u)_\delta\|_{V^1_{-\beta}(\Omega)}.$$

Hence (29) yields the assertion of our lemma ∎

Theorem 3. *Suppose that the function h has the representation*

$$h(x) = g(x_1, x_2)\cdot x_1^\beta$$

in the neighbourhood U' of x°, where $g \in C^\infty(\overline{\Omega})$ with $|g(x)| \ge g_0 > 0$. Then the solution $u \in H$ of the boundary value problem (18) admits the decomposition

$$\chi u = \chi\,\ell_1\,g(0, x_2)\,\frac{x_1^{\beta+1}}{\beta+1} + u^{(1)}$$

where $u^{(1)} \in V^1_{-\beta-1}(\Omega)$ if $\beta > \frac{1}{2}$ and $u^{(1)} \in V^1_{-2\beta-\frac{1}{2}+\varepsilon}(\Omega)$ if $\beta \le \frac{1}{2}$. Here $\ell_1 = k_A\,W_2$ is the first component of the vector ℓ and ε is an arbitrary small positive number.

Proof. Rewriting (26) we have

$$\frac{\partial}{\partial x_1}\left(x_1^{-2\beta}\,\frac{\partial(\psi u)}{\partial x_1}\right) = x_1^{-2\beta} G$$

where

$$G = F - \frac{\partial^2(\psi u)}{\partial x_2^2} + 2g^{-1}\left(\nabla g, \nabla(\psi u)\right).$$

Hence

$$\frac{\partial(\psi u)}{\partial x_1} = -x_1^{2\beta}\int_{x_1}^{+\infty} \xi^{-2\beta}\,G(\xi, x_2)\,dx + x_1^{2\beta}\,C(x_2). \tag{30}$$

We write G in the form

$$G(x) = \psi(x)\,\ell_1\,\beta\,g(0, x_2)\,x_1^{\beta-1} + G_0(x)$$

where

$$G_0(x) = \psi(x)\Big(\ell_1\,\beta\,\big(g(x_1,x_2)-g(0,x_2)\big)\,x_1^{\beta-1} + (\ell,\nabla g)\,x_1^{\beta}\Big)$$
$$+\,2\,(\nabla\psi,\nabla u) + u\,\Delta\psi - 2\beta\,x_1^{-1}\,\frac{\partial\psi}{\partial x_1}\,u$$
$$-\,2g^{-1}\,(\nabla g,\nabla\psi)\,u - \frac{\partial^2(\psi u)}{\partial x_2^2} + 2g^{-1}\,\psi\,(\nabla g,\nabla u).$$

Since $\psi u \in V^1_{-\beta}(\Omega)$ and $\frac{\partial(\psi u)}{\partial x_2} \in V^1_{-\beta}(\Omega)$, the function G_0 belongs to the space $V^0_{-\beta}(\Omega)$. Hence for arbitrary $\varepsilon > 0$ the Hardy inequality yields

$$\int_\Omega x_1^{-1+2\max(\varepsilon,\beta-\frac12)}\left|\int_{x_1}^{+\infty}\xi^{-2\beta}\,G_0(\xi,x_2)\,d\xi\right|^2 dx$$
$$\le c\int_\Omega x_1^{1+2\max(\varepsilon,\beta-\frac12)}\left|x_1^{-2\beta}\,G_0(x_1,x_2)\right|^2 dx$$
$$\le c\int_\Omega x_1^{-2\beta}\left|G_0(x_1,x_2)\right|^2 dx.$$

Therefore, the function $x_1^{-2\beta}\,G_0(\xi,x_2)$ belongs to the space $V^0_\kappa(\Omega)$, where $\kappa = -\beta-1$ if $\beta > \frac12$ and $\kappa = -2\beta - \frac12 + \varepsilon$ if $\beta \le \frac12$. Let a_1 and a_2 be the functions defined by

$$a_1(x_2) = \max\big\{\xi:\ \psi(\xi,x_2) = 1\big\} \quad\text{and}\quad a_2(x_2) = \max\big\{\xi:\ \psi(\xi,x_2) \ne 0\big\}.$$

Then

$$x_1^{2\beta}\int_{x_1}^{+\infty}\beta\,\xi^{\beta-1}\,\psi(\xi,x_2)\,\ell_1\,g(0,x_2)\,d\xi$$
$$= \beta\,\ell_1\,x_1^{2\beta}\,g(0,x_2)\left(\int_{x_1}^{a_1(x_2)}\xi^{\beta-1}\,d\xi + \int_{a_1(x_2)}^{a_2(x_2)}\psi(\xi,x_2)\,\xi^{\beta-1}\,d\xi\right)$$
$$= \ell_1\,x_1^{2\beta}\,g(0,x_2)\,\big(-x_1^{\beta} + c_1(x_2)\big)$$

for all x_2 satisfying the condition $\psi(0,x_2) = 1$. Multiplying (30) by χ we obtain

$$\chi\frac{\partial u}{\partial x_1} = \chi\left(\ell_1\,g_0(0,x_2)\,x_1^{\beta} + c_2(x_2)\,x_1^{2\beta}\right) + R$$

where $R \in V^0_\kappa(\Omega)$. It is evident that $c_2 \in L_2(\partial\Omega\cap U)$, since $\chi\,\frac{\partial u}{\partial x_1} - R - \ell_1\,g(0,x_2)\,x_1^{\beta} \in V^0_{-\beta}(\Omega)$. This implies $\chi\,c_2(x_2)\,x_1^{2\beta} \in V^0_{-\beta-1}(\Omega)$. Consequently,

$$\chi\,\frac{\partial u}{\partial x_1} = \chi\,\ell_1\,g(0,x_2)\,x_1^{\beta} + R_1$$

where $R_1 \in V_\kappa^0(\Omega)$. Using Lemma 5 from this we can conclude that

$$\nabla\left(\chi u - \chi\,\ell_1\, g(0,x_2)\,\frac{x_1^{\beta+1}}{\beta+1}\right) \in V_\kappa^0(\Omega).$$

Since $\chi u - \chi\,\ell_1\, g(0,x_2)\,\frac{x_1^{\beta+1}}{\beta+1} = 0$ on $\partial\Omega$, the Hardy inequality implies

$$\left\|\chi u - \chi\,\ell_1\, g(0,x_2)\,\frac{x_1^{\beta+1}}{\beta+1}\right\|_{V_{\kappa-1}^0(\Omega)} \le c\left\|\nabla(\chi u - \chi\,\ell_1\, g(0,x_2)\frac{x_1^{\beta+1}}{\beta+1}\right\|_{V_\kappa^0(\Omega)}.$$

Thus, $\chi u - \chi\,\ell_1\, g(0,x_2)\,\frac{x_1^{\beta+1}}{\beta+1} \in V_\kappa^1(\Omega)$. This proves the theorem ∎

5. Appendix: The Ekman model

In an attempt to keep the paper self-contained we are going to outline the derivation of the partial differential equation (1) from continuum physics. However, for details compare [5, 17] and the papers cited there.

As our starting point we take the general 3-D turbulent Navier-Stokes equations

$$v_t + v\nabla v = G - \nabla p - 2\omega \times v + \operatorname{div}(A\nabla v) \tag{31}$$

in the 3-dimensional domain

$$\Omega^{3d} = \Big\{(x,z) \in \mathbb{R}^2 \times \mathbb{R} :\, x \in \Omega \text{ and } \xi(x) > z > -h(x)\Big\}. \tag{32}$$

Here we denoted v – the (3 dimensional) velocity field, $G = (0,0,-g)^T$ – the gravitation field, ω – the spin of Earth rotation, p – the pressure, ξ – the (a priori unknown) free surface and $A = \operatorname{diag}(A_H, A_H, A_V)$ – the tensor of turbulent impulse transition. In this section we use the symbols v, ∇ and div regardless of whether they are applied in the 3-dimensional or in the plane case. In the 3D case z is identified with x_3.

Using Einstein's convention, equation (31) may be rewritten as

$$\frac{\partial v_i}{\partial t} + v_j\frac{v_i}{\partial x_j} = G_i - \frac{\partial p}{\partial x_i} - 2\varepsilon_{ijk}\omega_j v_k + \frac{\partial}{\partial x_k}\left(A_{kj}\frac{\partial v_i}{\partial x_j}\right) \qquad (i = 1,2,3)$$

with $x_3 = z$, $A_{11} = A_{22} = A_H$, $A_{33} = A_V$, $A_{ij} = 0$ for $i \neq j$, and ε the Riccati tensor. Equation (31) is completed by the incompressibility condition

$$\operatorname{div} v = 0. \tag{33}$$

Here and in (31) the density is assumed constant and equal to 1. As boundary conditions we assume on $\partial\Omega^{3d}$

$$v(x,-h(x)) = 0 \tag{34}$$

(the so called "no slip" condition at the ground) and

$$A_V \frac{\partial v_i}{\partial z}(x, \xi(x)) = \kappa\, |W|\, W_i \qquad (i = 1, 2). \tag{35}$$

Here $W = (W_1, W_2)$ is the *wind vector* (usually taken at a height of $z = \xi + 10$ (in meters)) and $k_A > 0$ is an empirical constant describing the shear stress induced by the wind. Equation (35) expresses the continuity of the normal impulse flux through the interface between air and water (cf. [10]).

Now, following [5, 11, 17], several assumptions are introduced which lead to a scalar linear elliptic partial differential equation in the 2-dimensional domain Ω. We look for stationary solutions with small accelerations, and further consider only water bodies with a very small depth to length ratio. Thus we neglect v_t, $v\nabla v$, $\omega \times v$ and A_H as being small in comparison with the remaining terms. Following [5, 11, 17] we also assume A_V to depend only on x and not on z. Hence, we obtain from the third component of (31)

$$p(x, z) = g(\xi - z) \qquad \text{and} \qquad \frac{\partial^2 v_i}{\partial z} = \frac{g}{A_V}\frac{\partial \xi}{x_i} \quad (i = 1, 2). \tag{36}$$

Now, by integrating twice and substituting (34) and (35) into the second equation of (36), we obtain an explicit expression for v_1 and v_2 dependent on $\nabla\xi$ and z. Integrating once more, and denoting for $x \in \Omega$ the *velocity of mass transport* by

$$V(x) = (V_1, V_2)^T \qquad \text{with } V_i(x) = \int_{-h(x)}^{\xi(x)} v(z)\, dz \quad (i = 1, 2) \tag{37}$$

we arrive after elementary calculations at

$$V = \frac{\kappa}{2A_V}(\xi + h)^2 |W| W - \frac{g}{3A_V}(\xi + h)^3 \nabla\xi\,.$$

Finally, we assume $\xi << h$ and obtain

$$V = \frac{\kappa |W|}{2A_V} h^2 W - \frac{g h^3}{3A_V}\nabla\xi \tag{38}$$

or, equivalently,

$$\nabla\xi = \frac{3\kappa}{2gh}|W|W - \frac{3A_V}{gh^3}V\,.$$

Note that due to (33) the two-dimensional field V is again divergence free, i.e.

$$\operatorname{div} V = 0 \qquad \text{in } \Omega\,. \tag{39}$$

As a constitutive assumption in [5] the empirical formula

$$A_V = A_V^0 |W| h \tag{40}$$

was recommended. Assuming this we obtain

$$\nabla \xi = \frac{3\kappa}{2gh}|W|W - \frac{k_B|W|}{h^2}V \tag{41}$$

with $k_B = 3A_V^0/g$.

We observe that the velocity of mass transport V is a positive combination of directions of wind and of steepest descent of the surface level. We could have introduced equation (41) as a phenomenological assumption for wind-driven flow processes instead deriving it from (31) - (35), but this approach explains the form of the coefficient functions.

Now equation (1) turns out to be the integrability condition for the first order partial differential equation (41). In order to make this conspicuous we observe that for sufficiently smooth functions the operator $D = \operatorname{div} Q\nabla$ vanishes. Hence, multiplying (41) by Q and taking the divergence, we obtain for $|W|$ being constant on Ω

$$-k_B \operatorname{div}\left(\frac{QV}{h^2}\right) = \frac{3\kappa}{2gh^2}(\nabla h, QW). \tag{42}$$

We introduce the stream function u by

$$V = Q\nabla u. \tag{43}$$

Now, with $k_A = 3\frac{\kappa}{2gk_B}$, and because of $Q^2 = -I$, equation (42) reduces to equation (1). From (43) we obtain for the boundary values u_0 the interpretation

$$u_0(x) = \int q(s)\,ds \tag{44}$$

where the integration is carried out along the boundary $\partial\Omega$ and q is the mass flow through the boundary.

Remark 3. We are not interested here in local velocities $v = v(x,z)$, however they can be derived from (34) - (36) (v_1 and v_2) and from (33) (for v_3) by integration over $[-h(x), z]$ (cf. [5, 11, 17]).

The variational formulation (12) of problem (1) - (2) was first used for a Finite Element solution in [7, 9, 12]. For the discretization of (12) a conforming Finite Element Method based on (3) was used to calculate the stream functions u and velocity fields V for the Greifswalder Bodden and the Riga Bay of the Baltic Sea.

It is worthwhile to mention that the discretization of equation (5) instead of equation (1) allows us to work with positive definite symmetric matrices, while the Finite Difference Method applied in [17] yields non-symmetric systems of equations. Hence, the numerical calculations require only half the memory and time compared with [17]. Refined versions of the numerical method have been developped in [3, 6].

Recently, inverse problems have been concidered in [6, 8, 19]. For details of the numerical solutions and for applications we refer to [8, 12]. The Neumann problem and bounds for the discretization error will be considered in a forthcoming paper.

References

[1] Babuška, I. and A. K. Aziz: *Mathematical foundations of the finite element method.* In: The Mathematical Foundations of FEM with Applications to Partial Differential Equations (ed.: A. K. Aziz). New York - London: Acad. Press 1972.

[2] Bagirov, L. A.: *On the solutions of boundary value problems for elliptic equations degenerating on a part of the boundary.* Mat. Zametki 47 (1990), 29 - 38.

[3] Bockier, A.: *Zur hydrodynamischen Modellierung der Kirr-Bucht.* Dissertation. Rostock: University 1993.

[4] Ciarlet, P. G.: *The Finite Element Method for Elliptic Problems.* Amsterdam: North-Holland Publ. Comp. 1978.

[5] Felsenbaum, A. I.: *Theoretical Foundations and Methods for the Numerical Calculation of Stationary Marine Flows* (in Russian). Moscow: Acad. Nauk USSR 1960.

[6] Fennel, K.: *Zum direkten und inversen Ekman-Problem.* Dissertation. Rostock: University 1994.

[7] Frischmuth, K.: *Zur Modellierung und numerischen Lösung von Konfigurations-Fluß-Problemen.* Thesis. Rostock: University 1990.

[8] Frischmuth, K. and M. Hänler: *On the inverse problem for the Ekman equation.* Z. Angew. Math. Mech. (ZAMM) (submitted).

[9] Frischmuth, K., Roßmann, J. and R. Strauß: *Solvability and regularity results for the hydrodynamic Ekman model.* Technical Report. Rostock: University 1990.

[10] Hutter, K., Oman, G. and H.-G. Ramming: *Windbedingte Strömungen des homogenen Zürichsees.* Zürich: Eidgen. Techn. Hochschule (ETH)/ Versuchsanstalt für Wasserbau, Hydrologie und Glaziologie, Mitteilungs Nr. 61 (1982).

[11] Jankowski, A.: *Mathematical Modeling of the Flow Field in the Baltic Sea* (in Polish). Thesis. Wrocław: Ossolineum 1988.

[12] Koyalo, M., Martuzans, B., Frischmuth, K. and R. Strauß: *On the calculation of wind-induced flow in shallow waters* (in Russian). Izw. Acad. Nauk Latw. SSR/ Ser. Fiziko-Tekhn. Nauk (1990)1, 73 - 78.

[13] Kufner, A.: *Weighted Sobolev spaces* (Teubner-Texte zur Mathematik: Vol. 31). Leipzig: B. G. Teubner Verlagsges. 1980.

[14] Maeß, G.: *Vorlesungen über numerische Mathematik.* Vol. I. Berlin: Akademie-Verlag 1984.

[15] Mazja, V. G. and J. Roßmann: *Über die Asymptotik der Lösungen elliptischer Randwertaufgaben in der Umgebung von Kanten.* Math. Nachr. 138 (1988), 27 - 53.

[16] Roßmann, J.: *On two classes of weighted Sobolev-Slobodezkij spaces in a dihedral angle.* Banach Center Publ. 27 (1992), 399 - 424.

[17] Stoyan, G., Müller, W. and H. Baumert: *Numerische Simulation von wind- und durchflußinduzierten Strömungen in Flachgewässern auf der Basis des Ekman-Modells (Rechenprogramm LAKE).* Acta Hydrophys. 30 (1986), 51 - 67.

[18] Stein, E. M.: *Singular Integrals and Differentiability Properties of Functions.* Princeton (New Jersey): Univ. Press 1970.

[19] Voigtländer, Ch.: *Verallgemeinerte Spline-Approximation beim Ekman-Modell der Hydrodynamik.* Dissertation. Rostock: University 1992.

Received 24.02.1995; in revised form 27.02.1996