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On Absolute Summability Factors

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Abstract. By using for $\delta \geq 0$ so-called $[\overline{N}, p_n; \delta]_k$-boundedness of series $\sum_{n=1}^{\infty} a_n$ and sequences $(\lambda_n)_{n=1}^{\infty}$ we prove $|\overline{N}, p_n; \delta|_k$-summability of the series $\sum_{n=1}^{\infty} a_n \lambda_n$. This result generalizes a known one related to $|\overline{N}, p_n|_k$-summability of series.

Keywords: *Absolute summability of series, summability factors, infinite series*

AMS subject classification: 40 F 05, 40 D 15, 40 G 05

1. Introduction

Let $\sum_{n=1}^{\infty} a_n$ be a given series and (s_n) its sequence of partial sums. We denote by (u_n^α) with $\alpha > -1$ the sequence of n-th Cesàro means of order α of (s_n). Let $k \geq 1$ and $\delta \geq 0$. The series $\sum a_n$ is said to be $|C, \alpha|_k$-*summable* if (see [6])

$$\sum_{n=1}^{\infty} n^{k-1} |u_n^\alpha - u_{n-1}^\alpha|^k < \infty,$$

and it is said to be $|C, \alpha; \delta|_k$-*summable* if (see [7])

$$\sum_{n=1}^{\infty} n^{\delta k + k - 1} |u_n^\alpha - u_{n-1}^\alpha|^k < \infty.$$

In the special case when $\delta = 0$ or $\alpha = 1$, the $|C, \alpha; \delta|_k$-summability is the same as the $|C, \alpha|_k$- or $|C, 1; \delta|_k$-summability, respectively.

Let (p_n) be any sequence of positive numbers such that

$$P_n = \sum_{\nu=0}^{n} p_\nu \to \infty \qquad \text{as} \quad n \to \infty.$$

The transformation defined by

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu s_\nu$$

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gives the sequence (t_n) of $(\overline{N}, p_n)$-means of a sequence (s_n), generated by the sequence of coefficients (p_n) (see [8]).

Let as before $k \geq 1$ and $\delta \geq 0$. The series $\sum a_n$ is said to be $|\overline{N}, p_n|_k$-*summable* if (see [1])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |t_n - t_{n-1}|^k < \infty,$$

and it is said to be $|\overline{N}, p_n; \delta|_k$-*summable* if (see [4, 5])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |t_n - t_{n-1}|^k < \infty.$$

In the special cases when $\delta \leq 0$ or $k = 1$ and $\delta \leq 0$, the $|\overline{N}, p_n; \delta|_k$-summability is the same as the $|\overline{N}, p_n|_k$- or $|\overline{N}, p_n|$-summability, respectively. The $|\overline{N}, p_n|_k$- and $|\overline{N}, p_n; \delta|_k$-summability methods are totally different from each other. As a matter of fact one can see that $|\overline{N}, p_n; \delta|_k$-summability methods are different for different values of δ. Also if we take $p_n = 1$ for all values of n, then $|\overline{N}, p_n; \delta|_k$-summability reduces to $|C, 1; \delta|_k$-summability.

At last, let again $k \geq 1$ and $\delta \geq 0$. The series $\sum a_n$ is said to be $[\overline{N}, p_n]_k$-*bounded* if (see [2])

$$\sum_{\nu=1}^{n} p_\nu |s_\nu|^k = O(P_n) \qquad \text{for } n \to \infty,$$

and it is said to be $[\overline{N}, p_n; \delta]_k$-*bounded* if (see [4, 5])

$$\sum_{\nu=1}^{n} \left(\frac{P_\nu}{p_\nu}\right)^{\delta k} p_\nu |s_\nu|^k = O(P_n) \qquad \text{for } n \to \infty.$$

The $[\overline{N}, p_n]_k$- and $[\overline{N}, p_n; \delta]_k$-boundedness are totally different from each other. In the special cases when $\delta \leq 0$ or $k = 1$ and $\delta \leq 0$, the $[\overline{N}, p_n; \delta]_k$-boundedness is the same as the $[\overline{N}, p_n]_k$- and $[\overline{N}, p_n]$-boundedness, respectively.

In [3] the following theorem for $|\overline{N}, p_n|_k$-summability factors of infinite series is proved.

Teorem A. *Let the series $\sum a_n$ be $[\overline{N}, p_n]_k$-bounded and let the sequences (λ_n) and (p_n) satisfy for $n \to \infty$ the conditions*

(i) $p_{n+1} = O(p_n)$

(ii) $\sum_{\nu=1}^{n} p_\nu |\lambda_\nu| = O(1)$

(iii) $P_n |\Delta \lambda_n| = O(p_n |\lambda_n|)$.

Then the series $\sum a_n P_n \lambda_n$ is $|\overline{N}, p_n|_k$-summable for $k \geq 1$.

2. The main result

Our aim is to generalize Theorem A to the case of $|\overline{N}, p_n; \delta|_k$-summability. Thus we shall prove the following theorem.

Theorem B. *Let the series* $\sum a_n$ *be* $[\overline{N}, p_n; \delta]_k$*-bounded and let the sequences* (λ_n) *and* (p_n) *satisfy the conditions* (i) - (iii) *of Theorem A. If*

$$\sum_{n=\nu+1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}} = O\left(\left(\frac{P_\nu}{p_\nu}\right)^{\delta k} \frac{1}{P_\nu}\right), \tag{1}$$

then the series $\sum a_n P_n \lambda_n$ *is* $|\overline{N}, p_n; \delta|_k$*-summable for* $k \geq 1$ *and* $\delta \geq 0$.

Note that for $\delta \leq 0$ Theorem B implies Theorem A. Because, in this case the $[\overline{N}, p_n; \delta]_k$-boundedness reduces to the $[\overline{N}, p_n]_k$-boundedness and condition (1) reduces to

$$\sum_{n=\nu+1}^{\infty} \frac{p_n}{P_n P_{n-1}} = O\left(\frac{1}{P_\nu}\right)$$

which always holds.

We need the following lemma for the proof of Theorem B.

Lemma (see [3]). *If the sequences* (λ_n) *and* (p_n) *satisfy conditions (ii) and (iii) of Theorem A, then* $P_n|\lambda_n| = O(1)$ *for* $n \to \infty$.

3. Proof of Theorem B

Without any loss of generality we can assume that $a_0 = s_0 = 0$. Let (T_n) denote the sequence of $(\overline{N}, p_n)$-means of the series $\sum a_n P_n \lambda_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu \sum_{i=0}^{\nu} P_i a_i \lambda_i = \frac{1}{P_n} \sum_{\nu=0}^{n} (P_n - P_{\nu-1}) P_\nu a_\nu \lambda_\nu.$$

Then, for $n \geq 1$,

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n} P_{\nu-1} P_\nu a_\nu \lambda_\nu.$$

Using the Abel transformation, we get

$$\begin{aligned} T_n - T_{n-1} &= -\frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} P_\nu p_\nu s_\nu \lambda_\nu + \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} P_\nu P_\nu \Delta\lambda_\nu s_\nu \\ &\quad - \frac{p_n}{P_n p_{n-1}} \sum_{\nu=1}^{n-1} P_\nu p_{\nu+1} s_\nu \lambda_{\nu+1} + p_n s_n \lambda_n \\ &=: T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}. \end{aligned}$$

To complete the proof of the theorem, by the Minkowski inequality for $k > 1$, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |T_{n,r}|^k < \infty$$

for $1 \leq r \leq 4$. Now applying the Hölder inequality with indices k and k' where $\frac{1}{k} + \frac{1}{k'} = 1$, we get

$$\begin{aligned}
&\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |T_{n,1}|^k \\
&\leq \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}} \left\{\sum_{\nu=1}^{n-1} (p_\nu|\lambda_\nu)^k p_\nu |s_\nu|^k\right\} \left\{\frac{1}{P_{n-1}} \sum_{\nu=1}^{n-1} p_\nu\right\}^{k-1} \\
&= O(1) \sum_{\nu=1}^{m} (P_\nu|\lambda_\nu|)^k p_\nu |s_\nu|^k \sum_{n=\nu+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}} \\
&= O(1) \sum_{\nu=1}^{m} (P_\nu|\lambda_\nu|)^{k-1} |\lambda_\nu| \left(\frac{P_\nu}{p_\nu}\right)^{\delta k} p_\nu |s_\nu|^k \\
&= O(1) \sum_{\nu=1}^{m} |\lambda_\nu| \left(\frac{P_\nu}{p_\nu}\right)^{\delta k} p_\nu |s_\nu|^k \\
&= O(1) \sum_{\nu=1}^{m-1} \Delta|\lambda_\nu| \sum_{i=1}^{\nu} \left(\frac{P_i}{p_i}\right)^{\delta k} p_i |s_i|^k + O(1)|\lambda_m| \sum_{\nu=1}^{m} \left(\frac{P_\nu}{p_\nu}\right)^{\delta k} p_\nu |s_\nu|^k \\
&= O(1) \sum_{\nu=1}^{m-1} |\Delta\lambda_\nu| P_\nu + O(1)|\lambda_m| P_m \\
&= O(1) \sum_{\nu=1}^{m-1} p_\nu |\lambda_\nu| + O(1)|\lambda_m| P_m \\
&= O(1) \quad \text{for } m \to \infty
\end{aligned}$$

by virtue of the hypotheses of the theorem and the Lemma. Since $P_\nu|\Delta\lambda_\nu| = O(p_\nu|\lambda_\nu|)$, by condition (iii) of Theorem A, as for $T_{n,1}$, we get

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |T_{n,2}|^k = O(1) \sum_{\nu=1}^{m} |\lambda_\nu| \left(\frac{P_\nu}{p_\nu}\right)^{\delta k} p_\nu |s_\nu|^k = O(1)$$

for $m \to \infty$. Again, since $p_{n+1} = O(p_n)$, by condition (i) of Theorem A, as for $T_{n,1}$, we have

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |T_{n,3}|^k = O(1) \sum_{\nu=1}^{m} |\lambda_{\nu+1}| \left(\frac{P_\nu}{p_\nu}\right)^{\delta k} p_\nu |s_\nu|^k = O(1)$$

for $m \to \infty$. Finally, as for $T_{n,1}$ we get

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |T_{n,4}|^k = O(1) \sum_{n=1}^{m} |\lambda_n| \left(\frac{P_n}{p_n}\right)^{\delta k} p_n |s_n|^k = O(1)$$

for $m \to \infty$. Summarizing we get $\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |T_{n,r}|^k = O(1)$ as $m \to \infty$, for $1 \le r \le 4$. Tis completes the proof of Theorem B ∎

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