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A Generalization of the Weierstrass Theorem

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Abstract. The well-known Weierstrass theorem stating that a real-valued continuous function f on a compact set $K \subset \mathbb{R}$ attains its maximum on K is generalized. Namely, the space of real numbers is replaced by a set Y with arbitrary preference relation p (in place of the inequality $\leq$), and the assumption of continuity of f is replaced by its monotonic semicontinuity (with respect to the relation p).

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1. Introduction

Let X be a locally convex space and $K \subset X$ a sequentially compact set (a formal definition of such sets is given in Section 2). Let $f : K \to Y$ be a function mapping K into Y, which is an abstract space equipped with arbitrary preference relation $p \subset Y^2$. By p^* we mean the *transitive closure* of p, i.e. xp^*y if and only if there are elements $y_1, \dots y_n \in Y$ such that $x = y_1, y_1py_2, y_2py_3, \dots, y_{n-1}py_n, y_n = y$. By a *$(p,p^*)$-maximal point* of f on K we will understand a point $x_0 \in K$ such that if for some $x \in X$ we have $f(x_0)pf(x)$, then also $f(x)p^*f(x_0)$. Equivalently, f will be said to *achieve* its (p,p^*)-maximal value on K. Our aim is to investigate sufficient conditions for the existence of (p,p^*)-maximal points of f on K.

2. The main result

First we recall the definition of a sequentially compact set.

Definition 1 (see [2: p. 261]). A subset Z of a topological space X is called *sequentially compact* if for every sequence $\{x_i\}_{i\geq 1} \subset Z$ there is a subsequence $\{x_{i_k}\}_{k\geq 1} \subset \{x_i\}_{i\geq 1}$ converging to some $x \in Z$.

A locally convex topological vector space is a vector space with a topology defined by some collection of seminorms. The following example shows that the notion of sequential compactness does not coincide with the notion of compactness, even in the case of locally convex topological spaces.

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Example 2 (see [8: p. 253/Exercise 38]). Let $B = [0,1]^d$ be the d-fold Cartesian product of $[0,1]$, where d has the cardinality of a real line. Let B_0 be the subset of all elements of B with at most countable non-zero coordinates. Then B_0 is sequentially compact but not compact and B is compact but not sequentially compact.

The topology of a locally convex topological vector space is metrizable if and only if it may be defined by a countable collection of seminorms. This is not always possible, however. Let us consider the following example.

Example 3 (see [6: p. 68/Counterexample 1.1.8]). Let $C_0(\mathbb{R})$ denote the vector space of continuous functions on the real line $\mathbb{R}$ with compact support. For any positive function ξ let

$$\|f\|_\xi = \sup_x \xi(x)|f(x)|.$$

The collection of seminorms $\{\|\cdot\|_\xi\}$ defines a topology in $C_0(\mathbb{R})$ such that $C_0(\mathbb{R})$ is a complete locally convex Hausdorff topological vector space, but is not metrizable.

Weak topology is permanently used in many important fields of applications. The following theorem shows when a weak topology on a vector space X makes X be a locally convex linear space.

Theorem 4 (see [7: p. 76/Theorem 3.10]). *Let X be a vector space and X' a vector space of linear functionals, which separate points in X. Then the topology in X induced by X' makes X be a locally convex space. Moreover the conjugate space X^* to the space X is equal to X'.*

The next theorem establishes the equivalence between compactness and sequential compactness with respect to weak topologies.

Theorem 5 (see [1: Theorem C8/p. 164]). *Let X be a Banach space. A subset E of X is weakly compact if and only if it is weakly sequentially compact (i.e. sequentially compact with respect to the weak topology on X).*

Definition 6 (see [5: p. 201]). We shall call f *monotonically semicontinuous* (with respect to the relation p) at $x_0 \in X$, if for every sequence $\{x_i\} \subset X$ converging to x_0 and such that

$$f(x_i)p^*f(x_{i+1}) \qquad \text{for all } i \in \mathbb{N}$$

the relation

$$f(x_i)p^*f(x_0) \qquad \text{for all } i \in \mathbb{N} \tag{1}$$

holds.

Definition 7 (see [4: p. 288]). X is called *countably orderable* with respect to the relation p if for every non-empty subset $W \subseteq X$ the existence of a relation η well ordering W and such that $\eta \subseteq p^* \cup \mathrm{id}$ implies that W is at most countable.

The following theorem extends the Weierstrass theorem due to Gajek and Zagrodny [5] to the case when K is a sequentially compact set.

Theorem 8. *Let X be a locally convex topological space and let p be an arbitrary preference relation on a set Y. Assume $f : X \to Y$ is monotonically semicontinuous on a sequentially compact (with respect to the induced topology) and convex set $K \subset X$. Then f achieves its (p, p^*)-maximal value on K.*

Proof. Define a relation $q \subseteq K^2$ in the following way:

$$xqy \iff \begin{cases} x \neq y, f(x)p^* f(y) \text{ and there is } r > 0 \text{ such} \\ \text{that } (r(K-K)+x) \cap \{z \in K \,|\, f(y)p^* f(z)\} = \emptyset. \end{cases}$$

Let μ_{K-K} denote the Minkowski functional related to the set $K - K$. We have, for every $x, y \in K$;

$$\mu_{K-K}(x-y) \begin{cases} \leq r & \text{when } y \in (r(K-K)+x) \\ \geq r & \text{when } y \notin (r(K-K)+x). \end{cases}$$

First we show that K is countably orderable with respect to the relation q. Let $W \subseteq K$ be well ordered with respect to some relation $\eta \subseteq q \cup \mathrm{id}$. Since W is well ordered, for every $w \in W$ there is an immediate successor of w, say $n(w)$. Since $\eta \subseteq q \cup \mathrm{id}$ there is an $r(w) > 0$ such that

$$\big(r(w)(K-K)+w\big) \cap \big\{z \in W \,\big|\, n(w)\eta z, n(w) \neq z\big\} = \emptyset.$$

For every $n \in \mathbb{N}$, define

$$X_n = \Big\{w \in W \,\Big|\, r(w) > \frac{1}{n}\Big\}.$$

Obviously, $W = \bigcup_{n \in \mathbb{N}} X_n$. If X_{n_0} were not finite for some $n_0 \in \mathbb{N}$, then X_{n_0} would contain a subset $\{x_i\}_{i \in \mathbb{N}}$ such that $x_1 \eta x_2 \eta x_3 \ldots$. Consider the sequence $\{x_k\}_{k\,odd}$. We have

$$\mu_{K-K}(x_i - x_j) \geq \frac{1}{n_0}, \qquad \text{for all } i, j \text{ odd and } i \neq j. \tag{2}$$

Since K is sequentially compact, there is a point $x_0 \in K$ and a subsequence $\{x_{k_i}\} \subset \{x_k\}$ such that $\{x_{k_i}\}$ converges to x_0. Consequently, by the definition of convergence, we infer that for any neighbourhood of x_0, in particular for $\frac{1}{4n_0}(V \cap (K-K)) + x_0$ where V is a convex absorbing neighbourhood of zero, there is an index k_j such that, for $k_m \geq k_j$,

$$x_{k_m} \in \frac{1}{4n_0} V \cap (K-K) + x_0$$

and consequently

$$\mu_{V \cap (K-K)}(x_{k_m} - x_0) \leq \frac{1}{4n_0}.$$

So we have

$$\begin{aligned} \mu_{K-K}(x_{k_m} - x_{k_n}) &\leq \mu_{V \cap (K-K)}(x_{k_m} - x_{k_n}) \\ &= \mu_{V \cap (K-K)}\left[(x_{k_m} - x_0) + (x_0 - x_{k_n})\right] \\ &\leq \mu_{V \cap (K-K)}(x_{k_m} - x_0) + \mu_{V \cap (K-K)}(x_{k_n} - x_0) \\ &\leq \frac{1}{4n_0} + \frac{1}{4n_0} = \frac{1}{2n_0} \end{aligned}$$

which contradicts (2). Therefore each X_{n_0} is finite which implies that W is at most countable. So K is countably orderable with respect to the relation q.

Now let us consider any sequence $\{x_i\} \subset K$ such that $x_i q x_{i+1}$ for all $i \in \mathbb{N}$. By the definition of q, there are $r_i > 0$ such that

$$\big(r_i(K-K)+x_i\big) \cap \big\{z \in K \big| f(x_{i+1})p^* f(z)\big\} = \emptyset. \tag{3}$$

Since K is sequentially compact, there is a point $x \in K$ and a subsequence $\{x_{i_k}\} \subset \{x_i\}$ converging to x. Applying the monotonic semicontinuity property at x, we conclude that $f(x_{i_k})p^* f(x)$ for all $k \in \mathbb{N}$. Consequently by (3), it follows that $x_{i_k} q x$ for all $k \in \mathbb{N}$. By [4: Theorem 3.7] there is a (q,q)-maximal element of K, say x_0. We will show that this implies the thesis. To this end consider any $y \in K$, $y \neq x_0$, for which $f(x_0)pf(y)$. If there is an $r_0 > 0$ such that

$$\big(r_0(K-K)+x_0\big) \cap \big\{z \in K \big| f(y)p^* f(z)\big\} = \emptyset, \tag{4}$$

then $x_0 q y$. By the (q,q)-maximality of x_0, $y q x_0$, which implies $f(y)p^* f(x_0)$. So assume that (4) does not hold for any $r > 0$. Then chosing $r_1 = 1$, there is an element

$$y_1 \in \big(r_1(K-K)+x_0\big) \cap \big\{z \in K \big| f(y)p^* f(z)\big\}.$$

If (4) holds for $y = y_1$, then again we can get the assertion. So assume that for $r_2 = \frac{1}{2}$ there is an element

$$y_2 \in \big(r_2(K-K)+x_0\big) \cap \big\{z \in K \big| f(y_1)p^* f(z)\big\}$$

for which (4) is not valid, and so on. In this way either the theorem holds directly or we can get a sequence $\{y_i\}$ such that

$$f(x_0)pf(y)p^* f(y_1)p^* f(y_2)p^* \dots . \tag{5}$$

Suppose that x_0 is not the limit of the sequence $\{y_i\}$. Then there is a convex symmetric neighbourhood of 0, say U_0, a subsequence $\{y_{i_n}\}$ and a number $N \in \mathbb{N}$ such that, for all $n \geq N$, $y_{i_n} \notin (U_0 \cap (K-K)) + x_0$. The way the subsequence $\{y_{i_n}\}$ was constructed implies that we can choose points $k_n \in (K-K)$ for which $y_{i_n} - x_0 = r_n k_n \notin U_0$ for every $n \geq N$. On the other hand, the set K is sequentially compact. So for some subsequence $\{k_{n_j}\}$, $k_{n_j} \to k \in (K-K)$. It means that for every j, l large enough, we have $\mu_{U_0}(k_{n_j} - k_{n_l}) \leq 1$. Hence for $n \to \infty$

$$+\infty \longleftarrow r_n^{-1} \leq \mu_{U_0}(k_{n_j}) \leq \mu_{U_0}(k_{n_l}) + \mu_{U_0}(k_{n_j} - k_{n_l}) \leq \mu_{U_0}(k_{n_l}) + 1 < +\infty$$

which is a contradiction. Therefore the sequence $\{y_i\}$ is converging to x_0. Consequently, by the monotonic semicontinuity property, we have $f(y_i)p^* f(x_0)$ for all $i \in \mathbb{N}$. This and (5) together imply that $f(y)p^* f(x_0)$ ∎

Remark 9. When K is countable we need not assume that K is convex.

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