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# On the Method of Backward Steps of Carathéodory-Tonelli

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**Abstract.** We study the convergence of a sequence of functions which has been introduced by Carathéodory and Tonelli in connection with the solvability of the Cauchy problem for ordinary differential equations.

**Keywords:** *Ordinary differential equations, Cauchy problems, Backward steps method, Carathéodory-Tonelli solution, Peano phenomenon*

**AMS subject classification:** 34 A 45

## 1. Introduction and historical remarks

Consider the Cauchy problem

$$\begin{aligned} y' &= f(t,y) \\ y(0) &= y_0 \end{aligned} \tag{1}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. The method of Carathéodory and Tonelli to prove the existence for the Cauchy problem (1) consists in the following. For $n \in \mathbb{N}$ consider the functions $y_n$ defined by

$$y_n(t) = \begin{cases} y_0 & \text{if } 0 \le t \le \frac{1}{n} \\ y_0 + \int_0^{t-\frac{1}{n}} f(\tau, y_n(\tau))\,d\tau & \text{if } \frac{1}{n} \le t \le 1. \end{cases} \tag{2}$$

Since the computation of $y_n(t)$ for $t \in \left[\frac{i}{n}, \frac{i+1}{n}\right)$ $(1 \le i \le n-1)$ requires the knowledge of $y_n(t)$ for $t \in \left[\frac{i-1}{n}, \frac{i}{n}\right)$, the procedure is known as the *method of backward steps*. The sequence $\{y_n\}_{n\in\mathbb{N}}$ from (2) is equibounded and equicontinuous at least in an interval $[0,l]$ with $0 \le l \le 1$ [11], and hence, by the Ascoli-Arzelà compactness criterion, one can always find a subsequence which converges uniformly to a solution of problem (1). In the sequel we shall assume, only for sake of simplicity, that $l = 1$ (for conditions necessary and sufficient to ensure that $l = 1$ see [1]).

Let us make some historical remarks on the method of backward steps. The method was proposed by Tonelli in the more general framework of Volterra functional equations

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([10, 11]; see also [4, 7]).The idea had already been used by Carathéodory [2] with certain modifications (the functions $y_n$ are chosen as step functions rather than continuous ones) and generalisations (the non-linearity $f = f(t,y)$ need not be jointly continuous, but only satisfy a weaker condition which is now called *Carathéodory condition*). This method has been employed as well for partial differential equations (see, e.g., [3, 5, 8]), and has been studied subsequently by Stampacchia [9], Pellicciaro [6] and Vidossich [12] in the more general setting of functional analysis. If we have uniqueness in the solution of problem (1), then the sequence $\{y_n\}_{n\in\mathbb{N}}$ defined by (2), which is precompact, has just one accumulation point (every convergent subsequence converges necessarily toward the unique solution of problem (1)) and, of course, the sequence given by (2) converges. In a short review (see Math. Reviews 1 (1948), p. 92), N. Levinson suggested that the whole sequence given by (2) is always convergent, even if uniqueness fails. Such a result would be of considerable interest not only from theorethical point of view (e.g., it would not be necessary to pass to a subsequence in (2)), but also for practical purposes (e.g., for applying numerical procedures to (2)). As a matter of fact, Levinson's claim has not been confirmed, to the best of our knowledge, in subsequent work on the Carathéodory-Tonelli method. As far as we know, counterexamples have not yet been found either. It is this "uncertainty" which has motivated the present note, which consists of other two sections. In Section 2, we provide a set of sufficient conditions which guarantee the convergence of the whole sequence $\{y_n\}_{n\in\mathbb{N}}$ given by (2), even in the case of non-uniqueness. In Section 3, we concentrate on the case of Hölder continuous right-hand sides in problem (1).

## 2. Some sufficient conditions for convergence

In the following theorem we collect some sufficient conditions for the convergence of the sequence $\{y_n\}_{n\in\mathbb{N}}$ given by (2).

**Theorem 1.** *The sequence of Carathéodory-Tonelli* $\{y_n\}_{n\in\mathbb{N}}$ *given by* (2) *converges to a global solution* $y$ *of problem* (1) *if:*

**a)** $f(t,y_0) = 0$ *for every* $t \in [0,1]$.

**b)** $f(t,y)$ *is increasing with respect to* $y$ *and* $f(t,y) \geq 0$ *for every* $(t,y) \in [0,1]\times\mathbb{R}$ *(in this case the sequence* $\{y_n\}_{n\in\mathbb{N}}$ *converges uniformly to the lower integral of problem* (1)*).*

**c)** $f(t,y)$ *is increasing with respect to* $y$ *and* $f(t,y) \leq 0$ *for every* $(t,y) \in [0,1]\times\mathbb{R}$ *(in this case the sequence* $\{y_n\}_{n\in\mathbb{N}}$ *converges uniformly to the upper integral of problem* (1)*).*

**d)** $\{y\}_{n\in\mathbb{N}}$ *converges in a point* $t_0$ *to a value* $\alpha$*, such that* $(t_0,\alpha)$ *is not a Peano point for the equation* $y' = f(t,y)$ *(i.e. the Cauchy problem* $y' = f(t,y)$, $y(t_0) = \alpha$ *has a unique solution).*

**Proof.** a) Every $y_n$ is equal to the constant solution of problem (1). b) For every $n \in \mathbb{N}$ and every $t \in [0, \frac{1}{n}]$ we have $y_n(t) \leq y_{n+1}(t)$. Indeed, since $f$ is non-negative and increasing with respect to $y$, we have for $t \in \left[\frac{1}{n}, \frac{2}{n}\right]$

$$\begin{aligned} y_{n+1}(t) - y_n(t) &= \int_0^{t-\frac{1}{n+1}} f(\tau, y_{n+1}(\tau))\, d\tau - \int_0^{t-\frac{1}{n}} f(\tau, y_n(\tau))\, d\tau \\ &\geq \int_0^{t-\frac{1}{n}} \Big( f(\tau, y_{n+1}(\tau)) - f_n(\tau, y_n(\tau)) \Big)\, d\tau. \\ &\geq 0 \end{aligned}$$

The same inequality holds for $t \in [0,1]$ and consequently the sequence $\{y_n\}_{n\in\mathbf{N}}$ is increasing. Let $y$ be an arbitrary solution of problem (1). We have for $n \in \mathbb{N}$ and $t \in \left[0, \frac{1}{n}\right]$

$$y(t) = y_0 + \int_0^t f(\tau, y(\tau))\, d\tau \geq y_0 = y_n(t). \tag{3}$$

If $t \in \left[\frac{1}{n}, \frac{2}{n}\right]$, then

$$y(t) - y_n(t) \geq \int_0^{t-\frac{1}{n}} \Big( f(\tau, y(\tau)) - f(\tau, y_n(\tau)) \Big)\, d\tau \geq 0.$$

Analogously, $y_n(t) \leq y(t)$ for every $t \in [0,1]$. The last inequality implies that $y_n$ is convergent to the lower integral of problem (1). c) The proof is the same of that in Part b). d) Let us consider the problem

$$\begin{aligned} y' &= f(t,y) \\ y(t_0) &= \alpha. \end{aligned} \tag{4}$$

Since $(t_0, \alpha)$ is not a Peano point, every convergent subsequence of the sequence $\{y_n\}_{n\in\mathbf{N}}$ must converge to the unique solution of problem (4). The precompactness of $\{y_n\}_{n\in\mathbf{N}}$ implies that $y_n$ is convergent.

**Remark 1.** The sequence of Carathéodory-Tonelli $\{y_n\}_{n\in\mathbf{N}}$ converges to a global solution $y$ of problem (1) if the following condition holds:

**a)** $f$ is increasing with respect to $y$

**b)** $f(t,y) \geq 0$ in an open neighbourhood $I$ of $(0, y_0)$

**c)** The points of $\partial I$ are not Peano points for the equation $y' = f(t,y)$ (this is verified, by the way, if $f$ is of Lipschitz type in $y$, uniformly in $t$, outside of $\partial I$).

## 3. The case of Hölder-continuous right-hand side

First of all we prove a lemma wich will be useful to treat the case of Hölder-continuity of the function $f$

**Lemma 1.** *Given $0 < \alpha < 1$ and $\beta, \gamma > 0$, consider the sequence $\{a_k\}_{k\in\mathbb{N}}$, $a_k = a_k(\alpha, \beta, \gamma)$ defined inductively by*

$$\begin{aligned} a_{k+1} &= a_k + \beta\, a_k^\alpha \\ a_0 &= \gamma. \end{aligned} \tag{5}$$

*Then*

$$\delta^{\frac{1}{\alpha-1}} b_k < a_k \le b_k \qquad (k \in \mathbb{N}) \tag{6}$$

*where*

$$\delta = \left[1 + \frac{\beta(1-\alpha)}{\gamma^{1-\alpha}}\right]^{\frac{\alpha}{1-\alpha}} \qquad \textit{and} \qquad b_k = \left[(1-\alpha)\left(\beta k + \frac{\gamma^{1-\alpha}}{1-\alpha}\right)\right]^{\frac{1}{1-\alpha}}.$$

**Proof.** If we consider a continuous version

$$\begin{aligned} \varphi'(t) &= \beta \varphi^\alpha(t) \\ \varphi(0) &= \gamma \end{aligned}$$

of problem (5) we find

$$\varphi(t) = \left[(1-\alpha)\left(\beta t + \frac{\gamma^{1-\alpha}}{1-\alpha}\right)\right]^{\frac{1}{1-\alpha}} \qquad (0 \le t \le +\infty).$$

Since $\frac{1}{1-\alpha} > 1$, the function $\varphi = \varphi(t)$ is increasing and convex on $[1, +\infty)$. Since $b_k = \varphi(k)$, we have

$$b_k - b_{k-1} = \varphi(k) - \varphi(k-1) = \varphi'(c) \ge \varphi'(k-1) = \beta\, b_{k-1}^\alpha$$

for a suitable $c \in (k-1, k)$ and consequently

$$b_k \ge b_{k-1} + \beta\, b_{k-1}^\alpha. \tag{7}$$

Since $b_0 = \varphi(0) = a_0 = \gamma$, we derive from (7), by induction, $a_k \le b_k$ for every $k \in \mathbb{N}$.

From the other side

$$b_k - b_{k-1} = \varphi'(c) \le \varphi'(k) = \beta\, b_k^\alpha. \tag{8}$$

The sequence $\left\{\frac{b_k^\alpha}{b_{k-1}^\alpha}\right\}_{k\in\mathbb{N}}$ is decreasing with respect to $k$ and so has its maximum for $k = 1$. This maximum is equal to the $\delta$ in the statement of the lemma. It follows from (8) that $b_k \le b_{k-1} + \beta\delta\, b_{k-1}^\alpha$. Since $\delta > 1$, we have $b_0 < a_0\, \delta^{\frac{1}{1-\alpha}}$, and by induction

$$\begin{aligned} b_k &\le b_{k-1} + \beta\delta\, b_{k-1}^\alpha < a_{k-1}\delta^{\frac{1}{1-\alpha}} + \beta\delta\, a_{k-1}^\alpha \delta^{\frac{\alpha}{1-\alpha}} \\ &= \delta^{\frac{1}{1-\alpha}}(a_{k-1} + \beta\, a_{k-1}^\alpha) = \delta^{\frac{1}{1-\alpha}} a_k \end{aligned}$$

and the lemma is proved ∎

**Theorem 2.** *Let the function* $f = f(t,y)$ *in problem* (1) *be such that, for every* $y_1, y_2 \in \mathbb{R}$,

$$|f(t,y_1) - f(t,y_2)| \le L\,|y_1 - y_2|^\alpha \qquad (0 \le t \le 1)$$

*with* $L > 0$ *and* $0 < \alpha < 1$ *fixed. Then two limit functions* $\bar{y}_1$ *and* $\bar{y}_2$ *for the sequence* $\{y_n\}_{n\in\mathbb{N}}$ *defined by* (2) *satisfy the inequality*

$$|\bar{y}_1(t) - \bar{y}_2(t)| \le L\,(1-\alpha)^{\frac{1}{1-\alpha}} \qquad (0 \le t \le 1), \tag{9}$$

*i.e. they cannot differ too much.*

**Proof.** The sequence $\{y_n\}_{n\in\mathbb{N}}$ is equibounded and we denote by $M$ a number such that, for every $n \in \mathbb{N}$ and $t \in [0,1]$, $|f(t,y_n(t))| \le M$. Given $n,p \in \mathbb{N}$, we have $|y_{n+p}(t) - y_n(t)| \le \frac{M}{n} =: a_0$ for $0 \le t \le \frac{1}{n}$. If $\frac{1}{n} \le t \le \frac{2}{n}$, it follows from $0 \le \tau \le t - \frac{1}{n} \le \frac{1}{n}$ that

$$\begin{aligned}
|y_{n+p}(t) - y_n(t)| &\le \int_0^{t-\frac{1}{n}} \Big|f(\tau, y_{n+p}(\tau)) - f(\tau, y_n(\tau))\Big|\,d\tau + \int_{t-\frac{1}{n}}^{t-\frac{1}{n+1}} \big|f(\tau, y_{n+p}(\tau))\big|\,d\tau \\
&\le L \int_0^{t-\frac{1}{n}} |y_{n+p}(\tau) - y_n(\tau)|^\alpha d\tau + M\left(\frac{1}{n} - \frac{1}{n+p}\right) \\
&\le \frac{M}{n} + \frac{L}{n}\left(\frac{M}{n}\right)^\alpha \\
&= a_0 + \beta\, a_0^\alpha =: a_1
\end{aligned}$$

where $\beta = \frac{L}{n}$. Now it is easy to verify by induction that, for $\frac{k}{n} \le t \le \frac{k+1}{n}$, the inequality

$$\Big|y_{n+p}(t) - y_n(t)\Big| \le a_k = a_k\left(\alpha, \frac{M}{n}, \frac{L}{n}\right)$$

is true where the meaning of the symbols is the same as in Lemma 1. Since the sequence $\{a_k\}_{k\in\mathbb{N}}$ is increasing, we have

$$\Big|y_{n+p}(t) - y_n(t)\Big| \le a_n\left(\alpha, \frac{M}{n}, \frac{L}{n}\right) \qquad (0 \le t \le 1).$$

For $n \to +\infty$, $a_n \to L\,(1-\alpha)^{\frac{1}{1-\alpha}}$ from Lemma 1, whence our statement follows ∎

When the sequence $\{y_n\}_{n\in\mathbb{N}}$ given by (2) converges, we call its limit function $y = y(t)$ the Carathéodory-Tonelli solution for problem (1).

We conclude with a remark on this definition.

**Remark 2.** The Carathéodory-Tonelli solution $y = y(t)$ for problem (1) does not depend continuously on the function $f = f(t,y)$ and on the initial value $y(0) = y_0$. In fact, for the problem

$$y' = \sqrt{|y| + \varepsilon}$$
$$y(0) = \varrho$$

with $\varepsilon \geq 0$ and $\varrho \geq 0$, the Carathéodory-Tonelli solution is given by

$$y(t;\varepsilon,\varrho) = \frac{t^2}{4} + (\sqrt{\varrho + \varepsilon})t + \varrho$$

when at least one of the numbers $\varepsilon$ and $\varrho$ is different from 0. Moreover, $y(t;0,0) = 0$. So we can observe that

$$\lim_{\varepsilon \to 0} y(t;\varepsilon,0) \neq y(t;0,0) \qquad \text{and} \qquad \lim_{\varrho \to 0} y(t;0,\varrho) \neq y(t;0,0).$$

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