
The notable configuration of inscribed equilateral triangles in a triangle

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1 Introduction

Many notable configurations for a given triangle have been described by different geometers in the past. Among the most famous classic configurations are the Euler and Simson lines, the Gergonne, Lemoine, Brocard and Nagel concurrences, the Feuerbach, Brocard, Lemoine, Tucker and Taylor circumferences, etc. More modern configurations can be found in the Soddy circles [9], the Euler-Gergonne-Soddy triangle [8], the Torricelli configuration [10], etc. In fact, one can see in [3] four hundred configurations (triangle centers), in [4] even eight hundred configurations are picked up. Recently, new configurations continue to appear, for example Morley triangles related to the Feuerbach circumferences [5], associated rectangular hyperbolas [1], triangle centers associated with a rhombus [6], Euler lines concurrent on the nine-point circle [7], etc.

Among modern configurations we focus our attention on the Lucas circles. Edouard Lucas studied the configuration given by three inscribed squares in a triangle with a side parallel to each one of the sides of the triangle. With them appear three circles. Recently [2], Yiu and Hatzipolakis have been continuing the study of this configuration and, in particular, have shown recently that the Lucas circles are tangent to each other.

Bekanntlich gibt es eine Vielzahl bereits untersuchter Konfigurationen zu einem gegebenen ebenen Dreieck. Der Autor des nachfolgenden Beitrags knüpft an eine von E. Lucas studierte Konfiguration an, die ein Dreieck mit drei eingeschriebenen Quadraten zum Ausgangspunkt hat und zu den sogenannten Lucas-Kreisen führt. Hier wird nun ein Dreieck mit drei eingeschriebenen gleichseitigen Dreiecken, von denen jeweils genau eine Seite zu einer der Seiten des Ausgangsdreiecks parallel ist, untersucht. Bei dieser Konfiguration zeigt sich unter anderem, dass jeweils vier der insgesamt neun Eckpunkte der eingeschriebenen Dreiecke auf drei Kreisen liegen, die sich in einem Punkt schneiden.

In this paper we take on Lucas' idea and consider not the three inscribed squares, but the three inscribed equilateral triangles with one side parallel to each one of the sides of the triangle (see Fig. 1).

The notable configuration which appears is outlined below and will be summarized in the theorem of the following section.

We will show that the centers of these equilateral triangles are aligned (see Fig. 2). We will see also that the mid-points of the sides of the equilateral triangles are aligned three by three (see Fig. 2). We will find that the simple ratios of these mid-points and the simple ratios of the vertices are equal to the simple ratio of the centers. But the most remarkable feature will be the appearance of three circumferences which link the vertices four by four, and these three circles are concurrent in a point (see Figs. 3, 4).

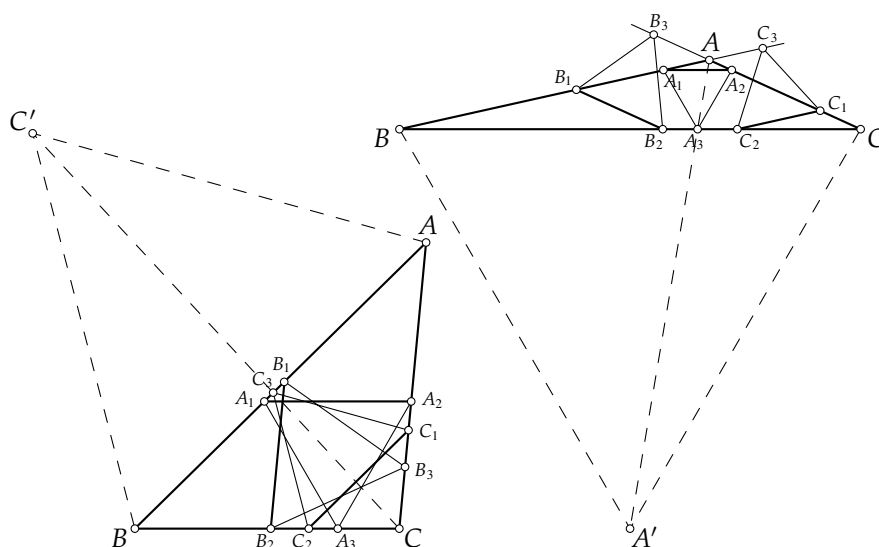


Fig. 1 Construction of the three inscribed equilateral triangles.

2 Configuration

First of all, in order to establish our notations we recall that the “simple ratio” (A, B, C) of three aligned points is $(A, B, C) = \pm \frac{AC}{BC}$ with a positive sign if C is not between A and B, and a negative sign otherwise.

We recall that in order to construct an inscribed equilateral triangle in $\triangle ABC$ with a side parallel to side AB, we consider the exterior equilateral triangle with side AB, $\triangle ABC'$. Let C_3 be the intersection of the line CC' with the line AB. The lines through the point C_3 parallel to AC' and BC' intersect with the lines AC and BC at the points C_1 and C_2 respectively. The triangle $\triangle C_1C_2C_3$ is the inscribed equilateral triangle (see Fig. 1).

The other two triangles $\triangle A_1A_2A_3$ and $\triangle B_1B_2B_3$, with sides parallel to BC and AC respectively, can be constructed in a similar way.

The description of the configuration can be summarized in the following theorem.

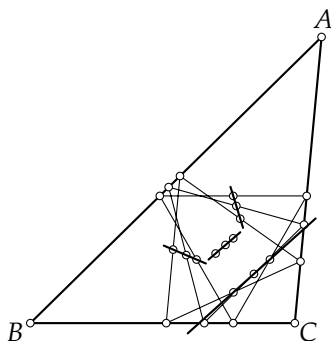


Fig. 2 The centers O_A, O_B, O_C are aligned. The lines AA_3, BB_3, CC_3 are concurrent (outward Fermat point $F = X_{13}$). The mid-points A_{12}, B_{13}, C_{13} are aligned; the mid-points A_{13}, B_{12}, C_{23} are aligned; the mid-points A_{23}, C_{12}, B_{23} are aligned. All the simple ratios coincide with the simple ratio (O_A, O_B, O_C) .

Theorem 1 Let $\triangle ABC$ be a triangle. Let $\triangle A_1A_2A_3, \triangle B_1B_2B_3, \triangle C_1C_2C_3$ be the three inscribed equilateral triangles with A_1A_2 parallel to BC , B_1B_2 parallel to AC , C_1C_2 parallel to AB and A_1, B_1, C_3 on the line AB .

Let O_A, O_B, O_C be the respective centers of the three equilateral triangles, and let A_{ij}, B_{ij}, C_{ij} be the respective mid-points of the sides A_iA_j, B_iB_j, C_iC_j .

Then we have:

1. The center points O_A, O_B, O_C are aligned.
2. The lines AA_3, BB_3, CC_3 are concurrent (in the outward Fermat point).
3. The mid-points A_{12}, B_{13}, C_{13} are aligned, the mid-points A_{13}, B_{12}, C_{23} are aligned and the mid-points A_{23}, C_{12}, B_{23} are also aligned.
4. The following simple ratios are equal:

$$\begin{aligned} (O_A, O_B, O_C) &= (A_2, B_3, C_1) = (A_1, B_1, C_3) = (A_3, B_2, C_2) \\ &= (A_{12}, B_{13}, C_{13}) = (A_{13}, B_{12}, C_{23}) = (A_{23}, B_{23}, C_{12}). \end{aligned}$$

5. The points A_3, C_2, A_1, C_3 are concyclic, the points A_3, B_2, A_2, B_3 are concyclic and the points C_3, B_1, C_1, B_3 are also concyclic.
6. The above circumferences $A_3C_2A_1C_3, A_3B_2A_2B_3, C_3B_1C_1B_3$ are concurrent in a point η .

Proof. Take a system of Cartesian coordinates in such a way that

$$A = (a, b) \text{ with } 0 \leq a, b \leq 1, B = (0, 0), C = (1, 0).$$

A straightforward computation proves the parts 1 and 3 of the theorem (see Fig. 2).

Part 2 of the theorem is well-known. The concurrence of the lines AA_3, BB_3, CC_3 is the outward Fermat point F (X_{13} in [4]).

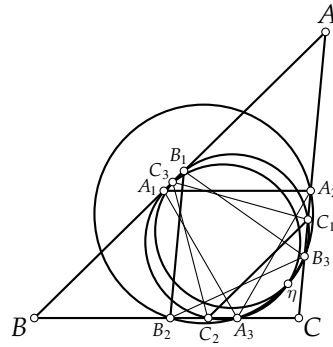


Fig. 3 The points A_3, C_2, A_1 and C_3 are concyclic, the points A_3, B_2, A_2 and B_3 are concyclic and the points C_3, B_1, C_1 and B_3 are concyclic. Also, the three circumferences are concurrent at the point η .

Part 4 can be found with a straightforward computation. The value of the simple ratio obtaining in every case is the same expression:

$$(O_A, O_B, O_C) = \frac{-1}{\sqrt{3}} \frac{(3a - \sqrt{3}b)(3a^2 - 6a + 3b^2 + 2\sqrt{3}b + 3)}{(2b + \sqrt{3})(3a^2 - 3a + 3b^2 - \sqrt{3}b)}.$$

Part 5 (see Figs. 3, 4) can be found with a long computation and we have that the points A_3, C_2, A_1, C_3 belong to the circumference of center O_{AC} given by

$$O_{ACx} = \frac{1}{2} \sqrt{3} \frac{3a^3 + 3\sqrt{3}a^2b + 3ab^2 + 3a^2 + 2\sqrt{3}ab + 3\sqrt{3}b^3 + 5b^2}{(3a^2 + 3b^2 + 2\sqrt{3}b)(2b + \sqrt{3})},$$

$$O_{ACy} = -\frac{1}{2} \frac{3a^3 - 5\sqrt{3}a^2b + 3ab^2 - 3a^2 + 2\sqrt{3}ab - 5\sqrt{3}b^3 - 9b^2}{(3a^2 + 3b^2 + 2\sqrt{3}b)(2b + \sqrt{3})},$$

and radius r_{AC} :

$$\sqrt{3 \frac{(a^2+b^2)(3a^4 - 4\sqrt{3}a^3b + 10a^2b^2 - 4\sqrt{3}ab^3 - 6a^3 + 8\sqrt{3}a^2b - 14ab^2 + 3a^2 - 4\sqrt{3}ab + 7b^4 + 8\sqrt{3}b^3 + 7b^2)}{(3a^2 + 3b^2 + 2\sqrt{3}b)^2(2b + \sqrt{3})^2}}.$$

Analogously, the points A_3, B_2, A_2, B_3 belong to the circumference of center O_{AB} given by:

$$O_{ABx} = \frac{1}{2} \sqrt{3} \frac{3a^3 + \sqrt{3}a^2b + 3ab^2 - 6a^2 + 3a + \sqrt{3}b^3 + 6b^2 + 3\sqrt{3}b}{(3a^2 - 6a + 3b^2 + 2\sqrt{3}b + 3)(2b + \sqrt{3})},$$

$$O_{ABy} = \frac{1}{2} \frac{3a^3 + 5\sqrt{3}a^2b + 3ab^2 - 6a^2 - 8\sqrt{3}ab + 3a + 5\sqrt{3}b^3 + 6b^2 + 3\sqrt{3}b}{(3a^2 - 6a + 3b^2 + 2\sqrt{3}b + 3)(2b + \sqrt{3})},$$

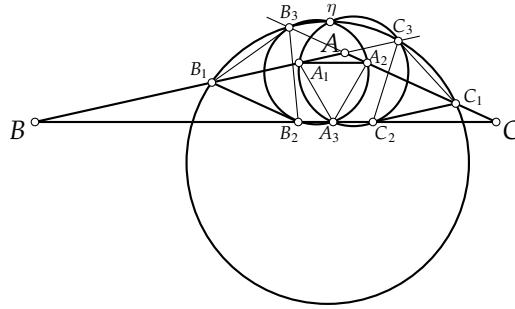


Fig. 4 Concurrency of the three circumferences.

and radius r_{AB} :

$$\sqrt{3 \frac{(a^2 - 2a + b^2 + 1)(3a^4 + 4\sqrt{3}a^3b + 10a^2b^2 + 4\sqrt{3}ab^3 - 6a^3 - 4\sqrt{3}a^2b - 6ab^2 + 3a^2 + 7b^4 + 4\sqrt{3}b^3 + 3b^2)}{(3a^2 - 6a + 3b^2 + 2\sqrt{3}b + 3)^2 (2b + \sqrt{3})^2}}.$$

Analogously the points B_1, C_3, B_3, C_1 belong to the circumference of center O_{BC} given by:

$$O_{BCx} = \frac{3}{2} \frac{3a^4 + 4\sqrt{3}a^3b + 6a^2b^2 + 4\sqrt{3}ab^3 - 6a^3 - 2\sqrt{3}a^2b - 2ab^2 + 3a^2 + 2\sqrt{3}ab + 3b^4 + 2\sqrt{3}b^3 + 5b^2}{(3a^2 + 2\sqrt{3}b + 3b^2)(3a^2 - 6a + 3 + 2\sqrt{3}b + 3b^2)},$$

$$O_{BCy} = -\frac{1}{2} \frac{\sqrt{3} (3a^2 - 3a + 3b^2 + \sqrt{3}b) (a - \sqrt{3}b - 1) (\sqrt{3}b + a)}{(3a^2 + 3b^2 + 2\sqrt{3}b) (3a^2 - 6a + 3b^2 + 2\sqrt{3}b + 3)},$$

and radius r_{BC} :

$$\sqrt{9 \frac{(a^2 + b^2)(a^2 - 2a + b^2 + 1)(3a^4 + 6a^2b^2 - 6a^3 - 4\sqrt{3}a^2b - 6ab^2 + 3a^2 + 4\sqrt{3}ab + 3b^4 + 4\sqrt{3}b^3 + 7b^2)}{(3a^2 + 3b^2 + 2\sqrt{3}b)^2 (3a^2 - 6a + 3b^2 + 2\sqrt{3}b + 3)^2}}.$$

Finally, part 6 corresponds to the above circumferences

$$A_3C_2A_1C_3, \quad A_3B_2A_2B_3, \quad C_3B_1C_1B_3$$

are concurrent in a point η (see Figs. 3, 4). To prove this statement let η be the image of point A_3 under the axial symmetry with respect to the line $O_{AB}O_{AC}$. We obtain

$$\eta = \left(\frac{1}{2} \frac{\varsigma}{\tau}, -\frac{1}{6} \sqrt{3} \frac{\nu}{\phi} \right)$$

with

$$\begin{aligned} \varsigma &= 3a^6 + 8\sqrt{3}a^5b + 9a^4b^2 + 16\sqrt{3}a^3b^3 + 9a^2b^4 + 8\sqrt{3}ab^5 - 3a^5 \\ &\quad - 11\sqrt{3}a^4b - 6a^3b^2 - 14\sqrt{3}a^2b^3 - 3ab^4 - 3a^4 - 2\sqrt{3}a^3b \\ &\quad - 10\sqrt{3}ab^3 + 3a^3 + 5\sqrt{3}a^2b + 3ab^2 + 3b^6 - 3\sqrt{3}b^5 - 5b^4 + 5\sqrt{3}b^3, \\ \tau &= 3a^6 + 9a^4b^2 + 9a^2b^4 + 9\sqrt{3}a^4b - 9a^5 - 18a^3b^2 + 10\sqrt{3}a^2b^3 \\ &\quad - 9ab^4 + 12a^4 - 18\sqrt{3}a^3b + 18a^2b^2 - 10\sqrt{3}ab^3 - 9a^3 + 9\sqrt{3}a^2b \\ &\quad + 3a^2 - 9ab^2 + 3b^6 + \sqrt{3}b^5 - 2b^4 + \sqrt{3}b^3 + 3b^2, \\ v &= (3a^2 - 3a + 3b^2 - \sqrt{3}b) (3a^4 - 2a^2b^2 - 6a^3 + 2ab^2 + 3a^2 - 5b^4 + 3b^2), \\ \phi &= 3a^6 + 9a^4b^2 + 9a^2b^4 - 9a^5 + 9\sqrt{3}a^4b - 18a^3b^2 + 10\sqrt{3}a^2b^3 - 9ab^4 \\ &\quad + 12a^4 - 18\sqrt{3}a^3b + 18a^2b^2 - 10\sqrt{3}ab^3 - 9a^3 - 9ab^2 + 9\sqrt{3}a^2b + 3a^2 \\ &\quad + 3b^6 + \sqrt{3}b^5 - 2b^4 + \sqrt{3}b^3 + 3b^2. \end{aligned}$$

This point η , by construction, belongs to the two circumferences $A_3C_2A_1C_3$, $A_3B_2A_2B_3$. It only remains to prove that η belongs to the circumference $C_3B_1C_1B_3$. But when we compute the distance between η and O_{BC} we obtain exactly the same expression obtained above for r_{BC} .

This completes the proof. \square

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