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Lifting Theorem as a Special Case of Abstract Interpolation Problem

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Abstract. Using properties of the de Branges-Rovnyak spaces we include the classical lifting problem into the general scheme of the abstract interpolation problem.

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0. Introduction

This note describes the inclusion of the lifting problem [5] into the general scheme of the abstract interpolation problem [2, 3]. We will simultaneously refer to the Sz.Nagy-Foiaş and de Branges-Rovnyak functional models. The use of these two models of the same object is explained by the fact that we join together "a priori" independent results and we prefer to keep to the way they were originally stated. We would like to note that one can find a unified approach to functional models in [4, 7].

Let us briefly recall the Sz.Nagy-Foiaş construction. Let E_1 and E_2 be Hilbert spaces, θ be an $[E_1, E_2]$-valued analytic contractive function on the unit disk $\mathbb{D}$. Let us define an $[E_1, E_1]$-valued function $\Delta = (1 - \theta^*\theta)^{\frac{1}{2}}$ on the unit circle $\mathbb{T}$, and let us compose the operator-valued function

$$\Phi = \begin{bmatrix} \theta \\ \Delta \end{bmatrix} : \; E_1 \longrightarrow \begin{bmatrix} E_2 \\ E_1 \end{bmatrix}.$$

We define the space $\mathfrak{K} = \left[\frac{H^2(E_2)}{\Delta L^2(E_1)}\right]$ and its subspace

$$K_\theta = \mathfrak{K} \ominus \begin{bmatrix} \theta \\ \Delta \end{bmatrix} H^2(E_1) = \begin{bmatrix} \dfrac{H^2(E_2)}{\Delta L^2(E_1)} \end{bmatrix} \ominus \begin{bmatrix} \theta \\ \Delta \end{bmatrix} H^2(E_1). \tag{0.1}$$

We will denote by P_θ the operator of orthogonal projection from $\mathfrak{K}$ to K_θ. The mapping T, defined on K_θ by the formula

$$Tz = P_\theta tz \qquad (z \in K_\theta) \tag{0.2}$$

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is a completely non-unitary contraction [6: Chapter 6, Section 3]. Conversely, for given a completely non-unitary contraction one can choose Hilbert spaces E_1 and E_2 and an $[E_1, E_2]$-valued analytic contractive function θ such that the operator T, defined by formula (0.2), is unitarily equivalent to the original contraction [6: Chapter 6, Section 2].

Let us pass now to the description of the de Branges-Rovnyak model [1]. We define the matrix-valued function

$$\Sigma_\theta = \begin{bmatrix} 1_{E_2} & \theta \\ \theta^* & 1_{E_1} \end{bmatrix}^{\frac{1}{2}} : \begin{bmatrix} E_2 \\ E_1 \end{bmatrix} \longrightarrow \begin{bmatrix} E_2 \\ E_1 \end{bmatrix}.$$

Let us consider the subspace

$$L^2_\theta = \overline{\Sigma_\theta L^2} = \operatorname{clos}\{\Sigma_\theta f : f \in L^2\}$$

of the Hilbert space $L^2 = \left[\begin{smallmatrix} L^2(E_2) \\ L^2(E_1) \end{smallmatrix}\right]$. The subspaces

$$\left\{\Sigma_\theta \begin{bmatrix} H^2_-(E_2) \\ 0 \end{bmatrix}\right\} \qquad \text{and} \qquad \left\{\Sigma_\theta \begin{bmatrix} 0 \\ H^2(E_1) \end{bmatrix}\right\}$$

are isometrical embeddings of the spaces $H^2_-(E_2)$ and $H^2(E_1)$ into L^2_θ. We define the de Branges-Rovnyak space as the orthogonal complement

$$H_\theta = L^2_\theta \ominus \left\{\Sigma_\theta \begin{bmatrix} H^2_-(E_2) \\ 0 \end{bmatrix} \oplus \Sigma_\theta \begin{bmatrix} 0 \\ H^2(E_1) \end{bmatrix}\right\}$$

of the direct sum of these spaces. In this functional model a completely non-unitary contraction $\tilde{T}$ is represented as

$$\tilde{T}z = \tilde{P}_\theta t z \qquad (z \in H_\theta)$$

where $\tilde{P}_\theta$ denotes the operator of orthogonal projection from L^2_θ to H_θ.

Remark 0.1 A vector $f = \left[\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right] \in L^2_\theta$ lies in the space H_θ if and only if the vector $\left[\begin{smallmatrix} f_+ \\ f_- \end{smallmatrix}\right] := \Sigma_\theta f$ belongs to $\left[\begin{smallmatrix} H^2(E_2) \\ H^2_-(E_1) \end{smallmatrix}\right]$. We will be keeping this system of notations throughout the paper.

The spaces K_θ and H_θ, defined above, are isometrically isomorphic. An isomorphism, for example, may be given by the mapping

$$f \longrightarrow \begin{bmatrix} 1_{E_2} & 0 \\ \theta^* & \Delta \end{bmatrix}^{[-1]} \Sigma f \in K_\theta \qquad (f \in H_\theta).$$

1. Formulation of the lifting problem and the abstract interpolation problem

Let $\mathfrak{U}, \mathfrak{U}_*$ and $\mathfrak{U}', \mathfrak{U}'_*$ be separable Hilbert spaces, and let $1, 1_*$ and $1', 1'_*$ be the identity operators, defined on these spaces, respectively. Let θ and θ' be $[\mathfrak{U}, \mathfrak{U}_*]$- and $[\mathfrak{U}', \mathfrak{U}'_*]$-valued pure analytic contractive functions, defined on the unit disk $\mathbb{D}$. Assigning $E_1 = \mathfrak{U}, E_2 = \mathfrak{U}_*$ in formula (0.1), and then $E_1 = \mathfrak{U}', E_2 = \mathfrak{U}'_*$ and and $\theta = \theta'$, we construct the spaces

$$K_\theta = \begin{bmatrix} H^2(\mathfrak{U}_*) \\ \overline{\Delta L^2(\mathfrak{U})} \end{bmatrix} \ominus \begin{bmatrix} \theta \\ \Delta \end{bmatrix} H^2(\mathfrak{U}) \subset \mathfrak{K} = \begin{bmatrix} H^2(\mathfrak{U}_*) \\ \overline{\Delta L^2(\mathfrak{U})} \end{bmatrix}$$

and

$$K_{\theta'} = \begin{bmatrix} H^2(\mathfrak{U}'_*) \\ \overline{\Delta' L^2(\mathfrak{U}')} \end{bmatrix} \ominus \begin{bmatrix} \theta' \\ \Delta' \end{bmatrix} H^2(\mathfrak{U}') \subset \mathfrak{K}' = \begin{bmatrix} H^2(\mathfrak{U}'_*) \\ \overline{\Delta' L^2(\mathfrak{U}')} \end{bmatrix}$$

where $\Delta = (1 - \theta^*\theta)^{\frac{1}{2}}$ and $\Delta' = (1' - \theta'^*\theta')^{\frac{1}{2}}$. We denote by I and I' the identity operators on the spaces $\mathfrak{K}$ and $\mathfrak{K}'$, respectively.

Let us define the contractions in K_θ and $K_{\theta'}$ in analogy to (0.2) by the formulas

$$Tz = P_\theta tz \quad (z \in K_\theta) \qquad \text{and} \qquad T'z' = P_{\theta'} tz' \quad (z' \in K_{\theta'}). \tag{1.1}$$

We consider an operator X with $\|X\| \le 1$, acting from $K_{\theta'}$ into K_θ and intertwining the contractions T' and T

$$XT' = TX. \tag{1.2}$$

The *lifting problem* (further we will use the abbreviation "(L)- problem") consists in describing all operators $Y : \mathfrak{K}' \to \mathfrak{K}$, possessing the following properties:

$$X = P_\theta Y|_{K_{\theta'}} \tag{1.3}$$

$$Y\Phi' H^2(\mathfrak{U}') \subset \Phi H^2(\mathfrak{U}) \tag{1.4}$$

$$\|Y\| \le 1 \tag{1.5}$$

$$Y(tz') = t(Yz') \quad \text{for all } z' \in \mathfrak{K}'. \tag{1.6}$$

We will call an arbitrary operator Y, satisfying the conditions (1.3) - (1.6), a *lifting* of the operator X.

Now we are turning to the formulation of the abstract interpolation problem [2, 3] (further we will use the abbreviation "(AI)-problem"). Let

K be a linear space

E_1, E_2 be separable Hilbert spaces

T_1, T_2 be linear operators defined on K

M_1, M_2 be linear maps $M_1 : K \to E_1$ and $M_2 : K \to E_2$

D be a non-negative quadratic form defined on K.

The collection $\{T_1, T_2; M_1, M_2; D\}$ is called the *interpolation data*. The interpolation data must satisfy the identity

$$D(T_2x, T_2y) - D(T_1x, T_1y) = (M_1x, M_1y)_{E_1} - (M_2x, M_2y)_{E_2} \tag{1.7}$$

for all $x, y \in K$. The solutions of the (AI)-problem are defined as a set of pairs (ω, F), where ω is an $[E_1, E_2]$-valued contractive function, analytic in the unit disk $\mathbb{D}$, and F is a map from K into H_ω, having the properties

$$\begin{bmatrix} F_+T_1x \\ F_-T_1x \end{bmatrix} = t\begin{bmatrix} F_+T_2x \\ F_-T_2x \end{bmatrix} - \begin{bmatrix} 1_{E_2} & \omega \\ \omega^* & 1_{E_1} \end{bmatrix}\begin{bmatrix} -M_2x \\ M_1x \end{bmatrix} \tag{1.8}$$

and

$$\|Fx\|^2 \le D(x,x) \tag{1.9}$$

for all $x \in K$ where $\left[\begin{smallmatrix} F_+x \\ F_-x \end{smallmatrix}\right] = \Sigma_\omega Fx$. The map F, defined above, is called the *Fourier representation* of the space K.

2. Inclusion of the (L)-problem into the scheme of the (AI)-problem

In this section we will need an explicit formula for the orthoprojector $P_{\theta'} : \mathfrak{K}' \to K_{\theta'}$ defined by

$$P_{\theta'}z' = z' - \Phi' P'_+ \Phi'^* z' \qquad (z' \in \mathfrak{K}') \tag{2.1}$$

(see [8]). We have denoted by P'_+ the operator of orthogonal projection from $L^2(\mathfrak{U}')$ to $H^2(\mathfrak{U}')$. In particular, formula (2.1) shows that a vector $z' \in \mathfrak{K}'$ belongs to the space $K_{\theta'}$ if and only if $\Phi'^* z' \in H^2_-(\mathfrak{U}')$.

Let us compute the operator T' defined by (1.1) with the help of (2.1) in the form

$$T'z' = tz' - \Phi' P'_+ t\Phi'^* z' \qquad (z' \in K_{\theta'}). \tag{2.2}$$

The commutative relation (1.2), summed up with the formulas for the operators T' and T, allow us to establish an identity, similar to (1.7). By means of (2.2) let us calculate the vectors $T'x'$ and $T'y'$ for $x', y' \in K_{\theta'}$ and their scalar product as

$$(T'x', T'y') = (x', y') - (\Phi' P'_+ t\Phi'^* x', ty') \\ - (tx', \Phi' P_+ t\Phi'^* y') + (\Phi' P'_+ t\Phi'^* x', \Phi' P_+ t\Phi'^* y').$$

We transform the second term at the right side of that equality as

$$(\Phi' P'_+ t\Phi'^* x', ty') = (P'_+ t\Phi'^* x', t\Phi'^* y') = (P'_+ t\Phi'^* x', P'_+ t\Phi'^* y').$$

Doing the same to the third summand and observing that $\Phi'^*\Phi' = 1'$, we get

$$(T'x', T'y') = (x', y') - (P'_+ t\Phi'^* x', P'_+ t\Phi'^* y'). \tag{2.3}$$

Using identity (1.2) and formula (1.1) for the operator T, we obtain

$$XT'z' = TXz' = tXz' - \Phi P_+ t\Phi^* Xz' \qquad (z' \in K_{\theta'}). \tag{2.4}$$

Let us compute the vectors $XT'x'$ and $XT'y'$ for $x', y' \in K_{\theta'}$ and their scalar product with the help of (2.4) as

$$(XT'x', XT'y') = (Xx', Xy') - (P_+t\Phi^* Xx', P_+t\Phi^* Xy'). \tag{2.5}$$

Subtracting equality (2.5) from equality (2.3) we get

$$\begin{aligned}((I - X^*X)x', y') &- ((I - X^*X)T'x', T'y') \\ &= \left(P'_+t\Phi'^* x', P'_+t\Phi'^* y'\right)_{\mathfrak{U}'} - \left(P_+t\Phi^* Xx', P_+t\Phi^* Xy'\right)_{\mathfrak{U}}.\end{aligned} \tag{2.6}$$

The non-negativity of the form $\left((I - X^*X)x', y'\right)$ $(x', y' \in K_{\theta'})$ follows from the inequality $\|X\| \leq 1$.

We assign

$$\left.\begin{aligned} K &= K_{\theta'} \\ E_1 &= \mathfrak{U}',\ E_2 = \mathfrak{U} \\ M_1 z' &= P'_+t\Phi'^* z' : K_{\theta'} \to \mathfrak{U}',\ M_2 z' = P_+t\Phi^* Xz' : K_{\theta'} \to \mathfrak{U} \\ T_1 &= T',\ T_2 = id \ \text{ on } \ K_{\theta'} \\ D &= I - X^*X. \end{aligned}\right\} \tag{2.7}$$

Thus, we find ourselves in the conditions of the (AI)-problem. The new notations convert identity (2.6) into (1.7).

The set of the solutions of the (AI)-problem is formed by pairs (ω, F), where ω is a $[\mathfrak{U}', \mathfrak{U}]$-valued analytic contractive function and F is a Fourier representation of the space $K_{\theta'}$ with values in the de Branges-Rovnyak space H_ω. Each pair (ω, F) must satisfy the conditions

$$\begin{bmatrix} F_+T'x' \\ F_-T'x' \end{bmatrix} = t \begin{bmatrix} F_+x' \\ F_-x' \end{bmatrix} - \begin{bmatrix} 1 & \omega \\ \omega^* & 1' \end{bmatrix} \begin{bmatrix} -P_+t\Phi^* Xx' \\ P'_+t\Phi'^* x' \end{bmatrix} \tag{2.8}$$

and

$$\|Fx'\|^2 \leq \left((I - X^*X)x', x'\right) \tag{2.9}$$

for all $x' \in K_{\theta'}$.

An (AI)-problem with interpolation data (2.7) will be called (L')- *problem*.

3. One-to-one correspondence between solutions of the (L)- and (L′)- problems

Further we write Σ instead of Σ_ω. We will apply the following lemma during the proof of Proposition 3.1.

Lemma 3.1. *Let $f \in H_\omega$. Then*

$$\|f_+ + \omega h\|^2_{L^2(\mathfrak{U})} \le \|f\|^2 + \|h\|^2_{L^2(\mathfrak{U}')} \tag{3.1}$$

for all $h \in H^2(\mathfrak{U}')$.

Proof. By virtue of the inequality

$$\begin{bmatrix} 1 & \omega \\ \omega^* & 1' \end{bmatrix} \ge \begin{bmatrix} 1 & \omega \\ \omega^* & \omega^*\omega \end{bmatrix} = \begin{bmatrix} 1 & \omega \\ \omega^* & 1' \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & \omega \\ \omega^* & 1' \end{bmatrix}$$

for an arbitrary vector $g = \Sigma\tilde{g} \in L^2_\omega$ we have

$$\begin{aligned} \|g\|^2 &= (\Sigma\tilde{g}, \Sigma\tilde{g}) \\ &= \left(\begin{bmatrix} 1 & \omega \\ \omega^* & 1' \end{bmatrix}\tilde{g}, \tilde{g}\right) \\ &\ge \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & \omega \\ \omega^* & 1' \end{bmatrix}\tilde{g}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & \omega \\ \omega^* & 1' \end{bmatrix}\tilde{g}\right) \\ &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\Sigma g, \Sigma g\right). \end{aligned}$$

In particular, if we set $g = f + \Sigma\left[\begin{smallmatrix} 0 \\ h \end{smallmatrix}\right]$ with $f \in H_\omega$ and $h \in H^2(\mathfrak{U}')$, we obtain

$$\begin{aligned} &\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\left(\Sigma f + \begin{bmatrix} 1 & \omega \\ \omega^* & 1' \end{bmatrix}\begin{bmatrix} 0 \\ h \end{bmatrix}\right), \left(\Sigma f + \begin{bmatrix} 1 & \omega \\ \omega^* & 1' \end{bmatrix}\begin{bmatrix} 0 \\ h \end{bmatrix}\right)\right) \\ &\qquad = \left(\begin{bmatrix} f_+ + \omega h \\ 0 \end{bmatrix}, \begin{bmatrix} f_+ + \omega h \\ 0 \end{bmatrix}\right) = \|f_+ + \omega h\|^2_{L^2(\mathfrak{U})}. \end{aligned}$$

On the other hand, by definition of the de Branges-Rovnyak space,

$$\begin{aligned} \|g\|^2 &= \left(f + \Sigma\begin{bmatrix} 0 \\ h \end{bmatrix}, f + \Sigma\begin{bmatrix} 0 \\ h \end{bmatrix}\right) \\ &= (f, f) + \left(\begin{bmatrix} 1 & \omega \\ \omega^* & 1' \end{bmatrix}\begin{bmatrix} 0 \\ h \end{bmatrix}, \begin{bmatrix} 0 \\ h \end{bmatrix}\right) \\ &= \|f\|^2 + \|h\|^2_{L^2(\mathfrak{U}')} \end{aligned}$$

and the assertion is proved ∎

The following proposition associates a solution of the (L)-problem with such one of the (L′)-problem.

Proposition 3.1. *Let* (ω, F) *be a solution of the* (L′)*-problem. Then the map* $Y : \mathfrak{K}' \to \mathfrak{K}$ *given by the equalities*

$$Y|_{K_{\theta'}} = X + \Phi F_+ \qquad \text{and} \qquad Y|_{\Phi' H^2(\mathfrak{U}')} = \Phi\omega|_{H^2(\mathfrak{U}')} \tag{3.2}$$

is a lifting of the operator X.

Proof. Let us show that the operator Y, defined above, is a contraction. According to the definition of the space $K_{\theta'}$, we decompose an arbitrary vector z' from $\mathfrak{R}'$ as $z' = k' + \Phi' h'$, where $k' \in K_{\theta'}$ and $h' \in H^2(\mathfrak{U}')$. By definition (3.2) we have

$$\begin{aligned} \|Yz'\|^2 &= \|Y(k' + \Phi' h')\|^2 \\ &= \|Xk' + \Phi(F_+ k' + \omega h')\|^2 \\ &= \|Xk'\|^2 + \|F_+ k' + \omega h'\|^2_{L^2(\mathfrak{U})}. \end{aligned}$$

Using Lemma 3.1, we continue the above equalities by the inequality

$$\|Xk'\|^2 + \|F_+ k' + \omega h'\|^2_{L^2(\mathfrak{U})} \le \|Xk'\|^2 + \|Fk'\|^2 + \|h'\|^2_{L^2(\mathfrak{U}')}.$$

It follows immediately from (2.9) that

$$\|Xk'\|^2 + \|Fk'\|^2 \le \|k'\|^2$$

for any $k' \in K_{\theta'}$. Hence, we continue as

$$\|Xk'\|^2 + \|Fk'\|^2 + \|h'\|^2_{L^2(\mathfrak{U}')} \le \|k'\|^2 + \|h'\|^2_{L^2(\mathfrak{U}')} = \|k' + \Phi' h'\|^2 = \|z'\|^2.$$

Further, we verify that, indeed, the operator Y is defined as multiplication by a certain matrix-valued function. In order to prove this, it is sufficient to demonstrate that

$$Y\{tz'\} = t\{Yz'\}$$

for all $z' \in \mathfrak{R}'$. Firstly, comparing formula (2.2) with definition of M_1 at (2.7) we see that

$$tk' = T'k' + \Phi' M_1 k'$$

for all $k' \in K_{\theta'}$. With the help of this observation we start from the left side of the preceding equality

$$Y\{tz'\} = Y\{t(k' + \Phi' h')\} = Y\{T'k' + \Phi'(M_1 k' + th')\}.$$

Formula (3.2) allows us to pass from the operator Y to the maps X and F_+

$$Y\{T'k' + \Phi'(M_1 k' + th')\} = (X + \Phi F_+)T'k' + \Phi\omega(M_1 k' + th').$$

The commutative relations (2.8) and (2.4) give us

$$\begin{aligned} &(X + \Phi F_+)T'k' + \Phi\omega(M_1 k' + th') \\ &\quad = (tXk' - \Phi M_2 k') + \Phi(tF_+ k' + M_2 k' - \omega M_1 k') + \Phi\omega(M_1 k' + th') \\ &\quad = t(Xk' + \Phi F_+)k' + t\Phi\omega h'. \end{aligned}$$

Returning to the operator Y, we obtain

$$t(Xk' + \Phi F_+)k' + t\Phi\omega h' = tY\{k' + \Phi' h'\} = t\{Yz'\}.$$

To complete the proof we should observe that relations (1.3) and (1.4) are fulfilled in a trivial way ∎

Remark 3.1 (see [5: Section 7]). The matrix of the operator Y, defined in the preceding proposition, admits the block decomposition

$$Y(t) = \begin{bmatrix} A(t) & 0 \\ B(t) & C(t) \end{bmatrix} : \begin{bmatrix} H^2(\mathfrak{U}'_*) \\ \overline{\Delta' L^2(\mathfrak{U}')} \end{bmatrix} \longrightarrow \begin{bmatrix} H^2(\mathfrak{U}_*) \\ \overline{\Delta L^2(\mathfrak{U})} \end{bmatrix}$$

where the blocks $A(t), B(t)$ and $C(t)$ have the following properties:

(i) $A(t)$ is a $[\mathfrak{U}'_*, \mathfrak{U}_*]$-valued bounded analytic function.

(ii) $B(t)$ and $C(t)$ are respectively $[\mathfrak{U}'_*, \mathfrak{U}]$- and $[\mathfrak{U}', \mathfrak{U}]$-valued measurable bounded functions, satisfying a.e. on $\mathbb{T}$ the conditions

$$B(t)\mathfrak{U}'_* \subset \overline{\Delta(t)\mathfrak{U}} \qquad \text{and} \qquad C(t)\Delta'(t)\mathfrak{U}' \subset \overline{\Delta(t)\mathfrak{U}}.$$

(iii) $Y(t)\Phi'(t) = \Phi(t)A_0(t)$ a.e. on $\mathbb{T}$, where $A_0(t)$ is a $[\mathfrak{U}', \mathfrak{U}]$-valued bounded analytic function.

The following Proposition 3.2 is, in some sense, converse to Proposition 3.1.

Proposition 3.2. *Let the operator Y be a lifting of X. Then the pair (ω, F) defined as*

$$\omega = \Phi^* Y \Phi' \tag{3.3}$$

and

$$\begin{bmatrix} F_+ k' \\ F_- k' \end{bmatrix} = \begin{bmatrix} \Phi^* & 0 \\ 0 & \Phi'^* \end{bmatrix} \begin{bmatrix} I & Y \\ Y^* & I' \end{bmatrix} \begin{bmatrix} -Xk' \\ k' \end{bmatrix} \tag{3.4}$$

for $k' \in K_{\theta'}$ sets a solution of the (L′)*-problem.*

Proof. Let us verify that

$$\begin{bmatrix} F_+ k' \\ F_- k' \end{bmatrix} \in \begin{bmatrix} H^2(\mathfrak{U}) \\ H^2_-(\mathfrak{U}') \end{bmatrix}.$$

Indeed,

$$Xk' = P_\theta Y k' = Yk' - \Phi P_+ \Phi^* Y k' \qquad (k' \in K_{\theta'})$$

and, according to formula (3.4), we have

$$\begin{bmatrix} F_+ k' \\ F_- k' \end{bmatrix} = \begin{bmatrix} P_+ \Phi^* Y k' \\ \Phi'^* k' - \omega^* \Phi^* X k' \end{bmatrix}.$$

Hence, the first component $F_+ k'$ lies in $H^2(\mathfrak{U})$, and the second one belongs to $H^2_-(\mathfrak{U}')$.

Let us prove that the map (3.4) satisfies the inequality

$$\|Fk'\|^2 \le (Dk', k') \tag{3.5}$$

for all $k' \in K_{\theta'}$. Let $g \in L^2(\mathfrak{U})$ and $g' \in L^2(\mathfrak{U}')$. Let us consider the scalar product

$$\left\langle \begin{bmatrix} I & Y \\ Y^* & I' \end{bmatrix} \begin{bmatrix} -Xk' \\ k' \end{bmatrix}, \begin{bmatrix} \Phi g \\ \Phi' g' \end{bmatrix} \right\rangle. \tag{3.6}$$

We are going to estimate the square of the modulus of this scalar product by means of the Cauchy-Schwarz-Bunyakovskii inequality:

$$\left|\left\langle \begin{bmatrix} I & Y \\ Y^* & I' \end{bmatrix} \begin{bmatrix} -Xk' \\ k' \end{bmatrix}, \begin{bmatrix} \Phi g \\ \Phi' g' \end{bmatrix} \right\rangle\right|^2 \\ \leq \left\langle \begin{bmatrix} I & Y \\ Y^* & I' \end{bmatrix} \begin{bmatrix} -Xk' \\ k' \end{bmatrix}, \begin{bmatrix} -Xk' \\ k' \end{bmatrix} \right\rangle \left\langle \begin{bmatrix} I & Y \\ Y^* & I' \end{bmatrix} \begin{bmatrix} \Phi g \\ \Phi' g' \end{bmatrix}, \begin{bmatrix} \Phi g \\ \Phi' g' \end{bmatrix} \right\rangle . \tag{3.7}$$

Let us compute the scalar products, composing the preceding inequality:

$$\left\langle \begin{bmatrix} I & Y \\ Y^* & I' \end{bmatrix} \begin{bmatrix} -Xk' \\ k' \end{bmatrix}, \begin{bmatrix} \Phi g \\ \Phi' g' \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} F_+ k' \\ F_- k' \end{bmatrix}, \begin{bmatrix} g \\ g' \end{bmatrix} \right\rangle \tag{3.8}$$

and

$$\left\langle \begin{bmatrix} I & Y \\ Y^* & I' \end{bmatrix} \begin{bmatrix} -Xk' \\ k' \end{bmatrix}, \begin{bmatrix} -Xk' \\ k' \end{bmatrix} \right\rangle = (Dk', k')$$

$$\left\langle \begin{bmatrix} I & Y \\ Y^* & I' \end{bmatrix} \begin{bmatrix} \Phi g \\ \Phi' g' \end{bmatrix}, \begin{bmatrix} \Phi g \\ \Phi' g' \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 1 & \omega \\ \omega^* & 1' \end{bmatrix} \begin{bmatrix} g \\ g' \end{bmatrix}, \begin{bmatrix} g \\ g' \end{bmatrix} \right\rangle = \left\langle \Sigma \begin{bmatrix} g \\ g' \end{bmatrix}, \Sigma \begin{bmatrix} g \\ g' \end{bmatrix} \right\rangle .$$

Inequality (3.7) turns into

$$\left|\left\langle \begin{bmatrix} F_+ k' \\ F_- k' \end{bmatrix}, \begin{bmatrix} g \\ g' \end{bmatrix} \right\rangle\right|^2 \leq (Dk', k') \left\langle \Sigma \begin{bmatrix} g \\ g' \end{bmatrix}, \Sigma \begin{bmatrix} g \\ g' \end{bmatrix} \right\rangle .$$

Hence, expression (3.8) sets a bounded linear functional

$$\left\langle \begin{bmatrix} F_+ k' \\ F_- k' \end{bmatrix}, \begin{bmatrix} g \\ g' \end{bmatrix} \right\rangle = \left\langle Fk', \Sigma \begin{bmatrix} g \\ g' \end{bmatrix} \right\rangle ,$$

on L^2_ω and the square of its norm does not exceed (Dk', k'), therefore inequality (3.5) is fulfilled. The fulfillment of the commutative identity (2.8) for the map (3.4) follows immediately from relations (2.2) and (2.4) ∎

The following Proposition 3.3 points out a close link between the mappings, constructed in the two preceding propositions.

Proposition 3.3. *The maps* $(\omega, F) \to Y$ *and* $Y \to (\omega, F)$*, introduced by formulas* (3.2) – (3.4), *are inverse.*

Proof. Let Y be a lifting of the operator X. Then, by virtue of Remark 3.1, $Y\Phi' = \Phi A_0$ where A_0 is a $[\mathfrak{U}', \mathfrak{U}]$-valued bounded analytic function. Further,

$$X = P_\theta Y|_{K_{\theta'}} = (Y - \Phi P_+ \Phi^* Y)|_{K_{\theta'}} .$$

We set the pair (ω, F), associated with the operator Y by formulas (3.3) and (3.4), as

$$\omega = A_0 \qquad \text{and} \qquad \begin{bmatrix} F_+ k' \\ F_- k' \end{bmatrix} = \begin{bmatrix} P_+ \Phi^* Y k' \\ \Phi'^* k' - \omega^* \Phi^* X k' \end{bmatrix} .$$

The lifting $\tilde{Y}$ for the pair (ω, F), constructed by formula (3.2), has the form

$$\tilde{Y}|_{\Phi' H^2(\mathfrak{U}')} = \Phi\omega|_{H^2(\mathfrak{U}')} = Y|_{\Phi' H^2(\mathfrak{U}')}$$

and

$$\tilde{Y}|_{K_{\theta'}} = X + \Phi F_+ = \{(Y - \Phi P_+ \Phi^* Y) + \Phi P_+ \Phi^* Y\}|_{K_{\theta'}} = Y|_{K_{\theta'}}.$$

Let us prove the second part of the statement. We assume that the pair (ω, F) is a solution of the (L$'$)-problem. Using (3.2), we construct the lifting Y, and, next, we associate the pair $(\tilde{\omega}, \tilde{F})$ to this lifting. It follows directly from our reasoning that $\tilde{\omega} = \omega$ and $\tilde{F}_+ = F_+$. It only remains to observe that the difference $F_- - \tilde{F}_-$ satisfies the functional relation

$$(F_- - \tilde{F}_-)(T'k') = t(F_- - \tilde{F}_-)k'$$

for all $k \in K_{\theta'}$. Since the function $(F_- - \tilde{F}_-)k'$ lies in the space $H^2_-(\mathfrak{U}')$, all its negative Fourier coefficients are equal to zero. Thus, we get $(F_- - \tilde{F}_-) \equiv 0$ on $K_{\theta'}$. The proof is completed ∎

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