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Stability of Linear Evolution Equations in Lattice Normed Spaces

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Abstract. Linear evolution equations in a space with a generalized norm are considered. Stability conditions are obtained. In particular, the "freezing" method for ordinary differential equations is extended to equations in Banach spaces.

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1. Introduction and notation

Throughout this paper B is a Banach lattice with a positive cone B^+ and order continuous norm $\|\cdot\|_B$ (see [10, 13]), and $L(B)$ is the space of all bounded linear operators acting in B. Let X be a linear space and $M: X \to B^+$ be a mapping with properties

$$\begin{aligned} &M(x) > 0 \text{ if and only if } x \neq 0 \\ &M(\lambda x) = |\lambda|\, M(x) \text{ for every } \lambda \in \mathbb{C} \text{ and } x \in X \\ &M(x+y) \leq M(x) + M(y) \text{ for every } x, y \in X. \end{aligned} \tag{1.1}$$

Such a mapping was introduced by L. Kantorovich (see [15: p. 334]) who called M *generalized norm*. Since this notion can confuse the reader, we will call a mapping M satisfying properties (1.1) *normalizing*, and X will be called *space with a normalizing mapping*. Following [15], we shall call B *norming lattice*. Note that a space with a normalizing mapping is a particular case of a pseudo-metric space introduced by Kurepa (cf. [2: p. 51]). Clearly, a space X with a normalizing mapping $M: X \to B^+$ is a normed space with norm

$$\|h\|_X = \|M(h)\|_B \qquad (h \in X). \tag{1.2}$$

In the sequel, the topology in the space X is defined by the norm (1.2), and X is assumed to be a Banach space.

Throughout this paper $A(t)$ for $t \geq 0$ is a linear closed operator in X with dense domain $D(A(t))$. Let us consider the equation

$$\dot{u} = A(t)\, u \qquad (t \geq 0) \tag{1.3}$$

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where $\dot{u} = \frac{du}{dt}$. In the present paper two-side solution estimates for the equation (1.3) are derived under some assumption on normalizing mappings. Note that a normalizing mapping gives us much more informations about the equation than a usual (number) norm does. As an application of the estimates, new stability conditions for the equation (1.3) are derived. The estimates complement well-known results (cf. [8] and references therein). In particular, by the normalizing mapping, the "freezing" method for ordinary differential equations [1, 9, 14] is extended to equations in Banach spaces. In addition, our results generalize estimates of Wazewski and Lozinskii from the theory of ordinary differential equations (see [3, 9]). Moreover, main results from [5, 6] (in the linear case) are extended. Note that our stability conditions below are particularly formulated in terms of the Hurwitzness of auxiliary matrices. This fact allows stability criteria for matrices (for example, the Hurwitz criterion) to be applied to infinite dimensional systems.

We define a *solution* of the equation (1.3) to be a function $u : [0, +\infty) \to D(A(t))$ having a strong derivative and satisfying equation (1.3) for all $t > 0$ (cf. [11, 12]). We will say that the equation (1.3) is *stable* if for every $u_0 \in D(A(0))$ it has a solution $u = u(t)$ with $u(0) = u_0$ satisfying the relation

$$\|u(t)\|_X \leq c\,\|u(0)\|_X \qquad (t \geq 0)$$

with a constant c independent of $u(0)$. Our **Assumptions** are as follows:

1. $D(A(t))$ is constant, i.e. $D(A(t)) = D$ for all $t \geq 0$.
2. $A(t)$ is *continuously differentable* in t on D, i.e. $A(t)v$ is strongly continuously differentable for any $v \in D$.
3. $I - \delta A(t)$ is invertible for all $t \geq 0$ and small enough $\delta > 0$.

As for Assumption 1, in our reasoning below the existence of an admissible set of operators $\{A(t)\}_{t \geq 0}$ (see [11, 12]) is sufficient. Here and below $I = I_X$ is the unit operator in X. Further, for a partition

$$0 = t_0^{(n)} < t_1^{(n)} < \ \dots\ < t_n^{(n)} = t$$

of the segment $[0, t]$ let us denote

$$U_{n,k} = \left(I - A(t_n^{(n)})\delta_n\right)^{-1}\left(I - A(t_{n-1}^{(n)})\delta_{n-1}\right)^{-1} \cdots \left(I - A(t_{k+1}^{(n)})\delta_{k+1}\right)^{-1}$$

for $k < n$ and $U_{n,n} = I$ where $\delta_k = \delta_k^{(n)} = t_k^{(n)} - t_{k-1}^{(n)}$ $(k = 1, \dots, n)$. Recall that if for any finite t there is a constant $N = N(t)$ independent of n and k such that

$$\|U_{nk}\|_X \leq N, \tag{1.4}$$

then the family $\{A(t)\}_{t \geq 0}$ is called *stable* [11, 12]. The property (1.4) is very essential for the existence of a solution. But to check this condition is usually not easy. As shown below, the relation (1.4) can be checked using the normalizing mapping.

A few words about the contents of the paper. In Section 2 upper and lower estimates for the solution are derived. Section 3 is concerned with finite systems of evolution equations. In Section 4, by results from Section 3, the "freezing" method for ordinary differential equations is extended to systems of evolution equations in Banach spaces. Finally, in Section 5 an example is given.

2. Solution estimates

Let us suppose that, for small enough $\delta > 0$, there is a continuous operator-valued function $a : \mathbb{R}_+ \to L(B)$ such that

$$M\big((I_X - \delta A(t))^{-1}h\big) \le \big(I_B + a(t)\delta\big)M(h) \qquad (h \in X;\, t \ge 0) \tag{2.1}$$

where $\mathbb{R}_+ = [0, +\infty)$. We will need to consider the linear equation

$$\dot z(t) = a(t)z(t) \qquad (t \ge 0) \tag{2.2}$$

in B.

Theorem 2.1. *Let inequality* (2.1) *hold. Then for any initial vector* $u_0 \in D$, *the equation* (1.3) *has a solution* u. *Moreover, it satisfies the inequality*

$$M(u(t)) \le z(t) \qquad (t \ge 0) \tag{2.3}$$

where z *is a solution of the equation* (2.2) *with the initial condition* $z(0) = M(u(0))$.

Proof. First we check the existence of a solution. According to inequality (2.1) we easily get

$$M(U_{n,0}u_0) \le \overleftarrow{\prod_{1\le k\le n}} \big(I_B + a(t_k)\delta_k\big)M(u_0) \tag{2.4}$$

where the arrow over the product symbol means that the cofactor indices increase from the right to the left. With the notation $m = \max_{0\le s\le t}\|a(s)\|_B$ there holds

$$\|I_B + \delta a(t)\|_B \le 1 + m\delta \le e^{m\delta} \qquad (t \ge 0).$$

Due to (1.2) the latter inequality and (2.4) imply

$$\|U_{k,0}u_0\|_X = \|M(U_{n,0}u_0)\|_B \le e^{tm}\|M(u_0)\|_B \qquad (1 \le k \le n).$$

Thus by (1.2), condition (1.4) holds and consequently the equation (1.3) has a solution for every initial vector $u_0 \in D$ due to the well-known Corollary to Theorem 4.4.1 in [12: p. 102] (see also [11]).

Now we prove the relation

$$u(t) = \lim U_{n0}u_0 \qquad \text{as} \quad \max_k \delta_k^{(n)} \to 0 \tag{2.5}$$

in the sense of the norm of X. Indeed, from (1.3) the equality

$$\delta_j^{-1}\big(u(t_j) - u(t_{j-1})\big) = A(t_j)u(t_j) + h_{n,j} \qquad (t_j = t_j^{(n)})$$

follows where

$$h_{n,j} = u'(t_j) - \delta_j^{-1}\big(u(t_j) - u(t_{j-1})\big) \qquad (j \ge 1).$$

That is,

$$u(t_j) = \left(I - A(t_j)\delta_j\right)^{-1}\left(u(t_{j-1}) + \delta_j h_{n,j}\right) \qquad (j \geq 1).$$

By this relation, omitting simple calculations, we get

$$u(t_n) = U_{n,0}u_0 + \sum_{k=0}^{n} U_{n,k}h_{n,k}\delta_k^{(n)}. \tag{2.6}$$

But u is a solution of the equation (1.3). So it is a differentiable function, and therefore the expression

$$u'(s) - \delta^{-1}\left(u(s+\delta) - u(s)\right)$$

is uniformly bounded with respect to $\delta > 0$ and $s \leq t < +\infty$. Thus, $h_{n,k}$ are uniformly bounded with respect to $k \leq n$ and $n \geq 2$. Taking into account that $h_{n,k} \to 0$ as $\delta_k^{(n)} \to 0$, and employing the Lebesgue theorem on passing to the limit under the integral sign, we get

$$\sum_{k=1}^{n} \|h_{n,k}\|_X \delta_k^{(n)} \to 0 \qquad \text{as } n \to +\infty.$$

Now (1.4) and (2.6) yield the required relation (2.5). On the other hand, the limit in the strong topology of the operators

$$\overleftarrow{\prod_{1 \leq k \leq n}} \left(I_B + a(t_k)\delta_k\right) \qquad \text{as } \max_k \delta_k^{(n)} \to 0$$

is the Cauchy operator related to the equation (2.2) (see [4]). Now the desired assertion follows from the inequality (2.4) and the equality (2.5) ∎

Note, if in (2.1) $a(t) = a_0$ $(t \geq 0)$ is a constant bounded linear operator, then under the hypothesis of Theorem 2.1

$$M(u(t)) \leq e^{a_0 t} M(u(0)) \qquad (t \geq 0).$$

Let us consider the case $B = \mathbb{R}$. Then inequality (2.1) takes the form

$$\left\|\left(I - \delta A(t)\right)^{-1} h\right\|_X \leq (1 + a(t)\delta)\|h\|_X \qquad (t \geq 0;\ h \in X)$$

where a is a scalar-valued continuous function. Then due to Theorem 2.1 any solution u of the equation (1.3) with an initial vector $u_0 \in D$ satisfies the estimate

$$\|u(t)\|_X \leq \|u(0)\|_X \exp\left(\int_0^t a(\tau)\,d\tau\right) \qquad (t \geq 0). \tag{2.7}$$

Certainly, if equation (2.2) is stable, then by (1.2) and (2.3) the equation (1.3) is stable as well. So Theorem 2.1 gives stability conditions.

Now we are going to establish lower estimates for solutions of the equation (1.3).

Theorem 2.2. *Let for all $t \geq 0$ and small enough $\delta > 0$ the relation*

$$M\big((I_X + \delta A(t))^{-1}h\big) \leq \big(I_B + a_-(t)\delta\big)M(h) \qquad (h \in X;\, t \geq 0) \tag{2.8}$$

hold with a continuous operator-valued function $a_- : \mathbb{R}_+ \to L(B)$. Then any solution u of the equation (1.3) (if it exists) satisfies the inequality $M(u(t)) \geq w(t)$ $(t \geq 0)$, where w is a solution of the equation

$$\dot{w}(t) = -a_-(t)w(t) \tag{2.9}$$

with the initial condition $w(0) = M(u(0))$.

Proof. For a fixed $t_0 > 0$ and every $t \in [0, t_0]$, put in equation (1.3)

$$\tau = t_0 - t, \qquad A_1(\tau) = -A(t_0 - \tau), \qquad y(\tau) = u(t_0 - \tau). \tag{2.10}$$

Then equation (1.3) takes the form

$$\frac{dy}{d\tau} = A_1(\tau)\, y(\tau) \qquad (0 \leq \tau \leq t_0). \tag{2.11}$$

Condition (2.8) gives

$$M\big((I_X - \delta A_1(\tau))^{-1}h\big) \leq \big(I_B + a_-(t_0 - \tau)\delta\big)M(h) \qquad \text{for } \; h \in X.$$

Now Theorem 2.1 yields the following estimate for a solution y of the equation (2.11):

$$M(y(\tau)) \leq V(\tau)M(y(0)) \qquad (0 \leq \tau \leq t_0)$$

where $V(\tau)$ is the Cauchy operator of the equation

$$\frac{dw_1}{d\tau} = a_-(t_0 - \tau)w_1 \qquad (0 \leq \tau \leq t_0). \tag{2.12}$$

But according to (2.10), $y(\tau) = u(t)$, and

$$M(u(t)) \leq V(t_0 - t)M(u(t_0)) \qquad (0 \leq t \leq t_0).$$

For $t = 0$ that relation implies

$$V^{-1}(t_0)M(u(0)) \leq M(u(t_0)).$$

Since V is the Cauchy operator of the equation (2.12), V^{-1} is the Cauchy operator of the equation (2.9) (see [3]). Now taking into account that t_0 is arbitrary, we arrive at the stated result ∎

Note that if in equation (2.9) $a_- = b_0$ is a constant bounded linear operator, then Theorem 2.2 implies

$$M(u(t)) \geq e^{-b_0 t} M(u(0)) \qquad (t \geq 0).$$

Let us consider the case $B = \mathbb{R}$. Then inequality (2.8) takes the form

$$\left\| \left(I + \delta A(t)\right)^{-1} h \right\|_X \leq (1 + b_-(t)\delta) \left\| h \right\|_X \qquad (t \geq 0;\ h \in X,\ \|h\|_X = 1)$$

where b_- is a scalar-valued continuous function. Then by Theorem 2.2 every solution u of the equation (1.3) with the initial vector $u_0 \in D$ satisfies the estimate

$$\|u(t)\|_X \geq \|u(0)\|_X \exp\left(-\int_0^t b_-(\tau)\, d\tau\right) \qquad (t \geq 0).$$

Certainly, if equation (1.3) is stable, then by Theorem 2.2 the equation (2.9) is stable as well. So Theorem 2.2 gives necessary stability conditions.

Furthermore, let for instance $X = H$ be a Hilbert space with scalar product $(\cdot,\cdot)_H$, and $A(t) = iS(t) + T(t)$ $(t \geq 0)$, where $S(t)$ are selfadjoint operators with constant domain D. Besides, suppose that $T(t)$ is bounded. Then

$$\operatorname{Re}\left(A(t)h, h\right)_H = \operatorname{Re}\left(T(t)h, h\right)_H \geq -\|T(t)\|_H \qquad (h \in D;\ \|h\|_H = 1)$$

and

$$\begin{aligned} \left\| \left(I_H + \delta A(t)\right) h \right\|_H^2 &= 1 + 2\operatorname{Re}\left(A(t)h, h\right)_H \delta + \left(A(t)h, A(t)h\right)_H \delta^2 \\ &\geq 1 - 2\|T(t)\|_H \delta \qquad \left(h \in D; \|h\|_H = 1\right). \end{aligned}$$

Thus

$$\left\| \left(I_H + \delta A(t)\right)^{-1} \right\|_H \leq 1 + \|T(t)\|_H \delta + o(\delta) \qquad \text{as } \delta \downarrow 0.$$

Now thanks to Theorem 2.2

$$\|u(t)\|_H \geq \|u(0)\|_H \exp\left(-\int_0^t \|T(\tau)\|_H d\tau\right) \qquad (t \geq 0).$$

3. Systems of evolution equations in a Banach space

Let $X = \sum_{k=1}^n \oplus E_k$ be a direct sum of Banach spaces E_k with norms $\|\cdot\|_{E_k}$, and let $h = (h_k)_{k=1}^n$ be an element of X. Define in X the normalizing mapping by

$$M(h) = (\|h_k\|_{E_k})_{k=1}^n. \tag{3.1}$$

That is, $M(h)$ is the vector whose coordinates are $\|h_k\|_{E_k}$ $(1 \leq k \leq n)$. Furthermore, let $A_{jk}(t)$ be linear operators acting from E_k into E_j, and $A(t)$ be defined by the matrix $(A_{jk}(t))_{j,k=1}^n$. Thus, equation (1.3) takes the form

$$\dot{u}_j = \sum_{k=1}^n A_{jk}(t) u_k \qquad (t \geq 0;\ 1 \leq j \leq n;\ u = u(t)). \tag{3.2}$$

It is assumed that the operators $A_{kk}(t)$ have dense common domains $D_k \subset E_k$ and are continuously differentiable on D_k. For $j \neq k$ the operators $A_{jk}(t)$ are bounded and strongly continuously differentiable. Moreover,

$$\|A_{jk}(t)\|_{E_k \to E_j} < a_{jk}(t) \qquad (t \geq 0). \tag{3.3}$$

Besides, with every small enough $\delta > 0$,

$$\big\|\big(I_{E_k} - \delta A_{kk}(t)\big)h_k\big)\big\|_{E_k} \geq \big(1 - a_{kk}(t)\delta\big)\|h_k\|_{E_k} \qquad (h_k \in D_k) \tag{3.4}$$

where a_{jk} $(1 \leq j,k \leq n)$ are scalar-valued continuous functions. That is, $A_{jk}(t)$ are assumed to be bounded for $j \neq k$. Clearly, the domain D of $A(t)$ is the direct sum of D_j $(1 \leq j \leq n)$. Hence,

$$M_j\big((I_X - \delta A(t))h\big) = \big\|\big(I_{E_j} - A_{jj}(t)\delta\big)h_j + \delta\textstyle\sum_{j\neq k} A_{jk}(t)h_k\big\|_{E_j}$$

where $M_j\big((I_X - \delta A(t))h\big)$ is the coordinate of the vector $M\big((I_X - \delta A(t))h\big)$. Therefore,

$$\begin{aligned} M_j\big((I_X - \delta A(t))h\big) &\geq \big\|\big(I_{E_j} - A_{jj}(t)\delta\big)h_j\big\|_{E_j} - \delta\textstyle\sum_{j\neq k}\big\|A_{jk}(t)h_k\big\|_{E_j} \\ &\geq \big(1 - a_{jj}(t)\delta\big)\|h_j\|_{E_j} - \delta\textstyle\sum_{k\neq j} a_{jk}(t)\|h_k\|_{E_k}. \end{aligned}$$

According to (3.1) that inequality can be written in vector form

$$M\big((I_X - \delta A(t))h\big) > \big(I_B - a(t)\delta\big)M(h) \qquad (h \in D) \tag{3.5}$$

with the matrix $a(t) = (a_{jk}(t))$. It is simple to check that the invertibility of the operators $I_{E_k} - \delta A_{kk}(t)$ $(1 \leq k \leq n)$ implies that of the operator $I_X - \delta A(t)$ for small enough $\delta > 0$, because the $A_{jk}(t)$ $(j \neq k)$ are bounded. Employing (3.5) we easily get

$$M\big((I_X - \delta A(t))^{-1}v\big) < \big(I_{C^n} - a(t)\delta\big)^{-1}M(h) \qquad (v \in X). \tag{3.6}$$

But

$$\big(I_{C^n} - a(t)\delta\big)^{-1} = I_{C^n} + a(t)\delta + o(\delta) \qquad (\delta \downarrow 0).$$

This relation and (3.6) yield the condition (2.1). Now Theorem 2.1 immediately implies

Theorem 3.1. *Let conditions* (3.3) *and* (3.4) *be fulfilled. Then for every initial vector* $u_0 \in D$, *the system* (3.2) *has a solution* $u : \mathbb{R}_+ \to D$. *Moreover, it satisfies the inequality* (2.3), *where* z *is the solution of the equation* (2.2) *with a variable* $(n \times n)$-*matrix* $a(t) = (a_{jk}(t))$ *and the initial condition* $z(0) = (\|u(0)\|_{E_k})_{k=1}^n$.

Note that, similarly, by Theorem 2.2 lower estimates for the solutions of the system (3.2) can be obtained.

Corollary 3.2. *Let conditions* (3.3) *and* (3.4) *be fulfilled with constant entries* $a_{jk}(t) = a_{jk}$ $(1 \le j,k \le n)$ *and assume that* (a_{jk}) *is a Hurwitz matrix (that is, all its eigenvalues lie in the open left half-plane). Then the system* (3.2) *is stable.*

Denote by $\lambda(z)$ the Lyapunov exponent of a solution z of the equation (2.2). Lozinskii (see [9]) introduced the logarithm norm

$$L(t) = \limsup_{h \to +0} \frac{1}{h}\big(\|I + h\,a(t)\| - 1\big).$$

For every solution z of the equation (2.2), the logarithm norm implies the estimate $\lambda(z) \le \bar{L}$ where we used he notation

$$\bar{p} = \limsup_{t \to +\infty} \frac{1}{t}\int_0^t p(s)\,ds.$$

In particular, that estimate implies

$$\lambda(z) \le \overline{\max_{1 \le i \le n}\big(a_{ii} + \textstyle\sum_{k \ne i}|a_{ik}|\big)} \quad \text{and} \quad \lambda(z) \le \overline{\max_{1 \le k \le n}\big(a_{kk} + \textstyle\sum_{k \ne i}|a_{ik}|\big)}.$$

The estimate $\lambda(z) \le \bar{L}$ and Theorem 3.1 yield

Corollary 3.3. *Let conditions* (3.3) *and* (3.4) *be fulfilled. Let also* $\bar{L} \le 0$. *Then the system* (3.2) *is stable.*

Now we are going to specialize assumption (3.4) in the case of an orthogonal sum of Hilbert spaces. Namely, let $X = H = \sum_{k=1}^{n} \oplus E_k$ be an orthogonal sum of Hilbert spaces E_k with scalar products $(\cdot,\cdot)_{E_k}$ and norms $\|\cdot\|_{E_k} = \sqrt{(\cdot,\cdot)_{E_k}}$. For an element $h = (h_k)_{k=1}^{n} \in H$, let the normalizing mapping be defined by formula (3.1) as well.

Corollary 3.4. *Let conditions* (3.3) *and*

$$\operatorname{Re}\big(A_{kk}(t)h,h\big)_{E_k} \le a_{kk}(t)\|h\|^2_{E_k} \qquad (1 \le k \le n, h \in D_k) \tag{3.7}$$

be fulfilled. Then for every initial vector $u_0 \in D$, *the system* (3.2) *has a solution* u. *Moreover, it satisfies the inequality* (2.3) *where* z *is a solution of the equation* (2.2) *with a variable* $(n \times n)$*-matrix* $a(t) = (a_{jk}(t))$ *and the initial condition* $z(0) = (\|u(0)\|_{E_k})_{k=1}^{n}$.

Proof. Considering that

$$\begin{aligned}\big\|\big(I_{E_k} - \delta A_{kk}(t)\big)h\big\|^2_{E_k} &= 1 - 2\operatorname{Re}\big(A_{kk}(t)h,h\big)_{E_k}\delta + \big(A_{kk}(t)h, A_{kk}(t)h\big)_{E_k}\delta^2 \\ &\ge 1 - 2\,a_{kk}(t)\delta \qquad \big(h \in D_k;\ \|h\|_{E_k} = 1\big)\end{aligned}$$

we easily get the inequality (3.4). Now the result is due to Theorem 3.1 ∎

Certainly, Corollaries 3.2 and 3.3 are true replacing condition (3.4) by condition (3.7).

Note that in the case $n = 1$, that is under the condition

$$\operatorname{Re}\big(A(t)h,h\big)_H \le a(t)\|h\|^2_H \qquad (1 \le k \le n, h \in D)$$

where a is a scalar-valued function, Corollary 3.4 yields the estimate (2.7). Thus we have obtained the Wazewski inequality [9] established for solutions of ordinary differential systems.

4. Applications of the "freezing" method

Recall some results connected with the "freezing" method for ordinary differential equations. Let Q be an $(n \times n)$-matrix with eigenvalues $\lambda_k(Q)$ $(k = 1, \dots, n)$ counted with their multiplicities. The following quantity plays a key role in this section:

$$g(Q) = \left(N^2(Q) - \sum_{k=1}^{n} |\lambda_k(Q)|^2 \right)^{1/2} \tag{4.1}$$

where $N(Q)$ is the Frobenius (Hilbert-Schmidt) norm of Q, i.e. $N^2(Q) = \operatorname{trace}(QQ^*)$. If Q is a normal matrix: $QQ^* = Q^*Q$, then $g(Q) = 0$. The following relations are true:

$$g^2(Q) \le N^2(Q) - |\operatorname{trace} Q^2| \qquad \text{and} \qquad g^2(Q) \le \frac{1}{2} N^2(Q^* - Q)$$

(see [7: Section 1.1]).

Consider the equation (2.2) assuming that $a = a(t)$ $(t \ge 0)$ is an $(n \times n)$-matrix satisfying the conditions

$$v = \sup_{t \ge 0} g(a(t)) < +\infty \tag{4.2}$$

and

$$\|a(t) - a(s)\|_{C^n} \le q_0\, |t - s| \qquad \text{for } t, s \ge 0 \tag{4.3}$$

where q_0 is a constant. Denote by $z(q_0, v)$ the extreme right-hand (unique positive and simple) root of the algebraic equation

$$z^{n+1} = q_0 \sum_{k=0}^{n-1} \frac{(k+1)\, v^k}{\sqrt{k!}}\, z^{n-k-1}. \tag{4.4}$$

Now we use the following result (see [7: p. 213]):

Let the conditions (4.2) and (4.3) hold, and let $a(t) + z(q_0, v) I_{C^n}$ be a Hurwitz matrix for all $t \ge 0$. Then the equation (2.2) is stable.

Combining that result with Theorem 3.1, we obtain the following

Theorem 4.1. *Let conditions (3.3) and (3.4) be fulfilled and let the matrix $a(t)$ satisfy conditions (4.2) and (4.3). Then the system (3.2) is stable provided that $a(t) + z(q_0, v) I_{C^n}$ is a Hurwitz matrix for all $t \ge 0$.*

Put

$$w_n = \sum_{k=0}^{n-1} \frac{k+1}{\sqrt{k!}}.$$

Then the inequality

$$z(q_0, v) \le v^{1 - \frac{1}{n}} (q_0\, w_n)^{\frac{1}{n}} \qquad \text{if } q_0 w_n \le v$$

is true (c.f. [7: p. 213]). The latter inequality and Theorem 4.1 yield

Corollary 4.2. *Let conditions (3.3) and (3.4) be fulfilled and let the matrix $a(t)$ satisfy conditions (4.2) and (4.3). Then if $q_0 w_n \le v$, the system (3.2) is stable provided that*

$$a(t) + v^{1 - \frac{1}{n}} (q_0 w_n)^{\frac{1}{n}} I_{C^n}$$

is a Hurwitz matrix for all $t \ge 0$.

5. Example

Consider the problem

$$\left.\begin{aligned} &\frac{\partial u_j}{\partial t} = \frac{\partial}{\partial x}\phi_j(x,t)\frac{\partial u_j}{\partial x} + \sum_{k=1}^{n}\psi_{jk}(x,t)\,u_k \qquad (1 \le j \le n;\ 0 < x < 1) \\ &u(t,0) = u(t,1) = 0 \qquad (t \ge 0) \end{aligned}\right\} \tag{5.1}$$

where ϕ_j and ψ_{jk} are real-valued functions defined on $[0,1]\times[0,+\infty)$ being continuously differentiable in t. Set $\vec{u} = (u_j)_{j=1}^n$, and for X take the Hilbert space $H = L^2([0,1],\mathbb{C}^n)$ of square integrable $\mathbb{C}$-valued functions $\vec{v} = (v_k)_{k=1}^n$ on $[0,1]$, with

$$(\vec{v},\vec{w})_H = \sum_{k=1}^{n}\int_0^1 v_k(x)\bar{w}_k(x)\,dx \qquad \text{and} \qquad \|\vec{v}\|_H = \sqrt{(\vec{v},\vec{v})_H}$$

as scalar product and norm, respectively. Clearly, H is the n-times orthogonal sum $H = \sum_{k=1}^n \oplus E$ of the same Hilbert space $E = L^2[0,1]$ of square integrable $\mathbb{C}$-valued functions with usual scalar product and norm. Define on $H = L^2([0,1],\mathbb{C}^n)$ a *normalizing mapping* $M : H \to \mathbb{R}_+^n$ by $M(\vec{v}) = (\|v_k\|_E)_{k=1}^n$. Furthermore, put

$$D_1 = \left\{ v \in L^2[0,1] : v'' \in L^2[0,1] \text{ with } v(0) = v(1) = 0 \right\}$$

and $D = (D_1)^n$. According to (3.2) define operators $A_{jk}(t)$ by the formulae

$$(A_{jk}(t)v_k)(x) = \psi_{jk}(x,t)\,v_k(x) \qquad (k \ne j)$$

and

$$(A_{jj}(t)v_j)(x) = \frac{\partial}{\partial x}\phi_j(x,t)\frac{\partial v_j}{\partial x} + \psi_{jj}(x,t)\,v_j \qquad (v_j, v_k \in D_1;\ j,k = 1,\dots,n). \tag{5.2}$$

Assume that

$$a_{jk}(t) = \sup_{x\in[0,1]} |\psi_{jk}(x,t)| < \infty \qquad (k \ne j,\ t \ge 0) \tag{5.3}$$

and

$$b_j(t) = \sup_{x\in[0,1]} \psi_{jj}(x,t) < \infty \qquad \text{and} \qquad m_j(t) = \inf_{x\in[0,1]} \phi_j(x,t) > 0 \ \ (t \ge 0). \tag{5.4}$$

Hence

$$\|A_{jk}(t)\|_E \le a_{jk}(t) \qquad (j \ne k,\ t \ge 0).$$

Further, integration by parts and the condition $v_j(0) = v_j(1) = 0$ give

$$\left(\frac{d}{dx}\phi_j(\cdot,t)v_j', v_j\right)_E = -(\phi_j(\cdot,t)v_j', v_j')_E \le -m_j(t)\,(v_j', v_j')_E \le -m_j(t)\pi^2\|v_j\|_1^2$$

since $\beta(S) = \pi^2$ is the smallest eigenvalue of the operator S defined on D_1 by the formula

$$(Sv_j)(x) = -v_j''(x) \qquad (v_j \in D_1;\ j = 1,\dots,n).$$

Moreover, relations (5.2) and (5.3) imply

$$(A_{jj}(t)v_j, v_j)_E \le a_{jj}(t)(v_j,v_j)_E$$

with

$$a_{jj}(t) = -m_j(t)\pi^2 + b_j(t) \qquad (j = 1,\dots,n). \tag{5.5}$$

Now according to Reamark 4.3, Theorem 4.1 implies

Proposition 5.1. *Let conditions* (5.3) *and* (5.4) *be fulfilled. In addition, let the matrix* $a(t) = (a_{jk}(t))$ *defined by* (5.3) *and* (5.5) *satisfy conditions* (4.2) *and* (4.3). *Then the system* (5.1) *is stable provided that* $a(t) + z(q_0, v)I_{C^n}$ *is a Hurwitz matrix for all* $t \geq 0$.

Certainly, instead of the differentiating operator one can consider the Laplacian one.

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