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Tensor complexes: Multilinear free resolutions constructed from higher tensors

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Abstract. The most fundamental complexes of free modules over a commutative ring are the Koszul complex, which is constructed from a vector (i.e., a 1-tensor), and the Eagon–Northcott and Buchsbaum–Rim complexes, which are constructed from a matrix (i.e., a 2-tensor). The subject of this paper is a multilinear analogue of these complexes, which we construct from an arbitrary higher tensor.

Our construction provides detailed new examples of minimal free resolutions, as well as a unifying view on a wide variety of complexes including: the Eagon–Northcott, Buchsbaum–Rim and similar complexes, the Eisenbud–Schreyer pure resolutions, and the complexes used by Gelfand–Kapranov–Zelevinsky and Weyman to compute hyperdeterminants. In addition, we provide applications to the study of pure resolutions and Boij–Söderberg theory, including the construction of infinitely many new families of pure resolutions, and the first explicit description of the differentials of the Eisenbud–Schreyer pure resolutions.

1. Introduction

In commutative algebra, the Koszul complex is the mother of all complexes.
David Eisenbud

The most fundamental complex of free modules over a commutative ring R is the Koszul complex, which is constructed from a vector (i.e., a 1-tensor) $\mathbf{f} = (f_1, \dots, f_a) \in R^a$. The

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next most fundamental complexes are likely the Eagon–Northcott and Buchsbaum–Rim complexes, which are constructed from a matrix (i.e., a 2-tensor) $\widetilde{\psi} \in R^a \otimes R^b$.

In this paper we construct multilinear analogues of these complexes, which we refer to as *tensor complexes*. These complexes are constructed from an arbitrary higher tensor $\widetilde{\phi} \in R^a \otimes R^{b_1} \otimes \cdots \otimes R^{b_n}$, providing a unifying perspective on many of these previously known families—including Koszul, Eagon–Northcott, and Buchsbaum–Rim complexes—and leads to new such families of resolutions. This also supplies a new tool for producing and studying invariants of higher tensors.

While tensor complexes display remarkable numerical properties (for instance, all extremal rays of the cone of Betti diagrams can be generated by our construction; see §10), their structure is surprisingly simple. We provide explicit descriptions of these free resolutions from several different perspectives; in particular, each tensor complex can be pieced together from linear strands of a Koszul complex. This not only adds tensor complexes to the few families of free resolutions that are understood in detail, it also provides new such families that are uniformly minimal over $\mathbb{Z}$. (Uniformity over $\mathbb{Z}$ can be quite subtle; see [Has].)

To motivate our main result, we first recall some properties of the more familiar Eagon–Northcott complex. The Eagon–Northcott complex for an arbitrary matrix can be constructed as a pullback from the universal case. Namely, if we first build the Eagon–Northcott complex $\mathrm{EN}(\psi)_\bullet$ over the polynomial ring $\mathbb{Z}[x_{i,j}]$ for an $a \times b$ matrix $\psi = \psi^{a\times b} = (x_{i,j})$ of indeterminates, then the Eagon–Northcott complex of $\widetilde{\psi}$ is $\mathrm{EN}(\psi)_\bullet \otimes_{\mathbb{Z}[x_{i,j}]} R$. Several nice properties of the complex $\mathrm{EN}(\psi)_\bullet$ are illustrated in the following theorem.

Theorem 1.1 (Eagon–Northcott [EN]). *The Eagon–Northcott complex* $\mathrm{EN}(\psi)_\bullet$ *of a matrix of indeterminates* ψ *satisfies the following:*

(i) *It is a graded free resolution of a Cohen–Macaulay module.*
(ii) *It is uniformly minimal over* $\mathbb{Z}$*, i.e.,* $\mathrm{EN}(\psi)_\bullet \otimes_{\mathbb{Z}[x_{i,j}]} \Bbbk[x_{i,j}]$ *is a minimal free resolution for any field* $\Bbbk$.
(iii) *It is a pure resolution, i.e.,* $\mathrm{EN}(\psi)_i$ *is generated in a single degree for each* i.
(iv) *It respects the bilinearity of* ψ*, i.e.,* $\mathrm{EN}(\psi)_\bullet$ *is* $\mathbf{GL}_a \times \mathbf{GL}_b$*-equivariant.*

The Buchsbaum–Rim complex also satisfies the assertions of Theorem 1.1. In fact, the Eagon–Northcott and Buchsbaum–Rim complexes fit naturally into a sequence of bilinear complexes arising from the matrix ψ and a weight $w \in \mathbb{Z}^2$ [BE].[1] We refer to an element of this sequence as a *matrix complex*, although these are sometimes called "generalized Koszul complexes" (see [Buc, BR]). While such a complex exists for any w, an analogue of Theorem 1.1 holds only for a limited set of weights.

To construct the tensor complexes of an arbitrary tensor $\widetilde{\phi}$, we similarly take the pullback of the universal case. Let $a \in \mathbb{N}$ and $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{N}^n$. We define a universal

[1] [Eis, §A2.6] outlines the construction of matrix complexes, and we use this as our primary reference for these complexes. There, the complexes are parametrized by $\mathbb{Z}^1$, which corresponds to the second coordinate of our $w \in \mathbb{Z}^2$; the first coordinate of w simply allows a twist of the complex as a whole.

tensor $\phi := \phi^{a\times\mathbf{b}}$ over the symmetric algebra $S = \mathrm{S}^\bullet(\mathbb{Z}^a \otimes \mathbb{Z}^{b_1} \otimes \cdots \otimes \mathbb{Z}^{b_n})$, and in §2.3, we construct the *tensor complex* $F(\phi, w)_\bullet$ from this universal tensor and a weight $w \in \mathbb{Z}^{n+1}$. The following theorem illustrates how tensor complexes are a multilinear extension of the Eagon–Northcott complex and the other matrix complexes, as long as we limit the choice of w, requiring it to be a *pinching weight* (see Definition 5.1).

Theorem 1.2. *If w is a pinching weight for $\phi^{a\times\mathbf{b}}$, then $F(\phi, w)_\bullet$ satisfies the following:*

(i) *It is a graded free resolution of a Cohen–Macaulay module $M(\phi, w)$.*
(ii) *It is uniformly minimal over $\mathbb{Z}$.*
(iii) *It is a pure resolution.*
(iv) *It respects the multilinearity of ϕ, i.e., $F(\phi, w)_\bullet$ is $\mathbf{GL}_a \times \cdots \times \mathbf{GL}_{b_n}$-equivariant.*

A connection between tensors and free complexes has previously been observed in special cases, [GKZ, §14] and [Wey, §9.4]. For instance, [GKZ, Proposition 14.3.2] uses a free complex to express hyperdeterminants of the boundary format, and this is a special case of our construction (see Proposition 9.1). Hyperdeterminants also play an important role in the study of general tensor complexes. As shown in Theorem 1.6, the support of $M(\phi, w)$ is set-theoretically defined by an ideal of hyperdeterminants of certain sub-tensors of ϕ. In addition, each such variety is a resultant variety for a system of multilinear equations on a product of projective spaces (see Proposition 1.8).

Tensor complexes extend another important class of free resolutions: pure resolutions of Cohen–Macaulay modules. Such resolutions are central objects in Boij–Söderberg theory, as they provide the extremal rays of the cone of Betti diagrams. We show in Theorem 1.9 that there are an infinite number of different tensor complexes whose Betti diagrams lie on any such extremal ray. In addition, Theorem 10.1 shows that each Eisenbud–Schreyer pure resolution from [ES1, §5] is obtained by taking hyperplane sections of a tensor complex, thus providing the first explicit description of the differentials of these complexes.

Properties of higher tensors are the subject of much recent work (see [Lan2, Lan1] for surveys). A tensor complex for an arbitrary tensor $\widetilde{\phi}$ attaches new invariants to the tensor. In some small cases (see Example 9.7), these invariants detect the rank of the tensor. It would be interesting to pursue further connections.

1.1. Constructing tensor complexes

Perhaps the most important feature about the tensor complexes $F(\phi, w)_\bullet$ is that we can describe them explicitly. To underscore their essential properties, we present three different perspectives on these complexes.

Strands of the Koszul complex. In [Eis, §A2.6], matrix complexes are constructed by splicing together two strands of a Koszul complex. Tensor complexes are similar: if ϕ is an $(n+1)$-tensor and w is a pinching weight for ϕ, then $F(\phi, w)_\bullet$ can be built by splicing together n strands of a Koszul complex.

For example, consider the universal $7 \times (2,2)$ tensor $\phi = \phi^{7\times(2,2)}$. Let $A \cong \mathbb{Z}^7$, $B_1 \cong \mathbb{Z}^2 \cong B_2$, $X^{7\times(2,2)} := A \otimes B_1^* \otimes B_2^*$, and $S := \mathrm{S}^\bullet(X^{7\times(2,2)})$. For the choice of pinching weight $w = (0,1,4)$, the tensor complex $F(\phi, w)_\bullet$ is

$$S^{10} \leftarrow S^{28}(-1) \overset{\sigma}{\leftarrow} S^{70}(-3) \leftarrow S^{70}(-4) \overset{\sigma'}{\leftarrow} S^{28}(-6) \leftarrow S^{10}(-7) \leftarrow 0.$$

To illustrate the equivariant structure of this free resolution, in §2.1 we introduce a column notation for writing representations of $\mathbf{GL}(A) \times \mathbf{GL}(B_1) \times \mathbf{GL}(B_2)$, giving $F(\phi, w)_\bullet$ the form

$$\begin{bmatrix} \bigwedge^0 \\ \mathrm{S}^1 \\ \mathrm{S}^4 \end{bmatrix} \leftarrow \begin{bmatrix} \bigwedge^1 \\ \mathrm{S}^0 \\ \mathrm{S}^3 \end{bmatrix}(-1) \overset{\sigma}{\nwarrow} \begin{bmatrix} \bigwedge^3 \\ \widetilde{\mathrm{D}}^0 \\ \mathrm{S}^1 \end{bmatrix}(-3) \leftarrow \begin{bmatrix} \bigwedge^4 \\ \widetilde{\mathrm{D}}^1 \\ \mathrm{S}^0 \end{bmatrix}(-4) \overset{\sigma'}{\nwarrow} \begin{bmatrix} \bigwedge^6 \\ \widetilde{\mathrm{D}}^3 \\ \widetilde{\mathrm{D}}^0 \end{bmatrix}(-6) \leftarrow \begin{bmatrix} \bigwedge^7 \\ \widetilde{\mathrm{D}}^4 \\ \widetilde{\mathrm{D}}^1 \end{bmatrix}(-7) \leftarrow 0. \tag{1.3}$$

Here, for instance, the F_2 term of (1.3) denotes the graded free S-module $\bigwedge^3(A) \otimes_{\mathbb{Z}} \det(B_1^*) \otimes_{\mathbb{Z}} \mathrm{S}^1(B_2) \otimes_{\mathbb{Z}} S(-3)$. This complex arises from three separate strands—vertically separated in (1.3)—of a Koszul complex $K(\phi)_\bullet$ on the $\mathbb{Z}^3$-graded polynomial ring $\mathrm{S}^\bullet(X^{7\times(2,2)} \otimes B_1 \otimes B_2)$. While this mirrors the construction of matrix complexes in [Eis, §A2.6], it will be modified in our situation by certain local cohomology modules (see §6). We splice these strands together via the maps σ and σ' whose entries are expressions in the 2×2 minors of the flattening $\phi^\flat\colon A^* \otimes S \to B_1^* \otimes B_2^* \otimes S$. The fact that $F(\phi, w)_\bullet$ forms a complex then follows from a generalized Laplace expansion formula for the determinant of a singular matrix. Example 12.1 provides a detailed illustration of this fact in a similar example.

For a tensor complex, a new phenomenon arises that was not present in the case of matrix complexes: it is possible that two consecutive maps are splice maps. In fact, there will be many cases where none of the differentials $F(\phi, w)_\bullet$ consist of linear forms; each strand consists of a single free module and each differential is a splicing map.

Tensor complexes and representation theory. The above approach to $F(\phi, w)_\bullet$ makes little use of its multilinear symmetry. By incorporating ideas from representation theory, we are able to provide a simple description of the differentials of $F(\phi, w)_\bullet$.

Let us reconsider the map σ from (1.3). This map is determined by its degree 3 part $[\sigma]_3\colon [F_2]_3 \to [F_1]_3$, which is the following map of finite-rank free $\mathbb{Z}$-modules:

$$[\sigma]_3\colon \bigwedge^3 A \otimes \bigwedge^2 B_1^* \otimes \mathrm{D}^0 B_1^* \otimes \mathrm{S}^1 B_2 \to \left(\bigwedge^1 A \otimes \mathrm{S}^0 B_1 \otimes \mathrm{S}^3 B_2\right) \otimes \mathrm{S}^2(X^{7\times(2,2)}).$$

Recalling that $X^{7\times(2,2)} = A \otimes B_1^* \otimes B_2^*$, we express the map $[\sigma]_3$ entirely in terms of tensor products and adjoints of multiplication and comultiplication maps. Namely, we use the subrepresentation $\bigwedge^2 A \otimes \bigwedge^2 B_1^* \otimes \mathrm{D}^2 B_2^* \subseteq \mathrm{S}^2(X^{7\times(2,2)})$ and construct $[\sigma]_3$ via the following equivariant maps on each tensor factor:

$$[\sigma]_3 \leftrightarrow \begin{cases} \bigwedge^3 A \to \bigwedge^1 A \otimes \bigwedge^2 A & \text{by comultiplication,} \\ \bigwedge^2 B_1^* \otimes \mathrm{D}^0 B_1^* \to \mathrm{S}^0 B_1 \otimes \bigwedge^2 B_1^* & \text{by identifying } \mathrm{D}^0 B_1^* \cong \mathrm{S}^0 B_1, \\ \mathrm{S}^1 B_2 \to \mathrm{S}^3 B_2 \otimes \mathrm{D}^2 B_2^* & \text{by the adjoint of multiplication.} \end{cases}$$

This provides an explicit description of the differentials of $F(\phi, w)_\bullet$ (see §4) and proves that $F(\phi, w)_\bullet$ is a complex (see Lemma 4.8). For acyclicity, we take a third perspective.

The geometric method. The geometric method of Kempf–Lascoux–Weyman [Wey, §5] provides the most powerful perspective for studying the tensor complex $F(\phi, w)_\bullet$. Continuing with the universal $7 \times (2, 2)$ tensor example, we define a complex $\mathcal{K}(\phi, w)_\bullet$ on $\mathrm{Spec}(S) \times \mathbb{P}(B_1) \times \mathbb{P}(B_2)$ as the sheafy version of $K(\phi)_\bullet$, twisted by a line bundle determined by w. Taking the derived pushforward of $\mathcal{K}(\phi, w)_\bullet$ along the projection $\pi\colon \mathrm{Spec}(S) \times \mathbb{P}(B_1) \times \mathbb{P}(B_2) \to \mathrm{Spec}(S)$ also yields the tensor complex $F(\phi, w)_\bullet$; we use this as our primary definition of $F(\phi, w)_\bullet$ (see Definition 2.4).

The geometric method immediately provides the acyclicity of $F(\phi, w)_\bullet$. The disadvantage is that the geometric method does not provide a clear description of the differentials of the complex. To make use of the representation-theoretic description in §4, it suffices to show that the differentials can be chosen equivariantly. (This is not obvious, since the representation theory of $\mathbf{GL}_n(\mathbb{Z})$ is not semisimple.)

1.2. The algebra and geometry of tensor complexes

We now summarize some additional results on tensor complexes, as well as applications of our work to Boij–Söderberg theory. We begin with the functorial properties of $F(\phi, w)_\bullet$.

Proposition 1.4. *Let $a' \leq a$, and let w and w' be weights. Let $S := \mathbb{Z}[X^{a\times\mathbf{b}}]$ and $S' := \mathbb{Z}[X^{a'\times\mathbf{b}}]$. Given an inclusion $i\colon \mathbb{Z}^{a'} \to \mathbb{Z}^a$ and a polynomial of multi-degree $w - w'$ in $S' \otimes \mathrm{S}^\bullet(B_1) \otimes \cdots \otimes \mathrm{S}^\bullet(B_n)$, we have a degree zero map of complexes*

$$F(\phi^{a'\times\mathbf{b}}, w')_\bullet \otimes_{S'} S \to F(\phi^{a\times\mathbf{b}}, w)_\bullet.$$

This result is proven in §7 and is related to [BEKS, Theorem 1.2], as the maps considered in that result are special cases of the above construction.

We now turn to properties of the module $M(\phi, w)$ that is resolved by the tensor complex $F(\phi, w)_\bullet$; these statements are proved in §8.

Corollary 1.5. *Let $\phi = \phi^{a\times\mathbf{b}}$ be the universal tensor and w be a pinching weight for ϕ.*

- (i) *The support of $M(\phi, w)$ is an irreducible subvariety of $\mathbb{A}^{a\times\mathbf{b}}$ that is independent of w and has codimension $a - \sum_i (b_i - 1)$.*
- (ii) *$M(\phi, w)$ is generically perfect, i.e., it is Cohen–Macaulay and faithfully flat over $\mathbb{Z}$.*

(iii) *The multiplicity of $M(\phi, w)$ is independent of w. Specifically, it is given by the multinomial coefficient*

$$e(M(\phi, w)) = \frac{a!}{(a - \sum_i (b_i - 1))! \prod_{i=1}^{n} (b_i - 1)!}.$$

Hyperdeterminantal varieties. Based on Corollary 1.5(i), we denote the support of $M(\phi, w)$ by $Y(\phi)$ and call such a variety a *hyperdeterminantal variety*. These hyperdeterminantal varieties simultaneously extend the determinantal varieties defined by maximal minors of a matrix (when ϕ is a 2-tensor) and the hypersurfaces defined by hyperdeterminants of the boundary format (see [GKZ, §14.3]).

Theorem 1.6. *Let $\phi = \phi^{a\times \mathbf{b}}$ and $Y(\phi) \subseteq \mathbb{A}^{a\times \mathbf{b}}$ be the support variety of $M(\phi, w)$. If $a' := 1 + \sum_{i=1}^{n} (b_i - 1)$, then $Y(\phi)$ is set-theoretically defined by the ideal*

$$\langle \textit{hyperdeterminant of } \phi' \mid \phi' \textit{ is an } (a' \times \mathbf{b})\textit{-subtensor of } \phi\rangle. \qquad (1.7)$$

The ideal (1.7) can fail to be radical, as we illustrate in Example 12.2. Further, Remark 8.2 explains how the variety $Y(\phi)$ is a resultant variety for a system of multilinear equations on a product of projective spaces, yielding the following result.

Proposition 1.8. *For a field $\Bbbk$, let $\mathbf{f} = f_1, \dots, f_a$ be a collection of multilinear forms on $\mathbb{P}_{\Bbbk}^{b_1-1} \times \cdots \times \mathbb{P}_{\Bbbk}^{b_n-1}$. This gives a tensor $\phi_{\mathbf{f}} \in \Bbbk^a \otimes \Bbbk^{b_1} \otimes \cdots \otimes \Bbbk^{b_n}$ and thus a specialization map $q_{\mathbf{f}}\colon S \to \Bbbk$, which sends $\phi \mapsto \phi_{\mathbf{f}}$. Let w be any pinching weight for the universal tensor $\phi^{a\times \mathbf{b}}$, and let $\partial_\bullet$ denote the differential of $F(\phi^{a\times \mathbf{b}}, w)$. Denote by $\partial_1(\mathbf{f})$ the matrix obtained by specializing the entries of ∂_1 via the map $q_{\mathbf{f}}$. The following are then equivalent:*

(i) *The vanishing locus $V(f_1, \dots, f_a) \subseteq \mathbb{P}_{\Bbbk}^{b_1} \times \cdots \times \mathbb{P}_{\Bbbk}^{b_n-1}$ is nonempty (over any algebraic closure of $\Bbbk$).*
(ii) *The matrix $\partial_1(\mathbf{f})$ does not have full rank.*

We explore the geometry of hyperdeterminantal varieties in §9. In contrast to the case of determinantal varieties, we show the varieties $Y(\phi)$ are rarely normal or Cohen–Macaulay.

Applications to Boij–Söderberg theory. The construction of tensor complexes has significant implications for Boij–Söderberg theory (see [ES2] for a survey) and the study of pure resolutions. A sequence $d = (d_0, \dots, d_p) \in \mathbb{Z}^{p+1}$ is a *degree sequence* if $d_i < d_{i+1}$ for all i. For a degree sequence d, we say that $G_\bullet$ is a *pure resolution of type d* if for each i, G_i is generated in degree d_i.

Theorem 1.9. *Let $d = (d_0, \dots, d_p) \in \mathbb{Z}^{p+1}$ be a degree sequence. Then there exist infinitely many choices of a, $\mathbf{b}$, and w such that w is a pinching weight for $\phi^{a\times \mathbf{b}}$, $F(\phi^{a\times \mathbf{b}}, w)_\bullet$ is a pure resolution of type d, and $M(\phi^{a\times \mathbf{b}}, w)$ is a Cohen–Macaulay module that is flat over $\mathbb{Z}$.*

The pure resolutions of type d constructed in Theorem 1.9 are unrelated to one another; this yields infinitely many new families of pure resolutions of type d for every d. More precisely, we have the following:

Proposition 1.10. *Suppose that* $a \geq \sum_{j=1}^{n}(b_j - 1)$ *and that* w *is a pinching weight for* $\phi^{a\times\mathbf{b}}$. *Then* $M(\phi^{a\times\mathbf{b}}, w)$ *is indecomposable.*

Theorem 1.9 builds on previous work of [EFW, ES1]. Namely, two constructions of Cohen–Macaulay modules with a pure resolution of type d were previously known: in characteristic 0 given in [EFW, §§3,4] and a different construction that works in arbitrary characteristic in [ES1, §5]. The Eisenbud–Schreyer construction arises as a hyperplane section of a certain tensor complex (see Theorem 10.1). However, this is unsurprising, as our original motivation for this project was to understand a multilinear version of their work.

Our results thus provide the first explicit description of pure resolutions over a field of positive characteristic, as we produce a closed formula for their differentials (without the need to explicitly compute the pushforward of a complex). In characteristic zero, a similar explicit description for the pure resolutions of [EFW, §3] appears in [SW, §§1, 2]. In another direction, a recent algorithm of Eisenbud, based on the Bernstein–Gelfand–Gelfand correspondence, enables the computation of the differentials of the pushforward of a complex. This algorithm would compute the differentials of any specific pure resolution of [ES1, §5] and is implemented in [M2, `BGG` package, version 1.4].

Finally, we note that the construction of the tensor complex $F(\phi, w)_\bullet$ extends to any scheme. Namely, if $\widetilde{\phi}$ is a global section of a tensor product of vector bundles $\mathcal{A} \otimes \mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_n$ on a scheme X, then there is a natural $\mathcal{O}_X$-module version of the complex $F(\phi, w)_\bullet$.

1.3. Outline

We outline our notation in §2 and describe the general geometric construction of the complex $F(\phi, w)_\bullet$. In §3, we introduce a particularly nice class of tensor complexes, called "balanced tensor complexes", and discuss their basic properties. The differentials of these complexes are described explicitly using representation-theoretic methods in §4.

Beginning with §5, we turn our attention to the main construction of tensor complexes, proving Theorem 1.2 via Theorem 5.3. §6 describes the construction of tensor complexes from strands of the Koszul complex, and §7 illustrates the functorial properties of tensor complexes. §§8 and 9 examine properties of the modules $M(\phi, w)$ and their supports $Y(\phi)$, respectively.

In §10, we relate the Eisenbud–Schreyer construction of pure modules to balanced tensor complexes. Further applications of our main results to Boij–Söderberg theory, including the construction of new families of pure resolutions, can be found in §11.

Finally, §12 provides a detailed example of a tensor complex, including presentation matrices for the differentials. We have also provided Appendix A, which reviews some basic definitions and constructions in multilinear algebra, and Appendix B provides a rapid review of the facts we employ from the representation theory of the general linear group over a field of characteristic zero.

2. Notation and general construction of the complex $F(\phi, w)_\bullet$

Let $a \in \mathbb{N}$ and $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{N}^n$. Let A and B_j, $1 \le j \le n$, be free $\mathbb{Z}$-modules of rank a and b_j, $1 \le j \le n$, respectively. We define $\mathbf{B} := B_1 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} B_n$ and $X := X^{a\times\mathbf{b}} := A \otimes \mathbf{B}^*$.

For a free $\mathbb{Z}$-module V, we write $\mathbb{Z}[V]$ for the symmetric algebra on V. Throughout, S will denote the polynomial ring $S := \mathbb{Z}[X^{a\times\mathbf{b}}] = \mathbb{Z}[x_{i,J}]$, where $1 \le i \le a$ and $J = (j_1, \dots, j_n)$, $1 \le j_\ell \le b_\ell$. We endow S with the standard $\mathbb{Z}$-grading $\deg(x_{i,J}) = 1$. We write the universal tensor $\phi = \phi^{a\times\mathbf{b}}$ as

$$\phi = (x_{i,J}) \in S \otimes_{\mathbb{Z}} X^{a\times\mathbf{b}}.$$

Given an $(n+1)$-tensor, there are a number of ways to obtain a matrix by "flattening" this tensor. One such flattening is particularly useful for our purposes. Via the isomorphism $S \otimes_{\mathbb{Z}} X^{a\times\mathbf{b}} \cong \mathrm{Hom}_S(S \otimes_{\mathbb{Z}} A^*, S \otimes_{\mathbb{Z}} \mathbf{B}^*)$, ϕ induces a map of free S-modules

$$\phi^\flat : S \otimes_{\mathbb{Z}} A^* \to S \otimes_{\mathbb{Z}} \mathbf{B}^*.$$

We write $\mathbb{P}(B_j)$ for the projective space of 1-dimensional quotients of B_j, so that $\mathbb{P}(B_j) = \mathrm{Proj}(\mathbb{Z}[B_j])$. Let $\mathbb{P}(\vec{B}) := \mathbb{P}(B_1) \times \cdots \times \mathbb{P}(B_n)$ and $\mathbb{A}^{a\times\mathbf{b}} := \mathrm{Spec}(\mathbb{Z}[X^{a\times\mathbf{b}}])$.

2.1. Representation theory conventions

Let $G = \mathbf{GL}(A) \times \mathbf{GL}(B_1) \times \cdots \times \mathbf{GL}(B_n)$. For a free $\mathbb{Z}$-module V of finite rank, we use $\mathrm{S}^i(V)$ to refer to its ith symmetric power, $\mathrm{D}^i(V)$ for its ith divided power, and $\det(V)$ for its top exterior power. We are most interested in divided powers twisted by a copy of the determinant, so we set

$$\widetilde{\mathrm{D}}^i(V) := \mathrm{D}^i(V) \otimes \det(V).$$

We use the convention that $\mathrm{H}^0(\mathbb{P}^0_{\mathbb{Z}}, \mathcal{O}(d)) \cong \mathrm{S}^d(\mathbb{Z})$ for all d. Although $\mathbf{GL}_1(\mathbb{Z}) \cong \mathbb{Z}/2$ cannot distinguish between two different d of the same parity, these representations are distinct from a "functor of points" perspective, i.e., they are distinct over larger coefficient rings, such as $\mathbb{Q}$. Similar remarks apply to powers of the determinant representation in general. When V is a $\mathbb{Q}$-vector space, we use $\mathbf{S}_\lambda V$ to denote irreducible representations of $\mathbf{GL}(V)$. See Appendix B for a summary of representation theory results used in this paper.

We write the representations over G as columns, so that the order of the rows allows us to omit the reference to the free modules $A, B_1, \dots, B_n$. Inside the columns, we abbreviate $\widetilde{\mathrm{D}}^i(B^*)$ as $\widetilde{\mathrm{D}}^i$. A twist by $S(-i)$ is denoted by $(-i)$ next to the column. For example,

$$\begin{bmatrix} \bigwedge^0 \\ \mathrm{S}^1 \\ \mathrm{S}^4 \end{bmatrix} := \bigwedge^0(A) \otimes \mathrm{S}^1(B_1) \otimes \mathrm{S}^4(B_2) \otimes S \quad \text{and}$$

$$\begin{bmatrix} \bigwedge^3 \\ \widetilde{\mathrm{D}}^1 \\ \mathrm{S}^1 \end{bmatrix}(-2) := \bigwedge^3(A) \otimes \widetilde{\mathrm{D}}^1(B_1^*) \otimes \mathrm{S}^1(B_2) \otimes S(-2).$$

2.2. Free resolution conventions

Conventions for the graded Betti diagrams of graded free complexes are standard. Namely, let $L_\bullet$ be a graded free complex over S. The *graded Betti numbers* $\beta_{i,j}(L_\bullet)$ are defined as follows:

$$L_i = \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{i,j}(L_\bullet)}.$$

The *Betti diagram* of $L_\bullet$ is then

$$\beta(L_\bullet) = \begin{pmatrix} \vdots & \vdots & & \vdots \\ \beta_{0,-1} & \beta_{1,0} & \cdots & \beta_{p,p-1} \\ \beta_{0,0} & \beta_{1,1} & \cdots & \beta_{p,p} \\ \beta_{0,1} & \beta_{1,2} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \end{pmatrix}.$$

Betti diagrams have nonzero entries in only finitely many positions, so we omit the rows of zeroes in examples.

Definition 2.1. Let M be a finitely generated graded S-module and $L_\bullet$ a free resolution of M. We say that $L_\bullet$ is *uniformly minimal* if $L_\bullet \otimes_{\mathbb{Z}} \Bbbk$ is a minimal free resolution for every field $\Bbbk$. In this case, we define $\beta_{i,j}(M) := \beta_{i,j}(L_\bullet)$.

2.3. General construction of $F(\phi, w)_\bullet$

To construct the complex $F(\phi, w)_\bullet$ we apply a minor extension of the geometric method [Wey, Chapter 5], working over $\mathbb{Z}$ instead of an arbitrary field. For the reader unfamiliar with [Wey], this may be a rather opaque definition. Several concrete descriptions of these complexes are given later (see Proposition 3.3, Definition 4.5, Theorem 5.3, and §6).

To apply this extension of the geometric method, we observe that the lemmas in [Wey, §5.2] hold over $\mathbb{Z}$ if the sheaves involved are flat over $\mathbb{Z}$ and all of the relevant sheaf cohomology (as in (2.5)) is free over $\mathbb{Z}$. In our situation, this is the case. (Alternatively, one can prove acyclicity of the relevant complexes over $\mathbb{Z}$ by proving acyclicity over each finite field as well as $\mathbb{Q}$, in which case the results of [Wey, Chapter 5] apply directly.)

Recall that $\mathbb{P}(\vec{B}) = \mathbb{P}(B_1) \times \cdots \times \mathbb{P}(B_n)$, and view $\mathbb{A}^{a \times \mathbf{b}} \times \mathbb{P}(\vec{B})$ as the total space of the trivial bundle $\mathcal{E} := A^* \otimes_{\mathbb{Z}} \mathbf{B} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{P}(\vec{B})}$ over $\mathbb{P}(\vec{B})$. Consider the vector bundle $\mathcal{T} := A^* \otimes \mathcal{O}_{\mathbb{P}(\vec{B})}(1, 1, \ldots, 1)$ on $\mathbb{P}(\vec{B})$. There is a natural surjective map $\mathcal{E} \to \mathcal{T}$ induced by the natural maps $B_i \otimes \mathcal{O}_{\mathbb{P}(B_i)} \to \mathcal{O}_{\mathbb{P}(B_i)}(1)$. Let $\mathcal{S}$ be the kernel of this map, so that we have an exact sequence of vector bundles on $\mathbb{P}(\vec{B})$ of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{S} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{T} & \longrightarrow & 0 \\ & & & & \| & & \| & & \\ & & & & A^* \otimes_{\mathbb{Z}} \mathbf{B} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{P}(\vec{B})} & & A^* \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{P}(\vec{B})}(1, \ldots, 1) & & \end{array}$$

Explicitly, $\mathcal{S} = \mathcal{H}om((\mathbf{B}^* \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{P}(\vec{B})})/\mathcal{O}_{\mathbb{P}(\vec{B})}(-1,\dots,-1), A^* \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{P}(\vec{B})})$, and we let $Z(\phi) = Z(\phi^{a\times\mathbf{b}}) \subseteq \mathbb{A}^{a\times\mathbf{b}} \times \mathbb{P}(\vec{B})$ denote $\mathbb{V}_{\mathbb{P}(\vec{B})}(\mathcal{S})$, the total space of $\mathcal{S}$. The total space $\mathbb{V}_{\mathbb{P}(\vec{B})}(\mathcal{E})$ of the vector bundle $\mathcal{E}$ is $\mathbb{A}^{a\times\mathbf{b}} \times \mathbb{P}(\vec{B})$. Write $\pi : \mathbb{A}^{a\times\mathbf{b}} \times \mathbb{P}(\vec{B}) \to \mathbb{A}^{a\times\mathbf{b}}$ for the projection. Let $Y(\phi) = \pi(Z(\phi))$ scheme-theoretically. Note that $Z(\phi)$ and $Y(\phi)$ are integral schemes. We have a commutative diagram

$$\begin{array}{ccc} Z(\phi) = \mathbb{V}_{\mathbb{P}(\vec{B})}(\mathcal{S}) & \longrightarrow & \mathbb{A}^{a\times\mathbf{b}} \times \mathbb{P}(\vec{B}) = \mathbb{V}_{\mathbb{P}(\vec{B})}(\mathcal{E}) \\ \mu \downarrow & & \downarrow \pi \\ Y(\phi) & \longrightarrow & \mathbb{A}^{a\times\mathbf{b}} \end{array} \tag{2.2}$$

Let $\pi_2 \colon \mathbb{A}^{a\times\mathbf{b}} \times \mathbb{P}(\vec{B}) \to \mathbb{P}(\vec{B})$ be the natural projection, and consider the following Koszul complex on $\mathbb{A}^{a\times\mathbf{b}} \times \mathbb{P}(\vec{B})$:

$$\mathcal{K}(\phi)_\bullet : \quad \mathcal{O}_{\mathbb{A}^{a\times\mathbf{b}}\times\mathbb{P}(\vec{B})} \leftarrow \bigwedge^1(\pi_2^*\mathcal{T}^*) \leftarrow \cdots \leftarrow \bigwedge^{a-1}(\pi_2^*\mathcal{T}^*) \leftarrow \bigwedge^a(\pi_2^*\mathcal{T}^*) \leftarrow 0 \tag{2.3}$$

which resolves the sheaf $\mathcal{O}_{Z(\phi)}$.

Definition 2.4. Fix a *weight vector* $w = (w_0, \dots, w_n) \in \mathbb{Z}^{n+1}$. To define $F(\phi, w)_\bullet$, we follow the construction of [Wey, Theorem 5.1.2], with $\mathcal{O}_{\mathbb{P}(\vec{B})}(w_1, \dots, w_n)$ in place of $\mathcal{V}$. Additionally, we twist the resulting complex by $S(-w_0)$ to obtain the *tensor complex*, denoted $F(\phi, w)_\bullet$.

It follows immediately that $F(\phi, w)_\bullet$ is a graded, free complex of S-modules that is quasi-isomorphic to $\mathbf{R}\pi_*\big(\mathcal{K}(\phi)_\bullet \otimes \pi_2^*\mathcal{O}_{\mathbb{P}(\vec{B})}(w_1, \dots, w_n)\big) \otimes_S S(-w_0)$. The terms of $F(\phi, w)_\bullet$ are

$$\begin{aligned} F(\phi, w)_i &= \bigoplus_{j\ge 0} \mathrm{H}^j\big(\mathbb{P}(\vec{B}), \textstyle\bigwedge^{i+j}\mathcal{T}^* \otimes \mathcal{O}_{\mathbb{P}(\vec{B})}(w_1, \dots, w_n)\big) \otimes S(-i-j-w_0) \\ &= \bigoplus_{j\ge 0} \mathrm{H}^j\big(\mathbb{P}(\vec{B}), \mathcal{O}_{\mathbb{P}(\vec{B})}(w_1 - i - j, \dots, w_n - i - j)\big) \otimes \textstyle\bigwedge^{i+j} A \otimes S(-i-j-w_0). \end{aligned} \tag{2.5}$$

We write ∂_i for the differential $F(\phi, w)_i \to F(\phi, w)_{i-1}$. Let $M(\phi, w) := \operatorname{coker} \partial_1$.

There is a minor abuse of notation inherent in the above definition. Namely, to define the differentials of such a complex via the geometric method, we must explicitly compute a free complex to represent the quasi-isomorphism class of a pushforward of a complex, and there is some choice involved in building this complex (see [Wey, §5.5]). Thus the differentials ∂_i are not a priori determined by ϕ and w. We ignore this subtlety because our main cases of interest are when w is a pinching weight for ϕ, and in these cases, we may make a canonical choice for each differential (up to sign) via representation theory, as illustrated in Proposition 4.1 and Theorem 5.3.

Remark 2.6. Let $K(\phi)_\bullet$ denote the $\mathbb{Z}^{n+1}$-graded complex of graded free $\mathbb{Z}$-modules on $\mathbb{Z}[X^{a\times\mathbf{b}}] \otimes \mathbb{Z}[B_1] \otimes \cdots \otimes \mathbb{Z}[B_n] = \mathbb{Z}[x_{i,J}, y_{j_\ell}]$ corresponding to the Koszul complex of sheaves $\mathcal{K}(\phi)_\bullet$ from (2.3). For $J = (j_1, \ldots, j_n)$, set $y_J := y_{j_1} \cdots y_{j_n}$. Consider the multilinear forms

$$f_i := \sum_J x_{i,J}\, y_J, \qquad i = 1, \ldots, a.$$

Then $K(\phi)_\bullet$ is the Koszul complex on $(f_1, \ldots, f_a)$.

Remark 2.7. If we replace A by any $\mathbb{Z}/2$-graded free $\mathbb{Z}$-module and take care in using $\mathbb{Z}/2$-graded multilinear algebra (see, for example, [Wey, §2.4]), essentially all of our assertions about tensor complexes remain true, with one significant difference. If the odd part of A is nonzero, then S will be a graded commutative algebra, and the resulting complexes will be infinite in length in one direction. If the even part of A is 0, then we obtain pure resolutions over the exterior algebra.

3. Balanced tensor complexes

In §2 we defined $F(\phi, w)_\bullet$ for an arbitrary weight vector $w \in \mathbb{Z}^{n+1}$. To obtain free resolutions with nice properties, including those outlined in Theorem 1.2, we impose further conditions on the weight vector w. For clarity, we begin by introducing a particularly simple class of examples called balanced tensor complexes. The construction is sufficiently rich to produce tensor complexes that are pure resolutions of type d for every degree sequence d. In fact, this construction is closely modeled on the Eisenbud–Schreyer construction of pure resolutions [ES1, §5]. In §5, we extend the results of this section to more general tensor complexes.

Definition 3.1. We say that $F(\phi^{a\times\mathbf{b}}, w)_\bullet$ is a *balanced tensor complex* if it satisfies the following conditions:

(i) $a = b_1 + \cdots + b_n$.

(ii) $w_1 = 0$ and $w_i = b_1 + \cdots + b_{i-1}$ for $i = 2, \ldots, n$.

Set $d(w) := (w_0, w_2 + w_0, w_3 + w_0, \ldots, w_{n-1} + w_0, w_n + w_0, a + w_0) \in \mathbb{Z}^{n+1}$.

The condition (i) is less restrictive than it appears because we allow the possibility of tensoring with rank 1 free modules. For instance, there is a natural way to identify a $7 \times (3, 2)$ tensor with a $7 \times (3, 1, 2, 1)$ tensor or with a $7 \times (1, 1, 3, 2)$ tensor and so on. These identifications enable us to produce many examples of balanced tensor complexes. The following example illustrates this flexibility.

Example 3.2 (Complexes of [Eis, §A2.6]). Let $b \le a \in \mathbb{N}$. The matrix complexes $\mathcal{C}^0, \ldots, \mathcal{C}^{a-b}$ of [Eis, §A2.6] may be realized as examples of balanced tensor complexes. Fix $0 \le i \le a - b$. Let

$$\mathbf{b} := (\underbrace{1, \ldots, 1}_{i}, b, \underbrace{1, \ldots, 1}_{a-b-i}).$$

The corresponding balanced tensor complex $F(\phi^{a\times\mathbf{b}}, w)_\bullet$ is isomorphic to $\mathcal{C}^i$.

The following proposition proves a portion of Theorem 1.2 for balanced tensor complexes.

Proposition 3.3. *Suppose that $F(\phi, w)_\bullet$ is a balanced tensor complex. Write $d(w) = (d_0, \ldots, d_n)$. Then*

$$F(\phi, w)_i \cong S(-d_i) \otimes \bigwedge^{d_i - w_0} A \otimes \bigotimes_{j=1}^{i} \widetilde{\mathrm{D}}^{d_i - d_j}(B_j^*) \otimes \bigotimes_{j=i+1}^{n} \mathrm{S}^{d_{j-1} - d_i}(B_j). \tag{3.4}$$

In particular, $F(\phi, w)_\bullet$ is a pure resolution of type $d(w)$ and satisfies the conditions of Theorem 1.2(i)–(iii).

Proof. From (2.5), we must consider sheaves of the form $\bigwedge^l \mathcal{T}^* \otimes \mathcal{O}(w_1, \ldots, w_n) = \bigwedge^l A \otimes \mathcal{O}(w_1 - l, \ldots, w_n - l)$, which is nonzero only if $l \in [0, a]$. By the Künneth formula, this sheaf will have nonzero cohomology precisely when $l \notin [w_i + 1, w_i + b_i - 1]$ for all $i = 1, \ldots, n$. Since $w_{i+1} = w_i + b_i$, it immediately follows that $l \in \{0 = w_1, w_2, \ldots, w_n, w_n + b_n = a\}$.

Set $d' := (0, w_2, \ldots, w_n, w_n + b_n)$, and note that $d'_i + w_0 = d_i$ for all i. Computing the cohomology for $l = d'_i$ yields

$$F(\phi, w)_i = S(-d_i) \otimes \bigwedge^{d_i - w_0} A \otimes \bigotimes_{j=1}^{i} \mathrm{H}^{b_j - 1}(\mathbb{P}(B_j), \mathcal{O}(w_j - d'_i)) \\ \otimes \bigotimes_{j=i+1}^{n} \mathrm{H}^0(\mathbb{P}(B_j), \mathcal{O}(w_j - d'_i)),$$

which is (3.4). In particular, the complex has no terms in negative homological degrees, and hence [Wey, Theorem 5.1.2] implies that $F(\phi, w)_\bullet$ is a minimal free resolution of $M(\phi, w)$ and $M(\phi, w) \otimes_S S(w_0)$ is naturally isomorphic to $\mathrm{H}^0(\mathbb{P}(\vec{B}), \mathrm{S}^\bullet(\mathcal{S}^*) \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{P}(\vec{B})}(w_1, \ldots, w_n))$. Since the latter is free over $\mathbb{Z}$, $M(\phi, w)$ is also free over $\mathbb{Z}$, completing the proof of Theorem 1.2(ii).

We now prove that $M(\phi, w)$ is Cohen–Macaulay (i.e., Theorem 1.2(i)). Since we know that $\operatorname{pdim} M(\phi, w) = n \geq \operatorname{codim} M(\phi, w)$, it suffices to show that $\operatorname{codim} M(\phi, w) \geq n$. By [Wey, Theorem 5.1.2(b)], the support of $M(\phi, w)$ is the variety $Y(\phi)$ from (2.2). Recall that $Z(\phi)$ is the total space of $\mathcal{S}$. The codimension of $Z(\phi)$ in $\mathbb{A}^{a \times \mathbf{b}} \times \mathbb{P}(\vec{B})$ thus equals the rank of $\mathcal{T}$, which is a. Therefore

$$\dim Y(\phi) \leq \dim Z(\phi) = \dim \mathbb{A}^{a \times \mathbf{b}} + \dim \mathbb{P}(\vec{B}) - a = \dim \mathbb{A}^{a \times \mathbf{b}} - n,$$

so $\operatorname{codim} Y(\phi) \geq n$, as desired. □

We provide a more detailed description of the support of $M(\phi, w)$ in §8. We also note that for any degree sequence d there exists a unique balanced tensor complex $F(\phi, w)_\bullet$ that is a pure resolution of type d. This follows from Theorem 10.1.

Example 3.5. Take $a = 11$ and $\mathbf{b} = (3, 1, 3, 4)$. To obtain a balanced complex, we set $w = (0, 0, 3, 4, 7)$. Then $d(w) = (0, 3, 4, 7, 11)$ with the following free resolution:

$$S^{1800} \leftarrow S^{17325}(-3) \leftarrow S^{19800}(-4) \leftarrow S^{4950}(-7) \leftarrow S^{675}(-11) \leftarrow 0,$$

which we denote

$$\begin{bmatrix} \bigwedge^0 \\ \mathrm{S}^0 \\ \mathrm{S}^3 \\ \mathrm{S}^4 \\ \mathrm{S}^7 \end{bmatrix} \leftarrow \begin{bmatrix} \bigwedge^3 \\ \widetilde{\mathrm{D}}^0 \\ \mathrm{S}^0 \\ \mathrm{S}^1 \\ \mathrm{S}^4 \end{bmatrix}(-3) \leftarrow \begin{bmatrix} \bigwedge^4 \\ \widetilde{\mathrm{D}}^1 \\ \widetilde{\mathrm{D}}^0 \\ \mathrm{S}^0 \\ \mathrm{S}^3 \end{bmatrix}(-4) \leftarrow \begin{bmatrix} \bigwedge^7 \\ \widetilde{\mathrm{D}}^4 \\ \widetilde{\mathrm{D}}^3 \\ \widetilde{\mathrm{D}}^0 \\ \mathrm{S}^0 \end{bmatrix}(-7) \leftarrow \begin{bmatrix} \bigwedge^{11} \\ \widetilde{\mathrm{D}}^8 \\ \widetilde{\mathrm{D}}^7 \\ \widetilde{\mathrm{D}}^4 \\ \widetilde{\mathrm{D}}^0 \end{bmatrix}(-11) \leftarrow 0.$$

4. Explicit differentials for balanced tensor complexes

Since our definition of $F(\phi, w)_\bullet$ involves an application of the geometric method, we would a priori need to explicitly compute the pushforward of a complex in order to define a specific differential. In this section, we use a representation-theoretic argument to illustrate that such a computation is unnecessary for balanced tensor complexes. Definition 4.5 describes the equivariant differential, and the main result of this section is Proposition 4.1. In §5, we extend this proposition to the more general setting of Theorem 1.2.

Proposition 4.1. *Let $F(\phi, w)_\bullet$ be a balanced tensor complex. Up to sign, there is a unique differential $\partial_\bullet$, defined explicitly in Definition 4.5, which makes $F(\phi, w)_\bullet$ into a G-equivariant free resolution. In particular, $(F(\phi, w)_\bullet, \partial_\bullet)$ satisfies Theorem 1.2(iv).*

For a free module F_i we use $[F_i]_e$ to denote the degree e piece of F_i; for a map of free modules $f\colon F_i \to F_{i-1}$ we use $[f]_e$ to denote the induced map $[f]_e\colon [F_i]_e \to [F_{i-1}]_e$. To define the G-equivariant differentials $\partial_i\colon F(\phi, w)_i \to F(\phi, w)_{i-1}$, we define a G-equivariant map $[\partial_i]_{d_i}\colon [F(\phi, w)_i]_{d_i} \to [F(\phi, w)_{i-1}]_{d_i}$ on the generators of $F(\phi, w)_i$ and extend S-linearly.

By Proposition 3.3, the source of $[\partial_i]_{d_i}$ is given by

$$[F(\phi, w)_i]_{d_i} = \bigwedge^{d_i - w_0} A \otimes \bigotimes_{j=1}^{i} \widetilde{\mathrm{D}}^{d_i - d_j}(B_j^*) \otimes \bigotimes_{j=i+1}^{n} \mathrm{S}^{d_{j-1} - d_i}(B_j). \tag{4.2}$$

Noting that $b_i = d_i - d_{i-1}$, the corresponding decomposition for the target is

$$\begin{aligned} [F(\phi, w)_{i-1}]_{d_i} &= [F(\phi, w)_{i-1}]_{d_{i-1}} \otimes \mathrm{S}^{b_i}(X) \\ &= \Big(\bigwedge^{d_{i-1} - w_0} A \otimes \bigotimes_{j=1}^{i-1} \widetilde{\mathrm{D}}^{d_{i-1} - d_j}(B_j^*) \otimes \bigotimes_{j=i}^{n} \mathrm{S}^{d_{j-1} - d_{i-1}}(B_j)\Big) \otimes \mathrm{S}^{b_i}(X). \end{aligned} \tag{4.3}$$

By Appendix A, we have inclusions of G-modules

$$\begin{aligned}\mathrm{S}^{b_i}(X) &\supseteq \bigwedge^{b_i} A \otimes \bigwedge^{b_i}(B_1^* \otimes \cdots \otimes B_n^*) \supseteq \bigwedge^{b_i} A \otimes \det B_i^* \otimes \mathrm{D}^{b_i}\Big(\bigotimes_{j\neq i} B_j^*\Big)\\ &\supseteq \bigwedge^{b_i} A \otimes \det B_i^* \otimes \bigotimes_{j\neq i} \mathrm{D}^{b_i}(B_j^*).\end{aligned} \tag{4.4}$$

Definition 4.5 (Equivariant differentials on $F(\phi, w)_\bullet$). The map $[\partial_i]_{d_i} : [F(\phi, w)_i]_{d_i} \to [F(\phi, w)_{i-1}]_{d_i}$ is defined to be the composition of a map

$$\iota : [F(\phi, w)_i]_{d_i} \to [F(\phi, w)_i]_{d_{i-1}} \otimes \Big(\bigwedge^{b_i} A \otimes \det B_i^* \otimes \bigotimes_{j\neq i} \mathrm{D}^{b_i}(B_j^*)\Big)$$

with the inclusion obtained from (4.4). We define ι to be the tensor product $\iota = \iota_A \otimes \iota_{B_1} \otimes \cdots \otimes \iota_{B_n}$ where the components are defined below. For ι_A we take the comultiplication map

$$\iota_A : \bigwedge^{d_i - w_0} A \to \bigwedge^{d_{i-1} - w_0} A \otimes \bigwedge^{b_i} A.$$

For $j \le i-1$, we take the twist by $\det(B_j^*)$ of the dual of the multiplication map $\mathrm{S}^{b_i}(B_j) \otimes \mathrm{S}^{d_{i-1}-d_j}(B_j) \to \mathrm{S}^{d_i - d_j}(B_j)$, and set

$$\iota_{B_j} : \widetilde{\mathrm{D}}^{d_i - d_j}(B_j^*) \to \widetilde{\mathrm{D}}^{d_{i-1}-d_j}(B_j^*) \otimes \mathrm{D}^{b_i}(B_j^*).$$

For $j = i$, we choose an identification (unique up to sign) $\widetilde{\mathrm{D}}^0(B_i^*) \cong \mathrm{S}^0(B_i) \otimes \det(B_i^*)$. Finally, when $j \ge i+1$ we take the dual of the contraction map $\mathrm{D}^{d_{j-1}-d_{i-1}}(B_j^*) \otimes \mathrm{S}^{b_i}(B_j) \to \mathrm{D}^{d_{j-1}-d_i}(B_j^*)$, and set

$$\iota_{B_j} : \mathrm{S}^{d_{j-1}-d_i}(B_j) \to \mathrm{S}^{d_{j-1}-d_{i-1}}(B_j) \otimes \mathrm{D}^{b_i}(B_j^*).$$

We then define $\partial_i : F(\phi, w)_i \to F(\phi, w)_{i-1}$ as the S-linear extension of $[\partial_i]_{d_i}$. The map ∂_i is clearly G-equivariant.

We say that a map of free $\mathbb{Z}$-modules is *saturated* if its cokernel is also a free $\mathbb{Z}$-module.

Lemma 4.6. *The map $[\partial_i]_{d_i}$ is saturated and injective.*

Proof. Since $[\partial_i]_{d_i}$ is the tensor product of the maps ι_A and ι_{B_j}, it suffices to show that each of these maps is saturated and injective. For ι_A this follows from [ABW, Theorems III.1.4, III.2.4]. For $j \neq i$, the map ι_{B_j} is the dual of a surjective map of free $\mathbb{Z}$-modules, so it is saturated and injective. Finally, the map ι_{B_i} is an isomorphism. □

The following lemma is essential to the claim of uniqueness in Proposition 4.1.

Lemma 4.7 (Base change to $\mathbb{Q}$). *The $G(\mathbb{Q})$-representation*

$$[F(\phi, w)_i]_{d_i} \otimes \mathbb{Q} = \bigwedge^{d_i - w_0} A \otimes \bigotimes_{j=1}^{i} \widetilde{\mathrm{D}}^{d_i - d_j}(B_j^*) \otimes \bigotimes_{j=i+1}^{n} \mathrm{S}^{d_{j-1}-d_i}(B_j) \otimes \mathbb{Q}$$

appears with multiplicity 1 *inside* $[F(\phi, w)_{i-1}]_{d_i} \otimes \mathrm{S}^{b_i}(X) \otimes \mathbb{Q}$.

Proof. We first find the subrepresentations $W = \mathbf{S}_\lambda A \otimes \mathbf{S}_{\mu^1} B_1^* \otimes \cdots \otimes \mathbf{S}_{\mu^n} B_n^* \otimes \mathbb{Q}$ of $\mathrm{S}^{b_i}(X) \otimes \mathbb{Q}$ whose tensor product with $[F(\phi, w)_{i-1}]_{d_{i-1}} \otimes \mathbb{Q}$ contains $[F(\phi, w)_i]_{d_i} \otimes \mathbb{Q}$. By Pieri's rule (B.1) (see Appendix B), this only happens for $\lambda = \mu^i = (1^{b_i})$ and $\mu^j = (b_i)$ for $j \neq i$. By Schur–Weyl duality ((B.2) and (B.3)), W appears in $\mathrm{S}^{b_i}(X) \otimes \mathbb{Q}$ with multiplicity 1. □

There is a straightforward proof that $\partial^2 = 0$, which we include below.

Lemma 4.8. *For any $i \geq 1$ we have $\partial_i \partial_{i+1} = 0$. In particular, $(F(\phi, w)_\bullet, \partial_\bullet)$ is a complex.*

Proof. It is enough to verify that the composition $[F(\phi, w)_{i+2}]_{d_{i+2}} \to [F(\phi, w)_{i+1}]_{d_{i+2}} \to [F(\phi, w)_i]_{d_{i+2}}$ is 0 where $0 \leq i \leq n-2$. Since $F(\phi, w)_\bullet$ is a free complex, we may tensor with $\mathbb{Q}$ before checking that this map is 0. Thus, for the rest of the proof, we assume that all free $\mathbb{Z}$-modules have been tensored by $\mathbb{Q}$.

Since the maps ∂_i are G-equivariant, it suffices to show that any $G(\mathbb{Q})$-equivariant map $[F(\phi, w)_{i+2}]_{d_{i+2}} \to [F(\phi, w)_i]_{d_{i+2}}$ is zero. First write $e := d_{i+2} - d_i = b_{i+2} + b_{i+1}$. By (B.4),

$$\mathrm{S}^e X = \bigoplus_{\lambda \vdash e} \mathbf{S}_\lambda A \otimes \mathbf{S}_\lambda(\mathbf{B}^*).$$

By Pieri's rule (B.1), the generators of $[F(\phi, w)_{i+2}]_{d_{i+2}}$ can only appear in the tensor product of the generators of $[F(\phi, w)_i]_{d_{i+2}}$ with the direct summand with $\mathbf{S}_\lambda = \bigwedge^e$. Now by (B.4),

$$\bigwedge^e(\mathbf{B}^*) = \bigoplus_{\mu \vdash e} \mathbf{S}_\mu(B_1^* \otimes \cdots \otimes B_i^* \otimes B_{i+3}^* \otimes \cdots \otimes B_n^*) \otimes \mathbf{S}_{\mu'}(B_{i+1}^* \otimes B_{i+2}^*).$$

Using Pieri's rule (B.1) and Schur–Weyl duality ((B.2) and (B.3)), we only need to focus on the summand with $\mathbf{S}_\mu = \mathrm{S}^e$. By (B.4),

$$\bigwedge^e(B_{i+1}^* \otimes B_{i+2}^*) = \bigoplus_{\nu \vdash e} \mathbf{S}_\nu B_{i+1}^* \otimes \mathbf{S}_{\nu'} B_{i+2}^*,$$

so we must show that

$$(\mathbf{S}_\nu B_{i+1}^* \otimes \mathbf{S}_{\nu'} B_{i+2}^*) \otimes \mathrm{S}^{b_{i+1}} B_{i+2}$$

does not contain a copy of $(\mathrm{S}^{b_{i+2}} B_{i+1})^* \otimes \det B_{i+1}^* \otimes \det B_{i+2}^*$ for any $\nu \vdash e$. Since $\operatorname{rank} B_{i+2}^* = b_{i+2}$, this happens precisely when ν' is the partition $(b_{i+1} + 1, 1^{b_{i+2}-1})$. However, in this case $\mathbf{S}_\nu B_{i+1}^* = 0$ because $\operatorname{rank} B_{i+1} = b_{i+1}$. □

Proposition 4.9. *The complex $(F(\phi, w)_\bullet, \partial_\bullet)$ is a free resolution of $M(\phi, w)$.*

Proof. To simplify notation, we drop reference to ϕ and w throughout this proof. Let $(F_\bullet, \epsilon_\bullet)$ be a uniformly minimal free resolution of M. We use $\epsilon_0 \colon F_0 \to M$ to denote the natural quotient map. From Lemma 4.8, $(F_\bullet, \partial_\bullet)$ is a free complex. We set $\partial_0 := \epsilon_0$.

We first claim that $\partial_0 \partial_1 = 0$. This can be checked after base changing to $\mathbb{Q}$. By [Wey, Theorem 5.4.1], the complex $F_\bullet \otimes \mathbb{Q}$ admits a $G(\mathbb{Q})$-equivariant differential $\epsilon'_\bullet$ which makes it acyclic. By Lemma 4.7, $[\partial_1]_{d_1} \otimes \mathbb{Q}$ is a nonzero scalar multiple of $[\epsilon'_1]_{d_1}$, and thus $\epsilon'_0\epsilon'_1 = 0 = \partial_0\partial_1$.

Now, since $(F_\bullet, \epsilon_\bullet)$ is a resolution of M and $(F_\bullet, \partial_\bullet)$ is a free complex mapping to M (by ∂_0), the identity $M \xrightarrow{\text{id}} M$ induces a map of complexes $a_\bullet : (F_\bullet, \partial_\bullet) \to (F_\bullet, \epsilon_\bullet)$ by [Eis, Lemma 20.3]. We claim that a_i is an isomorphism for each i, and we proceed by induction.

For $i = 0$, we may assume that a_0 is the identity. For the induction step, we assume that a_i is an isomorphism, so we have a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & [F_{i+1}]_{d_{i+1}} & \xrightarrow{[\epsilon_{i+1}]_{d_{i+1}}} & [F_i]_{d_{i+1}} & \longrightarrow & \operatorname{coker}([\epsilon_{i+1}]_{d_{i+1}}) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle [a_{i+1}]_{d_{i+1}}} & & \cong \uparrow{\scriptstyle [a_i]_{d_{i+1}}} & & \uparrow{\scriptstyle b} & & \\
0 & \longrightarrow & [F_{i+1}]_{d_{i+1}} & \xrightarrow{[\partial_{i+1}]_{d_{i+1}}} & [F_i]_{d_{i+1}} & \longrightarrow & \operatorname{coker}([\partial_{i+1}]_{d_{i+1}}) & \longrightarrow & 0
\end{array}$$

Since the middle arrow is an isomorphism, it follows that b is surjective. The cokernel of $[\epsilon_{i+1}]_{d_{i+1}}$ is a free $\mathbb{Z}$-module since the complex $(F_\bullet, \epsilon_\bullet)$ is a uniformly minimal resolution, and the cokernel of $[\partial_{i+1}]_{d_{i+1}}$ is a free $\mathbb{Z}$-module by Lemma 4.6. Thus, b is an isomorphism. By the five lemma, we conclude that $[a_{i+1}]_{d_{i+1}}$ is an isomorphism of $\mathbb{Z}$-modules, and hence a_{i+1} is an isomorphism of S-modules. □

We are now prepared to prove the main result of this section.

Proof of Proposition 4.1. Theorem 1.2(iv) follows from Proposition 4.9. For uniqueness, assume that $\epsilon_\bullet$ is another G-equivariant differential. Lemma 4.7, after a base-change to $\mathbb{Q}$, implies that ϵ_i and ∂_i differ by an integer scalar multiple. By uniform minimality, this integer cannot be divisible by any prime number, so it must be ± 1. □

Remark 4.10 (Kronecker coefficients). In characteristic 0, the acyclicity of the complex $(F(\phi, w)_\bullet, \partial_\bullet)$ imposes nonvanishing conditions on the Kronecker coefficients $g_{\lambda,\mu^1,\dots,\mu^n}$ (see Appendix B for the relevant definitions and results). For example, let $n = 2$ and consider the first differential

$$\bigwedge^{d_1} A \otimes \bigwedge^{d_1} B_1^* \otimes S(-d_1) \to \mathrm{S}^{d_1} B_2 \otimes S.$$

When $i < d_2 - d_1$, the G-equivariant map

$$\bigwedge^{d_1} A \otimes \bigwedge^{d_1} B_1^* \otimes \mathrm{S}^i(A \otimes B_1^* \otimes B_2^*) \to \mathrm{S}^{d_1} B_2 \otimes \mathrm{S}^{i+d_1}(A \otimes B_1^* \otimes B_2^*)$$

is injective. Now rewrite the left-hand side as

$$\bigwedge^{d_1} A \otimes \bigwedge^{d_1} B_1^* \otimes \bigoplus_{\lambda,\mu,\nu \vdash i} (\mathbf{S}_\lambda A \otimes \mathbf{S}_\mu B_1^* \otimes \mathbf{S}_\nu B_2^*)^{\oplus g_{\lambda,\mu,\nu}},$$

and the right-hand side as

$$\mathrm{S}^{d_1} B_2 \otimes \bigoplus_{\alpha,\beta,\gamma \vdash i+d_1} (\mathbf{S}_\alpha A \otimes \mathbf{S}_\beta B_1^* \otimes \mathbf{S}_\gamma B_2^*)^{\oplus g_{\alpha,\beta,\gamma}}.$$

It follows that if $g_{\lambda,\mu,\nu} \neq 0$, then for any partition α obtained from λ by adding a vertical strip of size d_1 and $\beta = (\mu_1 + 1, \dots, \mu_{d_1} + 1)$, there exists γ obtained from ν by adding a horizontal strip of size d_1 such that $g_{\alpha,\beta,\gamma} \neq 0$.

4.1. Writing the differentials via minors of flattenings

Definition 4.5 provides the following method for writing the differentials of $F(\phi, w)_\bullet$ explicitly in terms of minors of the flattening $\phi^\flat$. If we choose bases $\{f_k\}$ and $\{g_l\}$ of $F(\phi, w)_i$ and $F(\phi, w)_{i-1}$, then we may represent ∂_i by a matrix Ψ of polynomials of degree b_i. Consider the map

$$\alpha\colon [F(\phi, w)_i]_{d_i} \otimes [F(\phi, w)_{i-1}]^*_{d_{i-1}} \to \mathrm{S}^{b_i}(X)$$

which is adjoint to $[\partial_i]_{d_i}$. Note that $\alpha(f_k, g_l^*)$ is the (k, l)th entry of Ψ. Now consider the adjoint γ of the map ι given in Definition 4.5:

$$\gamma\colon [F(\phi, w)_i]_{d_i} \otimes [F(\phi, w)_{i-1}]^*_{d_{i-1}} \to \bigwedge^{b_i} A \otimes \det B_i^* \otimes \bigotimes_{j\neq i} \mathrm{D}^{b_i}(B_j^*).$$

Since $[\partial_i]_{d_i}$ was defined in terms of ι and the inclusion (4.4), it follows that α is given by γ and (4.4).

The first line of (4.4) corresponds to the inclusion of the $b_i \times b_i$ minors of $\phi^\flat$ into the space of all polynomials of degree b_i. Hence each entry of Ψ may be defined in terms of $b_i \times b_i$ minors of $\phi^\flat$, and we may write α explicitly via a formula for the inclusion

$$\bigwedge^{b_i} A \otimes \det B_i^* \otimes \bigotimes_{j\neq i} \mathrm{D}^{b_i}(B_j^*) \subseteq \bigwedge^{b_i} A \otimes \bigwedge^{b_i}(B_1^* \otimes \cdots \otimes B_n^*).$$

We obtain the necessary formula for this inclusion from repeated applications of the multilinear inclusions described in Appendix A.

Example 4.11. Let $a \times \mathbf{b} = 4 \times (2, 2)$ and $w = (0, 0, 2)$. The complex $F(\phi, w)_\bullet$ has the form

$$\begin{bmatrix} \bigwedge^0 \\ \mathrm{S}^0 \\ \mathrm{S}^2 \end{bmatrix} \xleftarrow{\partial_1} \begin{bmatrix} \bigwedge^2 \\ \widetilde{\mathrm{D}}^0 \\ \mathrm{S}^0 \end{bmatrix}(-2) \xleftarrow{\partial_2} \begin{bmatrix} \bigwedge^4 \\ \widetilde{\mathrm{D}}^2 \\ \widetilde{\mathrm{D}}^0 \end{bmatrix}(-4) \leftarrow 0.$$

Our goal is to write the differential ∂_1 explicitly.

Let $\{\alpha_1, \ldots, \alpha_4\}$ be a basis for A, $\{u_1, u_2\}$ be a basis for B_1^*, and $\{v_1, v_2\}$ be a basis for B_2^*. Also, let $\{v_1^*, v_2^*\}$ be the dual basis for B_2. To represent ∂_1 by a matrix, we choose the natural bases of $F(\phi, w)_1$ and $F(\phi, w)_0$ induced by our choice of bases for A, B_1^*, and B_2^*. Namely, our basis of $F(\phi, w)_1$ is given by the six elements of the form

$$f_{\{i_1,i_2\},\{1,2\},\emptyset} := (\alpha_{i_1} \wedge \alpha_{i_2}) \otimes (u_1 \wedge u_2) \otimes 1,$$

where $1 \le i_1 < i_2 \le 4$. Our basis of $F(\phi, w)_0$ is given by the three elements of the form

$$g_{\emptyset,\emptyset,(j_1,j_2)} := 1 \otimes 1 \otimes (v_1^{*j_1} v_2^{*j_2}),$$

where $(j_1, j_2) \in \mathbb{N}^2$ and $j_1 + j_2 = 2$. With notation as in this subsection, we have

$$\gamma(f_{\{i_1,i_2\},\{1,2\},\emptyset} \otimes g^*_{\emptyset,\emptyset,(j_1,j_2)}) = (\alpha_{i_1} \wedge \alpha_{i_2}) \otimes (u_1 \wedge u_2) \otimes (v_1^{(j_1)} v_2^{(j_2)}). \tag{4.12}$$

If we represent $\phi^\flat$ by the matrix of linear forms

$$\phi^\flat = \begin{array}{c} \\ u_1^*\otimes v_1^* \\ u_1^*\otimes v_2^* \\ u_2^*\otimes v_1^* \\ u_2^*\otimes v_2^* \end{array} \begin{array}{c} \begin{array}{cccc} \alpha_1^* & \alpha_2^* & \alpha_3^* & \alpha_4^* \end{array} \\ \begin{pmatrix} x_{1,(1,1)} & x_{2,(1,1)} & x_{3,(1,1)} & x_{4,(1,1)} \\ x_{1,(1,2)} & x_{2,(1,2)} & x_{3,(1,2)} & x_{4,(1,2)} \\ x_{1,(2,1)} & x_{2,(2,1)} & x_{3,(2,1)} & x_{4,(2,1)} \\ x_{1,(2,2)} & x_{2,(2,2)} & x_{3,(2,2)} & x_{4,(2,2)} \end{pmatrix} \end{array},$$

then combining (4.12) and (4.13) allows us to write the image of ∂_1 in terms of 2×2 minors of $\phi^\flat$.

For example, let us consider the entry of ∂_1 corresponding to $f_{\{1,2\},\{1,2\},\emptyset}$ and $g_{\emptyset,\emptyset,(2,0)}$. From Appendix A we see that the inclusion

$$\bigwedge^2 A \otimes \det B_1^* \otimes \mathrm{D}^2(B_2^*) \subseteq \bigwedge^2 A \otimes \bigwedge^2 (B_1^* \otimes B_2^*)$$

is given by

$$\begin{cases} (u_1 \wedge u_2) \otimes v_1^{(2)} \mapsto (u_1 \otimes v_1) \wedge (u_2 \otimes v_1), \\ (u_1 \wedge u_2) \otimes v_1 v_2 \mapsto (u_1 \otimes v_1) \wedge (u_2 \otimes v_2) + (u_1 \otimes v_2) \wedge (u_2 \otimes v_1), \\ (u_1 \wedge u_2) \otimes v_2^{(2)} \mapsto (u_1 \otimes v_2) \wedge (u_2 \otimes v_2). \end{cases} \tag{4.13}$$

Combining (4.12) and (4.13), we conclude that the entry of ∂_1 corresponding to $f_{\{1,2\},\{1,2\},\emptyset}$ and $g_{\emptyset,\emptyset,(2,0)}$ is given by $(\alpha_1 \wedge \alpha_2) \otimes (u_1 \otimes v_1) \wedge (u_2 \otimes v_1)$. Thus, we may write this entry of ∂_1 as the 2×2 minor of $\phi^\flat$ obtained by taking the determinant of the submatrix

$$\begin{array}{c} \\ u_1\otimes v_1 \\ u_2\otimes v_1 \end{array} \begin{array}{c} \begin{array}{cc} \alpha_1 & \alpha_2 \end{array} \\ \begin{pmatrix} x_{1,(1,1)} & x_{2,(1,1)} \\ x_{1,(2,1)} & x_{2,(2,1)} \end{pmatrix} \end{array}.$$

The other entries for ∂_1 may be obtained similarly. See Example 12.1 for a matrix representation of both ∂_1 and ∂_2 in this example.

5. Tensor complexes from pinching weights

We now introduce the notion of pinching weights for a tensor, which enables us to produce tensor complexes $F(\phi, w)_\bullet$ that satisfy the conditions of Theorem 1.2. In contrast with the case of balanced tensor complexes, there are often many possible pinching weights for a given tensor $\phi^{a \times \mathbf{b}}$.

The motivation behind the definition of a pinching weight is the following. Recall from (2.5) that the terms of $F(\phi, w)_\bullet$ can be written as direct sums of certain cohomology groups on $\mathbb{P}(\vec{B})$. Further, since the support $Y(\phi)$ of $M(\phi, w)$ is independent of w (Corollary 1.5(i)), the length of $F(\phi, w)_\bullet$ is at least codim $Y(\phi)$, and thus $F(\phi, w)_\bullet$ must be built from at least this many different nonzero cohomology groups. The weight w is a pinching weight precisely when $F(\phi, w)_\bullet$ is composed of this minimal number of cohomology groups.

Definition 5.1. A weight vector $w = (w_0, \dots, w_n) \in \mathbb{Z}^{n+1}$ is a *pinching weight* for $\phi^{a\times\mathbf{b}}$ if $w_1 < \cdots < w_n$ and if, for all $1 \le i \le n$, the intervals $[w_i + 1, w_i + b_i - 1]$ lie in $[0, a]$ and are pairwise disjoint.

The stipulation that $w_1 < \cdots < w_n$ is a matter of convention; it can be guaranteed by permuting the B_j. In addition, we note that if $F(\phi^{a\times\mathbf{b}}, w)_\bullet$ is a balanced tensor complex, then w is a pinching weight for $\phi^{a\times\mathbf{b}}$.

Notation 5.2. Let w be a pinching weight for $\phi^{a\times\mathbf{b}}$. Let $p = a - \sum_{j=1}^{n}(b_j - 1)$. We define two degree sequences and some constants in terms of w and the size of ϕ:

$$d'(w) := \Big([0, a] \setminus \bigcup_{i=1}^{n}[w_i + 1, w_i + b_i - 1]\Big) \in \mathbb{Z}^{p+1};$$
$$d(w) := d'(w) + (w_0, \dots, w_0) \in \mathbb{Z}^{p+1};$$
$$r_i := \min\{j \mid j > 0 \text{ and } w_j \ge d'_i\} \quad \text{for all } 0 \le i \le p.$$

Theorem 5.3. *If w is a pinching weight for ϕ, then $F(\phi, w)_\bullet$ is a free complex of length $a - \sum_{j=1}^{n}(b_j - 1)$, and the ith term of $F(\phi, w)_\bullet$ is*

$$F(\phi, w)_i = S(-d_i) \otimes \bigwedge^{d'_i} A \otimes \bigotimes_{j=1}^{r_i - 1} \widetilde{\mathrm{D}}^{d'_i - w_j - b_j}(B_j^*) \otimes \bigotimes_{j=r_i}^{n} \mathrm{S}^{w_j - d'_i}(B_j). \tag{5.4}$$

The tensor complex $F(\phi, w)_\bullet$ satisfies the conditions of Theorem 1.2. The choice of G-equivariant differential is unique, up to sign.

Proof. From the definition of pinching weights and an argument similar to the proof of Proposition 3.3, we conclude that

$$F(\phi, w)_i = S(-d_i) \otimes \bigwedge^{d_i - w_0} A \otimes \bigotimes_{j=1}^{r_i - 1} \mathrm{H}^{b_j - 1}(\mathbb{P}(B_j), \mathcal{O}(w_j - d'_i)) \otimes$$
$$\bigotimes_{j=r_i}^{n} \mathrm{H}^0(\mathbb{P}(B_j), \mathcal{O}(w_j - d'_i)).$$

This yields (5.4). The desired assertions of Theorem 1.2 then follow from minor variants of the arguments in the proofs of Propositions 3.3 and 4.1, where we replace the expression (3.4) by (5.4). The uniqueness (up to sign) of a G-equivariant differential follows by a similar variant of the proof of Proposition 4.1. □

Example 5.5. Let $a \times \mathbf{b} = 7 \times (2, 2)$ and $w = (w_0, 1, 4)$ for any w_0. This is a pinching weight for $\phi^{7\times(2,2)}$, since the intervals $[w_1 + 1, w_1 + 2 - 1] = [2, 2]$ and $[w_2 + 1, w_2 + 2 - 1] = [5, 5]$ are disjoint and both belong to the interval $[0, 7]$. The corresponding complex $F(\phi, w)_\bullet$ equals the tensor product of the complex (1.3) with $S(-w_0)$.

When $w = (0, -1, 6)$, the complex $F(\phi, w)_\bullet$ equals the linear complex

$$\begin{bmatrix} \bigwedge^1 \\ \widetilde{D}^0 \\ S^5 \end{bmatrix}(-1) \leftarrow \begin{bmatrix} \bigwedge^2 \\ \widetilde{D}^1 \\ S^4 \end{bmatrix}(-2) \leftarrow \begin{bmatrix} \bigwedge^3 \\ \widetilde{D}^2 \\ S^3 \end{bmatrix}(-3) \leftarrow \begin{bmatrix} \bigwedge^4 \\ \widetilde{D}^3 \\ S^2 \end{bmatrix}(-4) \leftarrow \begin{bmatrix} \bigwedge^5 \\ \widetilde{D}^4 \\ S^1 \end{bmatrix}(-5) \leftarrow \begin{bmatrix} \bigwedge^6 \\ \widetilde{D}^5 \\ S^0 \end{bmatrix}(-6) \leftarrow 0.$$

When $w = (-4, 1, 2)$, the complex $F(\phi, w)_\bullet$ equals

$$\begin{bmatrix} \bigwedge^0 \\ S^1 \\ S^2 \end{bmatrix}(4) \leftarrow \begin{bmatrix} \bigwedge^1 \\ S^0 \\ S^1 \end{bmatrix}(3) \leftarrow \begin{bmatrix} \bigwedge^4 \\ \widetilde{D}^1 \\ \widetilde{D}^0 \end{bmatrix}(0) \leftarrow \begin{bmatrix} \bigwedge^5 \\ \widetilde{D}^2 \\ \widetilde{D}^1 \end{bmatrix}(-1) \leftarrow \begin{bmatrix} \bigwedge^6 \\ \widetilde{D}^3 \\ \widetilde{D}^2 \end{bmatrix}(-2) \leftarrow \begin{bmatrix} \bigwedge^7 \\ \widetilde{D}^4 \\ \widetilde{D}^3 \end{bmatrix}(-3) \leftarrow 0.$$

Remark 5.6. Instead of setting w to be a pinching weight for ϕ, consider the case where the intervals $[w_i+1, w_i+b_i-1]$ are pairwise disjoint, but where we drop the requirement that all the intervals $[w_i + 1, w_i + b_i - 1]$ lie in $[0, a]$. In this case, $M(\phi, w)$ is a non-Cohen–Macaulay module with a pure resolution. For instance, with $w = (-4, 1, -3)$, $F(\phi^{7\times(2,2)}, w)_\bullet$ is

$$\begin{bmatrix} \bigwedge^0 \\ S^1 \\ \widetilde{D}^1 \end{bmatrix}(4) \leftarrow \begin{bmatrix} \bigwedge^1 \\ S^0 \\ \widetilde{D}^2 \end{bmatrix}(3) \leftarrow \begin{bmatrix} \bigwedge^3 \\ \widetilde{D}^0 \\ \widetilde{D}^4 \end{bmatrix}(1) \leftarrow \begin{bmatrix} \bigwedge^4 \\ \widetilde{D}^1 \\ \widetilde{D}^5 \end{bmatrix}(0) \leftarrow \begin{bmatrix} \bigwedge^5 \\ \widetilde{D}^2 \\ \widetilde{D}^6 \end{bmatrix}(-1) \leftarrow \begin{bmatrix} \bigwedge^6 \\ \widetilde{D}^3 \\ \widetilde{D}^7 \end{bmatrix}(-2) \leftarrow \begin{bmatrix} \bigwedge^7 \\ \widetilde{D}^4 \\ \widetilde{D}^8 \end{bmatrix}(-3) \leftarrow 0.$$

Since codim $M(\phi, w) = 5$, it is not Cohen–Macaulay.

6. Strands of the Koszul complex

We now provide a more elementary description of $F(\phi, w)_\bullet$ as a complex constructed by splicing strands of a Koszul complex together, extending the study of matrix complexes in [BE] and [Eis, §A2.6]. The purpose of this section is expository, so we focus on the example of the universal $7 \times (2, 2)$ tensor with pinching weight $w = (0, 1, 4)$ described in (1.3); the general case can be treated in a similar fashion. By Proposition 5.3,

$$\beta(F(\phi, w)_\bullet) = \begin{pmatrix} 10 & 28 & - & - & - & - \\ - & - & 70 & 70 & - & - \\ - & - & - & - & 28 & 10 \end{pmatrix}.$$

We now express $F(\phi, w)_\bullet$ in terms of three linear strands arising from a Koszul complex. As discussed in §1.1, these are:

Strand 1: Strand 2: Strand 3:

$$\begin{bmatrix}\bigwedge^0\\ \mathrm{S}^1\\ \mathrm{S}^4\end{bmatrix} \leftarrow \begin{bmatrix}\bigwedge^1\\ \mathrm{S}^0\\ \mathrm{S}^3\end{bmatrix}(-1), \quad \begin{bmatrix}\bigwedge^3\\ \widetilde{\mathrm{D}}^0\\ \mathrm{S}^1\end{bmatrix}(-3) \leftarrow \begin{bmatrix}\bigwedge^4\\ \widetilde{\mathrm{D}}^1\\ \mathrm{S}^0\end{bmatrix}(-4), \quad \begin{bmatrix}\bigwedge^6\\ \widetilde{\mathrm{D}}^3\\ \widetilde{\mathrm{D}}^0\end{bmatrix}(-6) \leftarrow \begin{bmatrix}\bigwedge^7\\ \widetilde{\mathrm{D}}^4\\ \widetilde{\mathrm{D}}^1\end{bmatrix}(-7).$$

To obtain them from a Koszul complex, we consider the space $\mathbb{A}^{a\times\mathbf{b}} \times \mathbb{P}(B_1) \times \mathbb{P}(B_2)$ and let $T := S \otimes_{\mathbb{Z}} \mathbb{Z}[B_1] \otimes_{\mathbb{Z}} \mathbb{Z}[B_2]$ with the induced $\mathbb{Z}^3$-grading. If $\mathbf{k} := (k, k, k) \in \mathbb{Z}^3$, then

$$K(\phi)_\bullet : \bigwedge^0 T^7 \leftarrow \bigwedge^1 T^7(-\mathbf{1}) \leftarrow \cdots \leftarrow \bigwedge^7 T^7(-\mathbf{7}) \leftarrow 0$$

is the $\mathbb{Z}^3$-graded Koszul complex from Remark 2.6. For any $\alpha, \beta \in \mathbb{Z}$, the subcomplex $[K(\phi)_\bullet]_{(*,\alpha,\beta)} := \bigoplus_{\gamma\in\mathbb{Z}}[K(\phi)_\bullet]_{(\gamma,\alpha,\beta)}$ of $K(\phi)_\bullet$ is a graded complex of S-modules. In particular, Strand 1 arises as the $(*, 1, 4)$ subcomplex of $K(\phi)_\bullet$:

$$\begin{aligned}[K(\phi)_\bullet]_{(*,1,4)} &= [\textstyle\bigwedge^0 T^7]_{(*,1,4)} \leftarrow [\bigwedge^1 T^7(-\mathbf{1})]_{(*,1,4)} \leftarrow \cdots \leftarrow [\bigwedge^7 T^7(-\mathbf{7})]_{(*,1,4)} \leftarrow 0\\ &= \begin{bmatrix}\bigwedge^0\\ \mathrm{S}^1\\ \mathrm{S}^4\end{bmatrix} \leftarrow \begin{bmatrix}\bigwedge^1\\ \mathrm{S}^0\\ \mathrm{S}^3\end{bmatrix}(-1) \leftarrow 0 \leftarrow \cdots \leftarrow 0.\end{aligned}$$

Strand 2 also arises from the Koszul complex $K(\phi)_\bullet$, but in a more subtle manner. Let $\widetilde{\mathrm{D}}^\bullet(B_1^*) := \bigoplus_{i=0}^\infty \widetilde{\mathrm{D}}^i(B_1^*)$, which is naturally isomorphic as a graded module to the top local cohomology group of the $\mathbb{Z}$-algebra $\mathbb{Z}[B_1]$ with support in the prime ideal generated by B_1. Let $T^{\{1\}}$ be the T-module $S \otimes \widetilde{\mathrm{D}}^\bullet(B_1^*) \otimes \mathrm{S}^\bullet(B_2)$ and set $K_\bullet^{\{1\}} := K(\phi)_\bullet \otimes_T T^{\{1\}}$. We then obtain Strand 2 as the $(*, 1, 4)$ subcomplex of $K^{\{1\}}$:

$$[K_\bullet^{\{1\}}]_{(*,1,4)} = 0 \leftarrow 0 \leftarrow 0 \leftarrow \begin{bmatrix}\bigwedge^3\\ \widetilde{\mathrm{D}}^0\\ \mathrm{S}^1\end{bmatrix}(-3) \leftarrow \begin{bmatrix}\bigwedge^4\\ \widetilde{\mathrm{D}}^1\\ \mathrm{S}^0\end{bmatrix}(-4) \leftarrow 0 \leftarrow 0 \leftarrow 0.$$

Finally, Strand 3 is obtained through a similar process. If $T^{\{1,2\}} := S \otimes \widetilde{\mathrm{D}}^\bullet(B_1^*) \otimes \widetilde{\mathrm{D}}^\bullet(B_2^*)$ and $K_\bullet^{\{1,2\}} := K(\phi)_\bullet \otimes_T T^{\{1,2\}}$, then Strand 3 arises as the $(*, 1, 4)$ subcomplex of $K_\bullet^{\{1,2\}}$.

Remark 6.1. The construction outlined in this section provides a slightly different view from [Eis, §A2.6] of building matrix complexes from strands of the Koszul complex, and we now contrast these approaches. Let us consider the case of a 7×2 matrix with $w = (0, 3)$. The complex $F(\phi, w)_\bullet$ then corresponds to the complex $\mathcal{C}^2$ of [Eis, §A2.6]. Incorporating the appropriate twists by determinants into $\mathcal{C}^2$ (as suggested by the footnotes in [Eis, §A2.6]), we see that the complexes $\mathcal{C}^2$ and $F(\phi, w)_\bullet$ are equal. In both cases, the Betti diagram of the free resolution is

$$\begin{pmatrix} 3 & 14 & 21 & - & - & - & - \\ - & - & - & 35 & 42 & 21 & 4 \end{pmatrix}.$$

However, the construction of $F(\phi, w)_\bullet$ differs from the construction of $\mathcal{C}^2$. We obtain the first strand of each construction in the same manner, as the $(*, 2)$ subcomplex of $K(\phi)_\bullet$. However, the second strands come from slightly different sources.

Strand 2 of $\mathcal{C}^2$ is obtained by peeling off the $(*, 3)$ subcomplex of $K(\phi)_\bullet$, which has

$$\beta\big([K(\phi)_\bullet]_{(*,3)}\big) = \begin{pmatrix} - & - & - & - & - & - & - \\ 4 & 21 & 42 & 35 & - & - & - \end{pmatrix},$$

and then dualizing that strand (and twisting by the appropriate determinants). Note that strand 2 of $\mathcal{C}^2$ originates in homological degrees 0, 1, 2 and 3 of the complex $K(\phi)_\bullet$, and then duality is used to turn this strand around.

By contrast, Strand 2 of $F(\phi, w)_\bullet$ comes from homological degrees 4, 5, 6 and 7 of a different complex $K_\bullet^{\{1\}}$:

$$\beta\big([K_\bullet^{\{1\}}]_{(*,2)}\big) = \begin{pmatrix} - & - & - & - & - & - & - \\ - & - & - & 35 & 42 & 21 & 4 \end{pmatrix}.$$

These strands coincide, at least up to a twist by determinants, because of the self-duality properties of the Koszul complex $K(\phi)_\bullet$.

7. Functoriality properties of tensor complexes

We now prove Proposition 1.4, which describes the functorial properties of the construction of tensor complexes. We also consider the relation to the complexes considered in [BEKS].

Proof of Proposition 1.4. We have $a' \le a$, $w, w' \in \mathbb{Z}^{n+1}$, and an inclusion $i\colon \mathbb{Z}^{a'} \to \mathbb{Z}^a$. First assume that $w = w'$. This induces a map of rings $S' \to S$ (where $S = \mathbb{Z}[X^{a\times \mathbf{b}}]$, $S' = \mathbb{Z}[X^{a'\times \mathbf{b}}]$) and a commutative diagram

$$\begin{array}{ccc} \mathbb{A}^{a\times\mathbf{b}} \times \mathbb{P}(\vec{B}) & \xrightarrow{\nu} & \mathbb{A}^{a'\times\mathbf{b}} \times \mathbb{P}(\vec{B}) \\ \big\downarrow{\scriptstyle\pi} & & \big\downarrow{\scriptstyle\pi'} \\ \mathbb{A}^{a\times\mathbf{b}} & \xrightarrow{\rho} & \mathbb{A}^{a'\times\mathbf{b}} \end{array}$$

On $\mathbb{A}^{a\times\mathbf{b}} \times \mathbb{P}(\vec{B})$ and $\mathbb{A}^{a'\times\mathbf{b}} \times \mathbb{P}(\vec{B})$, we have the Koszul complexes $\mathcal{K}(\phi)_\bullet$ and $\mathcal{K}(\phi')_\bullet$, respectively. The inclusion i induces a natural map $\nu^*\mathcal{K}(\phi')_\bullet \to \mathcal{K}(\phi)_\bullet$. We thus obtain a natural map $\mathbf{R}\pi_*(\nu^*\mathcal{K}(\phi')_\bullet) \to \mathbf{R}\pi_*\mathcal{K}(\phi)_\bullet$. By the projection formula [Har, Proposition II.5.6], there is a quasi-isomorphism

$$\mathbf{R}\pi_*(\nu^*\mathcal{K}(\phi')_\bullet) \cong \rho^*(\mathbf{R}\pi'_*\mathcal{K}(\phi')_\bullet) \tag{7.1}$$

(noting that $\mathbf{L}\rho^*$ and $\mathbf{L}\nu^*$ coincide with ρ^* and ν^*, since we apply them to a complex of locally free sheaves).

In fact, this map is an isomorphism of complexes. This follows from the claim that if $P_\bullet$ and $P'_\bullet$ are minimal (i.e., $\partial P_i \subseteq \mathfrak{m}P_{i-1}$, where $\mathfrak{m} \subseteq S$ is the ideal generated by the variables) bounded-below complexes of S-modules, then a quasi-isomorphism of $P_\bullet$ and $P'_\bullet$ induces an isomorphism of these complexes. To prove the claim, we first observe that there is a minimal bounded-below complex $\hat{P}_\bullet$ of free S-modules together with maps $P_\bullet \leftarrow \hat{P}_\bullet \rightarrow P'_\bullet$ that realizes the quasi-isomorphism. A map between bounded-below projective complexes which is a quasi-isomorphism is a homotopy equivalence, and a homotopy equivalence between minimal complexes of S-modules is an isomorphism, proving the claim.

We thus get a map $\rho^*(\mathbf{R}\pi'_*\mathcal{K}(\phi')_\bullet) \rightarrow \mathbf{R}\pi_*\mathcal{K}(\phi)_\bullet$. Note that $F(\phi', w)_\bullet$ is a minimal free resolution in the quasi-isomorphism class of $\mathbf{R}\pi'_*\mathcal{K}(\phi')_\bullet$, and $F(\phi, w)_\bullet$ is a minimal free resolution in the quasi-isomorphism class of $\mathbf{R}\pi_*\mathcal{K}(\phi)_\bullet$. The above map thus induces the desired map $f_w\colon F(\phi', w)_\bullet \otimes_{S'} S \rightarrow F(\phi, w)_\bullet$.

When $w \neq w'$, we fix a nonzero polynomial h of multidegree $w - w'$ on $S' \otimes \mathbb{Z}[B_1] \otimes \cdots \otimes \mathbb{Z}[B_n]$, assuming that one exists. Multiplication by h gives a morphism $\mathcal{K}(\phi')_\bullet(w') \rightarrow \mathcal{K}(\phi')_\bullet(w)$. By taking the global sections of the derived pushforward, we get a morphism $F(\phi', w')_\bullet \rightarrow F(\phi', w)_\bullet$. Tensoring with S and composing with the map f_w then yields the desired morphism $F(\phi', w')_\bullet \otimes_{S'} S \rightarrow F(\phi, w)_\bullet$. □

Example 7.2. Fix $\mathbf{b} = (2, 2)$ and $w = (0, 0, 2)$ and consider the tensor complexes $F(\phi^{a\times\mathbf{b}}, w)_\bullet$ for $a = 2, 3$ and 4. The Betti diagrams of these three complexes are

$$\begin{pmatrix} 3 & - & - \\ - & 1 & - \\ - & - & - \end{pmatrix}, \quad \begin{pmatrix} 3 & - & - \\ - & 3 & - \\ - & - & - \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 3 & - & - \\ - & 6 & - \\ - & - & 3 \end{pmatrix},$$

respectively. If we choose the multihomogeneous form 1, then the maps induced by Proposition 1.4 yield natural inclusions among these complexes.

Even when $w \neq w'$, the maps induced by Proposition 1.4 can be simple in special cases.

Lemma 7.3 (Homomorphism pushforward lemma). *With notation as in Proposition 1.4, assume that $F(\phi, w)_\bullet$ is a pure resolution of type $d = (d_0, \ldots, d_p)$ and that $F(\phi', w')_\bullet$ is a pure resolution of type $d' = (d'_0, \ldots, d'_q)$. Let h be the multihomogeneous form determining the morphism of complexes $h_\bullet\colon \mathcal{K}(\phi')_\bullet \rightarrow \mathcal{K}(\phi)_\bullet$ that induces the morphism of complexes $\nu_\bullet\colon F(\phi', w')_\bullet \rightarrow F(\phi, w)_\bullet$.*

Assume further that $d_i = d'_i$ for some i, and let $N := \sum_{j\leq r_i}(b_j - 1)$. Then the map ν_i may be chosen to be the induced map on cohomology, $\mathrm{H}^N(h_i)$, as in the following diagram:

$$\begin{array}{ccc} F(\phi', w')_i & \xrightarrow{\quad\nu_i\quad} & F(\phi, w)_i \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{H}^N(\mathbb{A}^{a\times\mathbf{b}} \times \mathbb{P}(\vec{B}), \mathcal{K}(\phi')_i) & \xrightarrow{\mathrm{H}^N(h_i)} & \mathrm{H}^N(\mathbb{A}^{a\times\mathbf{b}} \times \mathbb{P}(\vec{B}), \mathcal{K}(\phi)_i) \end{array}$$

Proof. As in [ES1, proof of Proposition 5.3], we may use the spectral sequence with $E_1^{k,-l} = \mathbf{R}^k\pi_*\mathcal{K}(\phi)_l$ to compute the complex $F(\phi, w)_\bullet$, along with a similar spectral sequence to compute $F(\phi', w')_\bullet$. We thus construct $\nu_\bullet$ by considering the map induced by $h_\bullet$ on these spectral sequences. Since $d_i = d'_i$, one may check that on the E_1 page, the induced map in position $(N, -i)$ is given by

$$\mathrm{H}^N(h_i)\colon \mathrm{H}^N(\mathbb{A}^{a\times\mathbf{b}} \times \mathbb{P}(\vec{B}), \mathcal{K}(\phi')_i) \to \mathrm{H}^N(\mathbb{A}^{a\times\mathbf{b}} \times \mathbb{P}(\vec{B}), \mathcal{K}(\phi)_i).$$

Since the terms of the complexes $F(\phi, w)_\bullet$ and $F(\phi', w')_\bullet$ come from the E_1 page of this spectral sequence, this map may be chosen as the map $\nu_i\colon F(\phi', w')_i \to F(\phi, w)_i$. □

Remark 7.4. The proof of [BEKS, Theorem 1.2] can be reinterpreted in terms of Proposition 1.4 and Lemma 7.3. Namely, the pure resolutions of [BEKS, Theorem 1.2] are specializations of certain tensor complexes of the form $a \times (2, \ldots, 2)$, and the morphisms constructed between two such resolutions are of the form of the morphisms given by Proposition 1.4. However, we note that Proposition 1.4 does not directly imply that result, as the above map of complexes could be null-homotopic. The essential step in the proof of [BEKS, Theorem 1.2] is checking that certain maps of complexes induce nonzero maps $M(\phi^{a'\times\mathbf{b}}, w') \to M(\phi^{a\times\mathbf{b}}, w)$, which requires analyzing the detailed description of $\nu_\bullet$ provided by Lemma 7.3.

8. Properties of the module $M(\phi, w)$

The goal of this section is to prove Corollary 1.5 and Propositions 1.8 and 1.10. We begin by discussing some facts about the support $Y(\phi)$ of $M(\phi, w)$. In §9 we explore the geometry of $Y(\phi)$ further.

Recall the diagram of (2.2). The scheme $Y(\phi)$ is integral since it is the scheme-theoretic image of the integral scheme $Z(\phi)$. Throughout this section we identify $\mathbb{A}^{a\times\mathbf{b}}$ with the space of $\mathbb{Z}$-linear maps $\psi\colon \mathbf{B}^* \to A^*$. For a linear subspace V of $\mathbf{B}^*$ we write $[V]$ for the corresponding subspace in $\mathbb{P}(\mathbf{B})$. So for any map $\psi \in \mathbb{A}^{a\times\mathbf{b}}$, we may think of $[\ker(\psi)]$ as a linear subspace of $\mathbb{P}(\mathbf{B})$. Let $\mathrm{Seg}(\mathbf{B})$ denote the image of the Segre embedding $\mathbb{P}(\vec{B}) \to \mathbb{P}(\mathbf{B})$.

Proposition 8.1. *The annihilator of $M(\phi, w)$ is the prime ideal that defines the integral scheme $Y(\phi)$. Under the identification $\mathbb{P}(\vec{B}) \cong \mathrm{Seg}(\mathbf{B})$, we have*

$$Z(\phi) = \{(\psi, y) \in \mathbb{A}^{a\times\mathbf{b}} \times \mathbb{P}(\vec{B}) \mid y \in [\ker(\psi)]\}.$$

We therefore have

$$Y(\phi) = \{\psi \in \mathrm{Hom}(\mathbf{B}^*, A^*) \mid [\ker(\psi)] \cap \mathrm{Seg}(\mathbf{B}) \neq \emptyset\} \subseteq \mathbb{A}^{a\times\mathbf{b}}.$$

Proof. The first assertion follows from [Wey, Theorems 5.1.2(b), 5.1.3(a)], which imply that $M(\phi, w)$ is a module over the normalization of $Y(\phi)$. Since $Z(\phi)$ is the total space of $\mathcal{S} = \mathcal{H}om((\mathbf{B}^* \otimes \mathcal{O}_{\mathbb{P}(\vec{B})})/\mathcal{O}_{\mathbb{P}(\vec{B})}(-\mathbf{1}), A^* \otimes \mathcal{O}_{\mathbb{P}(\vec{B})})$, we may think of $Z(\phi)$ as the set of maps $\psi\colon \mathbf{B}^* \otimes \mathcal{O}_{\mathbb{P}(\vec{B})} \to A^* \otimes \mathcal{O}_{\mathbb{P}(\vec{B})}$ whose kernel contains a rank 1 tensor, yielding the second assertion. The final assertion is now immediate. □

Remark 8.2. We now explain how Proposition 8.1 implies Proposition 1.8, which states that $Y(\phi)$ may be interpreted as a resultant variety for multilinear equations on $\mathbb{P}(\vec{B})$. As in Remark 2.6, we view a point in $\mathbb{A}^{a\times\mathbf{b}}$ as a collection $\widetilde{\mathbf{f}} = (\widetilde{f}_1, \dots, \widetilde{f}_a)$ of multilinear forms on $\mathbb{P}(\vec{B})$. Then Proposition 8.1 implies that $Z(\phi)$ is the incidence variety

$$Z(\phi) = \{(\widetilde{\mathbf{f}}, y) \in \mathbb{A}^{a\times\mathbf{b}} \times \mathbb{P}(\vec{B}) \mid y \in V_{\mathbb{P}(\vec{B})}(\widetilde{f}_1, \dots, \widetilde{f}_a)\},$$

and thus it follows that $Y(\phi)$ has the resultant interpretation

$$Y(\phi) = \{\widetilde{\mathbf{f}} \in \mathbb{A}^{a\times\mathbf{b}} \mid V_{\mathbb{P}(\vec{B})}(\widetilde{f}_1, \dots, \widetilde{f}_a) \neq \emptyset\}.$$

This yields Proposition 1.8 because, for any pinching weight w, the complex $F(\phi, w)_\bullet$ resolves a module whose support equals $Y(\phi)$. Hence the minors of ∂_1 cut out $Y(\phi)$ set-theoretically by [Eis, Proposition 20.7].

Remark 8.3. The map $\mu\colon Z(\phi) \to Y(\phi)$ restricts to an isomorphism over the (possibly empty) open subset of $Y(\phi)$ consisting of those ψ such that $\ker\psi$ contains a rank 1 tensor that is unique up to scalar multiple. Now, if $a > \sum_{i=1}^{n}(b_i - 1)$ (i.e., if $\operatorname{codim} Y(\phi) \geq 1$ by Corollary 1.5(i)), then such maps ψ exist, and hence μ is a birational morphism. We see this as follows. If $b_1 \cdots b_n \leq a$, then we may choose any rank 1 tensor and define ψ to be a map whose kernel is spanned by the chosen rank 1 tensor. Now suppose that $b_1 \cdots b_n > a$. Then, since $a > \sum_i (b_i - 1)$, there is a $(b_1 \cdots b_n - 1 - a)$-plane (i.e., a linear subvariety of codimension a) in $\mathbb{P}(\mathbf{B})$ that intersects $\mathrm{Seg}(\mathbf{B})$ in exactly one point. Define ψ to be a map with this linear subvariety as its kernel.

We are now prepared to prove Proposition 1.10 and Corollary 1.5.

Proof of Proposition 1.10. Note first that the sheaf

$$\widetilde{M}(\phi, w) = \mu_*\big(\mathcal{O}_{Z(\phi)} \otimes \pi_2^*(\mathcal{O}_{\mathbb{P}(\vec{B})}(w_1, \dots, w_n))\big) \otimes \mathcal{O}(-w_0)$$

is a twist of the pushforward of a line bundle. Using the fact that μ is birational by Remark 8.3, we see that $\widetilde{M}(\phi, w)$ is a rank 1 sheaf on $Y(\phi)$. Since $Y(\phi)$ is irreducible and $M(\phi, w)$ is Cohen–Macaulay, the indecomposability of $M(\phi, w)$ follows immediately. □

Proof of Corollary 1.5. For (i), the fact that the support of $M(\phi, w)$ does not depend on w follows from Proposition 8.1. The codimension formula follows from (5.4) and [Wey, Theorem 5.1.6(a)] (while this result is proven when the base ring is a field of characteristic 0, we may reduce to this case because $Y(\phi)$ is flat over $\mathbb{Z}$).

For (ii), recall that $M(\phi, w)$ is Cohen–Macaulay and has a uniformly minimal resolution over $\mathbb{Z}$ (Theorem 1.2(i) and (ii)). By uniform minimality, $\mathrm{Tor}_1^{\mathbb{Z}}(M(\phi, w), \mathbb{Z}/\ell) = 0$ and $\ell M(\phi, w) \neq M(\phi, w)$ for all primes ℓ. In other words, $M(\phi, w)$ is a faithfully flat $\mathbb{Z}$-module (see, for example, [Mat, Theorems 7.2, 7.8]), and hence $M(\phi, w)$ is generically perfect.

For (iii), we may assume that $w_0 = 0$. Set $p := a - \sum_j (b_j - 1)$, so $d(w) = d'(w) = (d'_0, \dots, d'_p)$. To simplify the notation, we use M to denote $M(\phi, w)$ throughout the rest of this proof. Since M is a Cohen–Macaulay module with a pure resolution of type $d(w)$ (by Theorem 1.2(i) and (iii)), we see from [HM, Theorem 1.2] that

$$e(M) = \frac{1}{(\operatorname{codim} M)!}\Big(\prod_{i=1}^{n}(d'_i - d'_0)\Big)\beta_{0,d'_0}(M).$$

(Huneke and Miller prove [HM, Theorem 1.2] only for cyclic Cohen–Macaulay modules, but [HM, (1.3)] can be modified by multiplying by $\beta_{0,d'_0}(M)$ to make the proof work for all Cohen–Macaulay modules with pure resolutions.)

We use (5.4) to compute $\beta_{0,d'_0}(M)$. Recall that $r_0 = \min\{j \mid w_j \geq d'_0\}$ and observe that, since w is a pinching weight, we have $[d'_0, a] = \{d'_0, d'_1, \ldots, d'_n\} \sqcup \bigsqcup_{j \geq r_0}[w_j + 1, w_j + b_j - 1]$. We may rewrite this as $[0, a - d'_0] = \{0, d'_1 - d'_0, \ldots, d'_n - d'_0\} \sqcup \bigsqcup_{j\geq r_0}[w_j + 1 - d'_0, w_j + b_j - 1 - d'_0]$. Since

$$\operatorname{rank}_{\mathbb{Z}}\Big(\bigotimes_{j=r_0}^{n} \mathrm{S}^{w_j - d'_0}(B_j)\Big) = \prod_{j\geq r_0} \frac{(w_j + 1 - d'_0)\cdots(w_j + b_j - 1 - d'_0)}{(b_j - 1)!},$$

this yields

$$\operatorname{rank}_{\mathbb{Z}}\Big(\bigotimes_{j=r_0}^{n} \mathrm{S}^{w_j - d'_0}(B_j)\Big) \cdot \prod_{j=1}^{n}(d'_j - d'_0) = \frac{(a - d'_0)!}{\prod_{j\geq r_0}(b_j - 1)!}.$$

In addition, we have $[0, d'_0 - 1] = \bigsqcup_{j<r_0}[w_j + 1, w_j + b_j - 1]$, since w is a pinching weight. Multiplying by -1 and adding d'_0, we obtain the equality $[1, d'_0] = \bigsqcup_{j<r_0}[d'_0 - w_j - b_j + 1, d'_0 - w_j - 1]$, and we similarly see that

$$\operatorname{rank}_{\mathbb{Z}}\Big(\bigotimes_{j=1}^{r_0 - 1} \widetilde{\mathrm{D}}^{d'_0 - w_j - b_j}(B_j^*)\Big) = \frac{d_0!}{\prod_{j<r_0}(b_j - 1)!}.$$

Finally, we combine these to get the multiplicity of M:

$$\begin{aligned} e(M) &= \frac{1}{(\operatorname{codim} M)!}\Big(\prod_{i=1}^{n}(d'_i - d'_0)\Big) \cdot (\operatorname{rank}_{\mathbb{Z}} F(\phi, w)_0) \\ &= \frac{1}{(\operatorname{codim} M)!}\binom{a}{d'_0}\Big(\frac{d_0!}{\prod_{j<r_0}(b_j - 1)!}\Big)\Big(\frac{(a - d'_0)!}{\prod_{j\geq r_0}(b_j - 1)!}\Big) \\ &= \frac{1}{(\operatorname{codim} M)!}\,\frac{a!}{\prod_{j=1}^{n}(b_j - 1)!}. \end{aligned}$$ □

See Remark 10.3 for a surprising consequence of the above formula for $e(M(\phi, w))$.

Remark 8.4. By imposing symmetry, we can obtain tensor complexes that are equivariantly self-dual. For example, reconsider the tensor complex from (1.3). Based on the representations that arise in the free resolution, the complex exhibits certain symmetries; but it is not a self-dual resolution of S-modules.

However, a variant of this complex is self-dual. Let $\Bbbk = \mathbb{Z}[1/2]$. Since $B_1 \cong B_2$, we may identify these free modules and consider $\mathrm{S}^2(B_1) \otimes_{\mathbb{Z}} \Bbbk \subseteq B_1 \otimes B_2 \otimes_{\mathbb{Z}} \Bbbk$. Let $S' := \Bbbk[A \otimes \mathrm{S}^2(B_1) \otimes_{\mathbb{Z}} \Bbbk]$ and ϕ' be the universal symmetric tensor in $A \otimes \mathrm{S}^2(B_1) \otimes S'$. By applying the above inclusion, we may view ϕ' as a tensor in $A \otimes B_1 \otimes B_2 \otimes S'$ and thus construct $F(\phi', w)_\bullet$ as a complex of S'-modules.

The complex $F(\phi', w)_\bullet$ is equivariantly self-dual as a complex of S'-modules. This self-duality is forced by the uniqueness of equivariant differentials, as discussed in §4. A similar construction works whenever $B_i \cong B_{n-i}$ for all i and $w_j + w_{n+1-j} = -b_j$ for all j.

9. Hyperdeterminantal varieties

There are two special cases where the supporting variety $Y(\phi)$ has been previously studied in some detail. First, if there is a unique i such that $b_i \neq 1$, then $Y(\phi)$ is the determinantal variety defined by the maximal minors of a universal matrix. Motivated by this example, we refer to $Y(\phi)$ as a *hyperdeterminantal variety*. The second case where hyperdeterminantal varieties have previously been studied is when $\operatorname{codim} Y(\phi) = 1$. As we prove in Proposition 9.1, in this case, $Y(\phi)$ is defined by a hyperdeterminant of the boundary format.

The main goal of this section is to prove Theorem 1.6, as well as describe other geometric properties of hyperdeterminantal varieties. Based on the two special cases above, one might wonder if the variety $Y(\phi)$ is Cohen–Macaulay in general. This turns out to be entirely false: Proposition 9.3 shows that $Y(\phi)$ is Cohen–Macaulay if and only if it is either a determinantal variety, a hypersurface, or all of $\mathbb{A}^{a \times \mathbf{b}}$. We consider the singular locus of $Y(\phi)$ in Proposition 9.4; in the hyperdeterminantal case, our result recovers a portion of [WZ, Theorem 0.5(a)].

To begin with hyperdeterminantal hypersurfaces, the tensor $\phi^{a \times \mathbf{b}}$ is said to have the *boundary format* when $a - \sum_{i=1}^{n}(b_i - 1) = 1$ [GKZ, §14.3]. In this case, there is a corresponding hyperdeterminant $\Delta_{a \times \mathbf{b}}$, which is generally defined over a field of characteristic 0. However, since $\Delta_{a \times \mathbf{b}}$ is unique up to scalar multiple, we view it as a polynomial over $\mathbb{Z}$ that is not divisible by any prime number ℓ, so that it is unique up to sign.

Proposition 9.1. *Let $\phi = \phi^{a \times \mathbf{b}}$ be of the boundary format and w be any pinching weight for ϕ. Then $F(\phi, w)_\bullet$ is a free resolution of length* 1, *and hence ∂_1 is a square matrix. Up to sign, the hyperdeterminant $\Delta_{a \times \mathbf{b}}$ equals* $\det(\partial_1)$.

Proof. We first show that $Y(\phi)$ equals the vanishing of the hyperdeterminant $\Delta_{a \times \mathbf{b}}$. By Corollary 1.5, we may choose any w to compute $Y(\phi)$. We set $w_0 := 0$, $w_1 := 1$, and $w_i := (\sum_{j<i} b_i) - (i - 2)$ for $i \geq 1$. We confirm that this yields a pinching weight for ϕ

by computing

$$[w_1 + 1, w_1 + b_1 - 1] = \begin{cases} [2, b_1], & i = 1, \\ [(3-i) + \sum_{j<i} b_j, (5-i) + \sum_{j\le i} b_j], & i \ge 2. \end{cases}$$

By Theorem 5.3, the resulting free resolution is a 2-term linear complex:

$$\begin{bmatrix} \bigwedge^0 \\ S^{w_1} \\ S^{w_2} \\ \vdots \\ S^{w_n} \end{bmatrix} \xleftarrow{\partial_1} \begin{bmatrix} \bigwedge^1 \\ S^{w_1 - 1} \\ S^{w_2 - 1} \\ \vdots \\ S^{w_n - 1} \end{bmatrix}(-1) \leftarrow 0,$$

where ∂_1 is a G-equivariant map.

The source and target of ∂_1 can naturally be associated with the source and target of the matrix ∂_A from [GKZ, Proposition 14.3.2], which is used to compute the hyperdeterminant $\Delta_{a\times\mathbf{b}}$. Clearly ∂_A is G-equivariant by definition. We claim that ∂_1 and ∂_A differ by ± 1. After passing to $\mathbb{Q}$, we see (by an argument similar to Lemma 4.7) that the map of representations $[\partial_1]_1 : [F_1]_1 \otimes \mathbb{Q} \to [F_0]_1 \otimes \mathbb{Q}$ is an injective map from an irreducible representation to a multiplicity-free representation. A similar statement holds for $[\partial_A]_1$, and hence $[\partial_1]_1$ and $[\partial_A]_1$ differ by an integer scalar. Hence it follows that $\det(\partial_1)$ is an integral scalar multiple of $\Delta_{a\times\mathbf{b}}$. However, since $Y(\phi)$ is irreducible, it follows that $\det(\partial_1)$ is also, up to sign, a power of an irreducible polynomial. This proves that $\det(\partial_1)$ and $\Delta_{a\times\mathbf{b}}$ are equal, up to sign.

Now let w be any pinching weight for ϕ, and let ∂_1 be the corresponding differential on the 2-term complex. Since $Y(\phi)$ does not depend on w, $\det(\partial_1)$ is a power of $\Delta_{a\times\mathbf{b}}$. Since $M(\phi, w)$ is Cohen–Macaulay of codimension 1, its multiplicity equals the degree of $\det(\partial_1)$. By combining Corollary 1.5(iii) and [GKZ, Corollary 14.2.6], it follows that $\deg\det(\partial_1) = \deg \Delta_{a\times\mathbf{b}}$, completing the proof. □

We note that [GKZ, Theorem 14.3.1] provides a resultant interpretation for a hyperdeterminant of the boundary format. As discussed in Remark 8.2, this interpretation generalizes to higher codimension, enabling us to prove Theorem 1.6.

Proof of Theorem 1.6. As it is enough to show this result after passing to an algebraically closed field $\Bbbk$, we replace $Y(\phi)$, etc., by their corresponding objects over $\mathrm{Spec}(\Bbbk)$. By Remark 8.2, we may then apply the resultant interpretation of $Y(\phi)$ to view the $\Bbbk$-points of $Y(\phi)$ as systems of multilinear equations $\widetilde{\mathbf{f}}$ that have a nonempty vanishing locus in $\mathbb{P}(\vec{B})$.

Recall that $a' = 1 + \sum_{i=1}^n (b_i - 1)$, and let I be the ideal of $a' \times \mathbf{b}$ hyperdeterminants from (1.7). We claim that set-theoretically, $V(I) = Y(\phi)$. Note that $Y(\phi) \subseteq V(I)$, since any collection of a' polynomials in the vector space $\langle \widetilde{f_1}, \dots, \widetilde{f_a} \rangle$ must have a common root, and thus all of the corresponding hyperdeterminants must vanish by [GKZ, Theorem 14.3.1].

For the reverse inclusion, suppose that there exists a point $\widetilde{\mathbf{f}} \in V(I) \setminus Y(\phi)$. Then $\widetilde{\mathbf{f}}$ has no common zero in $\mathbb{P}(\vec{B})$. Since $V(I)$ and $Y(\phi)$ are both G-equivariant, we may assume after a $\mathbf{GL}(A)$-change of coordinates that $\widetilde{f}_1, \dots, \widetilde{f}_{a'-1}$ intersect in a finite number of points $\{P_1, \dots, P_t\} \in \mathbb{P}(\vec{B})$. We now consider the vector space $W := \langle \widetilde{f}_{a'}, \dots, \widetilde{f}_a \rangle$ and choose $\widetilde{g} \in W$. Since every hyperdeterminant of every subtensor ϕ' of ϕ of size $a' \times \mathbf{b}$ vanishes on $\widetilde{\mathbf{f}}$, there must be some P_i that is a root of $\widetilde{g}$. Consequently, the incidence locus

$$\{(\widetilde{g}, P_i) \in W \times \{P_1, \dots, P_t\} \mid \widetilde{g}(P_i) = 0\}$$

is a closed sublocus of $W \times \{P_1, \dots, P_t\}$ that surjects onto W. It then follows that there is some connected component of this incidence locus that alone surjects onto W; in other words, there is some P_i that is simultaneously a root of all polynomials in W. This P_i is then also a common zero of $\widetilde{\mathbf{f}}$, contradicting our assumption that $\widetilde{\mathbf{f}} \notin Y(\phi)$. □

Remark 9.2. Bernd Sturmfels has pointed out that $Y(\phi^{a\times b\times 2})$ has a second interpretation as a resultant variety as well. For simplicity, we work over a field $\Bbbk$. By identifying points of $\mathbb{A}_{\Bbbk}^{a\times b\times 2}$ with maps in $\operatorname{Hom}(\Bbbk^a \otimes \Bbbk^2, \Bbbk^b)$, we may think of a point $\psi \in \mathbb{A}_{\Bbbk}^{a\times b\times 2}$ as a linear map

$$\iota_\psi : \mathbb{P}^{b-1} \to \mathbb{P}^{2a-1}.$$

The image of ι_ψ then intersects the Segre variety $\mathbb{P}^{a-1} \times \mathbb{P}^1$ if and only if ψ belongs to $Y(\phi)$. This can be checked directly as follows. Let $U_1, \dots, U_b$ be a sequence of $2 \times a$ matrices which span the image of ι_ψ. The image of ι_ψ intersects the Segre variety if and only if there exist nontrivial scalars λ_i and (α_1, α_2) such that (α_1, α_2) belongs to the kernel of $\sum_{i=1}^{b} \lambda_i U_i$. This is equivalent to the statement that the rank 1 tensor $(\lambda_i \alpha_j) \in \Bbbk^b \otimes \Bbbk^2$ belongs to the kernel of $\psi^\flat$, which is equivalent to $\psi \in Y(\phi)$ by Proposition 8.1.

We now provide a more detailed description of the geometry of $Y(\phi)$. When $b_i > 1$ for only one index i, $Y(\phi)$ is a determinantal variety defined by the maximal minors of a matrix of indeterminates. We thus investigate the situation when $b_i > 1$ for at least two indices i.

Proposition 9.3. *Suppose that $b_i > 1$ for at least two indices i and that $Y(\phi) \neq \mathbb{A}^{a\times\mathbf{b}}$. Then $Y(\phi)$ is not normal. If additionally* $\operatorname{codim} Y(\phi) \geq 2$, *then $Y(\phi)$ is not Cohen–Macaulay.*

Proof. Since $\mu\colon Z(\phi) \to Y(\phi)$ is birational (Remark 8.3), it suffices, by Zariski's connectedness theorem, to show that there is a fiber of μ that is not geometrically connected.

Let $\psi \in Y(\phi)$ be a generic map. We claim that $\ker(\psi) \cap \operatorname{Seg}(\mathbf{B})$ is a single point x. If $b_1 \cdots b_n \leq a$, then $\ker \psi$ is 1-dimensional; therefore, the intersection is a single point. If $b_1 \cdots b_n > a$, then the kernel of a map $\psi\colon \mathbf{B}^* \to A^*$ has codimension a. Since $a > \dim \operatorname{Seg}(\mathbf{B})$ and $[\ker \psi] \cap Y(\phi) \neq \emptyset$, we obtain the claim.

Let $\Bbbk$ be the algebraic closure of the residue field of ψ, so that x is $\Bbbk$-rational. Pick an additional $\Bbbk$-rational point y on $\operatorname{Seg}(\mathbf{B})$ but not on $[\ker \psi]$ such that the line joining x and y does not lie in $\operatorname{Seg}(\mathbf{B})$. (Here we use the hypothesis that $b_i > 1$ for at least two i. Note that if $b_i > 1$ for at most one i, then $\operatorname{Seg}(\mathbf{B})$ is a linear subvariety of $\mathbb{P}(\mathbf{B})$.) Pick a basis for $\mathbf{B}^*$ containing x and y, and let ψ' be a map that agrees with ψ on all basis elements except y and sends y to 0. Then $\psi' \in Y(\phi)$ and $[\ker \psi']$ intersects $\operatorname{Seg}(\mathbf{B})$ in finitely many points (but at least two). Hence the fiber over ψ' is not geometrically connected.

Now assume that codim $Y(\phi) = a - \sum_{i=1}^n (b_i - 1) \geq 2$. Then, by Proposition 9.4, $Y(\phi)$ is regular in codimension one. By the Serre criterion for normality [Eis, Theorem 11.5], $Y(\phi)$ does not satisfy the condition (S_2), so it is not Cohen–Macaulay. □

The following proposition provides a multilinear analogue of the classical fact that the singular locus of a determinantal variety consists of those maps whose kernel has dimension higher than the generic value.

Proposition 9.4. *Suppose that $b_i > 1$ for at least two indices i and that $Y(\phi) \neq \mathbb{A}^{a\times \mathbf{b}}$. Then the singular locus $Y(\phi)_{\mathrm{sing}}$ of $Y(\phi)$ coincides with the nonnormal locus $Y(\phi)_{\mathrm{nn}}$ of $Y(\phi)$. In particular,*

$$Y(\phi)_{\mathrm{sing}} = \{\psi \in Y(\phi) \mid [\ker(\psi)] \cap \mathrm{Seg}(\mathbf{B}) \textit{ is not a single reduced point}\}.$$

Furthermore, $Y(\phi)_{\mathrm{sing}}$ is irreducible of codimension $a - \sum_{i=1}^n (b_i - 1)$ in $Y(\phi)$.

Proof. Let $Y_1 := \{\psi \in Y(\phi) \mid [\ker(\psi)] \cap \mathrm{Seg}(\mathbf{B})$ is not a single reduced point$\}$. We first show that Y_1 is irreducible. Let $\Delta \subset \mathrm{Seg}(\mathbf{B}) \times \mathrm{Seg}(\mathbf{B})$ be the diagonal subscheme and $U := (\mathrm{Seg}(\mathbf{B}) \times \mathrm{Seg}(\mathbf{B})) \smallsetminus \Delta$. Write q_1 and q_2 for the two projection morphisms $\mathrm{Seg}(\mathbf{B}) \times \mathrm{Seg}(\mathbf{B}) \to \mathrm{Seg}(\mathbf{B})$. Note that $\mathcal{L} := (q_1^*\mathcal{O}(-1,\ldots,-1) \oplus q_2^*\mathcal{O}(-1,\ldots,-1))|_U$ is naturally a subbundle of the trivial bundle $\mathbf{B}^* \otimes \mathcal{O}_U$. Let Z' be the total space of $\mathcal{H}om((\mathbf{B}^* \otimes \mathcal{O}_U)/\mathcal{L}, A^* \otimes \mathcal{O}_U)$; note that Z' is an irreducible subvariety of $\mathbb{A}^{a\times\mathbf{b}} \times U$, which is the total space of $\mathcal{H}om(\mathbf{B}^* \otimes \mathcal{O}_U, A^* \otimes \mathcal{O}_U)$. A point $\psi \in \mathbb{A}^{a\times\mathbf{b}}$ lies in the image of Z' if and only if $[\ker\psi] \cap \mathrm{Seg}(\mathbf{B})$ consists of more than one point.

Hence, every point of Y_1 lies in the closure of the image of Z' (which is irreducible), except possibly the loci of ψ such that $[\ker(\psi)] \cap \mathrm{Seg}(\mathbf{B})$ consists of a single nonreduced point. Thus, to complete our argument that Y_1 is irreducible, we must show that any such ψ lies in the closure of Y_1. Fix some ψ_0 such that $[\ker\psi_0] \cap \mathrm{Seg}(\mathbf{B})$ is a single nonreduced point, and write $[\ker\psi_0]$ as a sum of lines $L_1 + \cdots + L_r$ such that $L = L_1$ is a tangent line to $\mathrm{Seg}(\mathbf{B})$ at x. Since a tangent line at a smooth point is a limit of secant lines, there is a family of secant lines L_t that have L as their limit, and we write $H_t := L_t + L_2 + \cdots + L_r$. There is then a compatible family of ψ_t such that $[\ker\psi_t] = H_t$ and ψ_t limits to ψ_0. Since L_t is a secant line, it follows that $H_t \cap \mathrm{Seg}(\mathbf{B})$ is supported on more than point, and hence $\psi_t \in Y_1$. Since ψ_0 is in the closure of the family ψ_t, it follows that ψ_0 also lies in Y_1, as desired.

We next compute the codimension of Y_1 in $Y(\phi)$. The map $Z' \to Y_1$ is a 2-to-1 map over the dense open subset of Y_1 where $[\ker\psi]$ intersects $\mathrm{Seg}(\mathbf{B})$ in two points. Therefore

$$\begin{aligned}\dim Y_1 &= \dim Z' = 2\dim\mathrm{Seg}(\mathbf{B}) + a(b_1\cdots b_n - 2)\\ &= \dim Y(\phi) - \mathrm{rank}\, A + \dim\mathrm{Seg}(\mathbf{B}).\end{aligned}$$

Hence the codimension of Y_1 in $Y(\phi)$ is $a - \sum_{i=1}^n (b_i - 1)$.

Finally, we claim that Y_1 coincides with both the singular locus $Y(\phi)_{\mathrm{sing}}$ and the nonnormal locus $Y(\phi)_{\mathrm{nn}}$ of $Y(\phi)$. As noted in Remark 8.3, $\mu\colon Z(\phi) \to Y(\phi)$ is birational over the open set $Y(\phi) \setminus Y_1$. Since $Z(\phi)$ is smooth, we see that $Y(\phi)_{\mathrm{sing}} \subseteq Y_1$. Now, since $b_i > 1$ for at least two indices i, $\mathrm{Seg}(\mathbf{B})$ is not a linear subvariety of $\mathbb{P}(\mathbf{B})$. Thus, as argued in the proof of Proposition 9.3, there exist $\psi \in Y_1$ such that $[\ker(\psi)] \cap \mathrm{Seg}(\mathbf{B})$ set-theoretically consists of at least two reduced points. Since Y_1 is irreducible, it follows that a general point of Y_1 has this property. Any such point is a nonnormal point of $Y(\phi)$, and since both Y_1 and $Y(\phi)_{\mathrm{nn}}$ are closed, we conclude that $Y_1 \subseteq Y(\phi)_{\mathrm{nn}}$. Of course, the nonnormal locus always sits in the singular locus, and we thus obtain the chain

$$Y(\phi)_{\mathrm{sing}} \subseteq Y_1 \subseteq Y(\phi)_{\mathrm{nn}} \subseteq Y(\phi)_{\mathrm{sing}},$$

proving that these loci coincide. □

Remark 9.5. In the case when $\phi^{a\times\mathbf{b}}$ is a tensor of the boundary format, Proposition 9.4 recovers the first part of [WZ, Theorem 0.5(a)], which says that the singular locus of a hyperdeterminantal hypersurface (of the boundary format) is irreducible and has codimension 1. In these cases, $Y(\phi)$ is Cohen–Macaulay since it is a hypersurface, but it fails to be normal.

Remark 9.6. A conjecture of M. Hochster asserts that every complete local domain has a finitely generated maximal Cohen–Macaulay module [Hoc2, Conjecture 6, p.10]. This is known to be true in only a handful of cases [Hoc1, Gri, Kat, Sch]. By combining Theorem 1.2 and Proposition 9.3, we can construct finitely generated maximal Cohen–Macaulay modules $M(\phi, w)$ with non-Cohen–Macaulay supports $Y(\phi)$. At all points y where the completion of $\mathcal{O}_{Y,y}$ is a domain (i.e., at the unibranched points of Y) we get new examples where Hochster's conjecture holds. As far as we know, these examples are not covered by any previously known results. For instance, we could take y to be the $\mathbb{Z}/p$-point lying over the origin of $\mathbb{A}^{a\times\mathbf{b}}$.

Example 9.7. Consider the case $a \times \mathbf{b} = 3 \times (2, 2)$ and $w = (0, 0, 1)$. Then $F(\phi, w)_\bullet$ is a 2-term complex $S^2(-3) \xrightarrow{\partial_1} S^2$. By the method for writing out ∂_1 described in §4.1, we see that each entry of ∂_1 corresponds to a specific 3×3 minor of $\phi^\flat$.

Now, let $\widetilde{\phi} \in \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ denote a $\mathbb{C}$-point of $\mathbb{A}^{a\times\mathbf{b}}$. By [LW, Theorem 1.1], the border rank of the tensor $\widetilde{\phi}$ is less than 3 if and only if the 3×3 minors of $\phi^\flat$ vanish when evaluated at $\widetilde{\phi}$. This is equivalent to asking that the specialization of ∂_1 at $\widetilde{\phi}$ yields the zero matrix. Thus in this case, the border rank of the tensor $\widetilde{\phi}$ is determined by the homological properties of the specialization of the tensor complex. It would be interesting to study whether similar connections hold in more generality.

10. Eisenbud–Schreyer pure resolutions are balanced tensor complexes

The existence of pure resolutions of type d for an arbitrary degree sequence d was originally conjectured in [BS, Conjecture 2.4]. The first construction of such pure resolutions in arbitrary characteristic appears in [ES1, §5]. Theorem 10.1 below implies that each of these Eisenbud–Schreyer pure resolutions can be realized as the specialization of some balanced tensor complex. Each of these resolutions is constructed from a sequence of sufficiently generic multilinear forms $\mathbf{g} := g_1, \ldots, g_a$ on $\mathbb{A}^n_{\Bbbk} \times \mathbb{P}(\vec{B})$, where $\Bbbk$ is any field; set $R := \Bbbk[x_1, \ldots, x_n]$ and denote the corresponding pure resolution of R-modules by $\mathrm{ES}(\mathbf{g}, d)_\bullet$.

Theorem 10.1. *Let $d = (d_0, \dots, d_n)$ be a degree sequence, and $\mathrm{ES}(\mathbf{g}, d)_\bullet$ be an Eisenbud–Schreyer pure resolution. Let $a := d_n - d_0$, $b_i := d_i - d_{i-1}$, and $w := (d_0, 0, d_1, d_2, \dots, d_{n-1})$. Then there exists a map $\mathbb{Z}[X^{a\times\mathbf{b}}] \to R$ such that*

$$\mathrm{ES}(\mathbf{g}, d)_\bullet \cong F(\phi^{a\times\mathbf{b}}, w)_\bullet \otimes_{\mathbb{Z}[X^{a\times\mathbf{b}}]} R.$$

Proof. Since each g_i is multilinear, we may write $g_i = \sum_J g_{i,J} y_J$, where the $g_{i,J}$ are linear forms on $\mathbb{A}^n$ and where y_J is a multilinear form on $\mathbb{P}(\vec{B})$. We then define a map $\mathbb{Z}[X^{a\times\mathbf{b}}] \to R$ by $x_{i,J} \mapsto g_{i,J}$. This yields a commutative diagram

$$\begin{array}{ccc} \mathbb{A}^n \times \mathbb{P}(\vec{B}) & \xrightarrow{\nu} & \mathbb{A}^{a\times\mathbf{b}} \times \mathbb{P}(\vec{B}) \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ \mathbb{A}^n & \xrightarrow{\rho} & \mathbb{A}^{a\times\mathbf{b}} \end{array}$$

By the projection formula [Har, Proposition II.5.6], we get a quasi-isomorphism

$$\mathbf{R}\pi'_*(\nu^*\mathcal{K}(\phi)_\bullet) \cong \rho^*(\mathbf{R}\pi_*\mathcal{K}(\phi)_\bullet)$$

(noting that $\mathbf{L}\rho^*$ and $\mathbf{L}\nu^*$ coincide with ρ^* and ν^*, since we apply them to a complex of locally free sheaves). The argument immediately following (7.1) yields an isomorphism of complexes. Using the notation of Remark 2.6, we have $\nu^*(f_i) = g_i$, so $\nu^*\mathcal{K}(\phi)_\bullet$ is the Koszul complex used in [ES1, Theorem 5.1] to construct the complex $\mathrm{ES}(\mathbf{g}, d)_\bullet$. □

Remark 10.2. In [ES1, Proposition 5.2], Eisenbud and Schreyer illustrate explicit multilinear forms over $\mathbb{Z}$ that satisfy the necessary genericity conditions. We note that Theorem 10.1 also holds when $R = \mathbb{Z}[x_1, \dots, x_n]$ and, in this case, $\mathrm{ES}(\mathbf{g}, d)_\bullet$ is a uniformly minimal resolution of a generically perfect module M of codimension n.

Remark 10.3. By combining Corollary 1.5(iii) and Theorem 10.1, we recover the curious fact that the multiplicity of the Eisenbud–Schreyer pure resolution of type $d = (d_0, \dots, d_n)$ depends only on the unordered (!) set of first differences $\{d_1 - d_0, \dots, d_n - d_{n-1}\}$. We first learned of this fact through a conversation with Eisenbud and Schreyer.

11. New families of pure resolutions

We have shown that a tensor $\phi^{a\times\mathbf{b}}$ and a pinching weight w yield a pure resolution $F(\phi^{a\times\mathbf{b}}, w)_\bullet$ of type $d(w)$ (Notation 5.2). Informally, we may think of this as a map $(a, \mathbf{b}, w) \mapsto d(w)$, where w is a pinching weight for $\phi^{a\times\mathbf{b}}$. From this perspective, the proof of Theorem 1.9 describes the fibers of this map.

Proof of Theorem 1.9. Let $d \in \mathbb{Z}^{p+1}$. We will describe all the choices of $a, \mathbf{b}$, and pinching weight w such that $F(\phi^{a\times\mathbf{b}}, w)_\bullet$ is a pure resolution of type d. (The module $M(\phi, w)$ is Cohen–Macaulay by Theorem 5.3.) Let $c \le d_0$ and $C \ge d_p$ be integers, and view d as a subsequence of $\{c, c+1, \dots, C\}$. Subdivide $\{c, c+1, \dots, C\} \setminus \{d_0, \dots, d_p\}$ into sequences $s^{(j)}$ of consecutive integers, where $1 \le j \le n$. We may assume that $\min(s^{(j+1)}) > \min(s^{(j)})$ for all j.

Let $a := C - c$ and $b_j := |s^{(j)}| + 1$ for $1 \leq j \leq n$. (Here $|\cdot|$ denotes the length of the sequence.) Let $w_0 := c$ and $w_j := \min(s^{(j)}) - c - 1$, $1 \leq j \leq n$. Since $s^{(j)} = \{w_j + c + 1, \dots, w_j + b_j + c - 1\}$, we see that the intervals $[w_j + 1, w_j + b_j - 1]$ are disjoint and contained in $[0, a]$. Therefore w is a pinching weight (Definition 5.1) for ϕ. Note that by construction, $d(w) = d$. Thus we have chosen a, $\mathbf{b}$, and w so that $F(\phi^{a \times \mathbf{b}}, w)_\bullet$ is a pure resolution of type d, and there are infinitely many such choices. □

Remark 11.1. If w is a pinching weight for ϕ (so that $F(\phi, w)_\bullet$ is a pure resolution of type $d(w)$), then the Betti diagram of $F(\phi, w)_\bullet$ is an integral multiple of the Betti diagram of the corresponding Eisenbud–Schreyer pure resolution. In particular, Theorem 1.9 has no implications for [EFW, Conjecture 6.1].

Table 1. Pure resolutions of type $d = (0, 3)$ with parameters $c = -2$ and $C = 4$.

Subdivision	$a \times \mathbf{b}$	w	$\beta(F(\phi, w)_\bullet)$
$(-2, -1), (1, 2), (4)$	$6 \times (3, 3, 2)$	$(-2, -1, 2, 5)$	$\begin{pmatrix} 60 & - \\ - & - \\ - & 60 \end{pmatrix}$
$(-2), (-1), (1, 2), (4)$	$6 \times (2, 2, 3, 2)$	$(-2, -1, 0, 2, 5)$	$\begin{pmatrix} 120 & - \\ - & - \\ - & 120 \end{pmatrix}$
$(-2, -1), (1), (2), (4)$	$6 \times (3, 2, 2, 2)$	$(-2, -1, 2, 3, 5)$	$\begin{pmatrix} 120 & - \\ - & - \\ - & 120 \end{pmatrix}$
$(-2), (-1), (1), (2), (4)$	$6 \times (2, 2, 2, 2, 2)$	$(-2, -1, 0, 2, 3, 5)$	$\begin{pmatrix} 240 & - \\ - & - \\ - & 240 \end{pmatrix}$

Example 11.2. Consider the degree sequence $d = (0, 3)$. Table 1 illustrates the various constructions of pure resolutions of type d with $c = -2$ and $C = 4$.

Example 11.3. The complexes $F_\bullet$ and $F'_\bullet$ in [BEKS, Example 6.5] are also specializations of tensor complexes; this follows from an argument similar to the proof of Corollary 10.1. Namely, the complex $F_\bullet$ is a specialization of the tensor complex for an $8 \times (2, 2, 2, 2)$ tensor with $w = (0, 0, 2, 6, 7)$; the complex $F'_\bullet$ is a specialization of the tensor complex for a $7 \times (2, 2, 2, 2)$ tensor with $w' = (0, -1, 2, 4, 5)$. We obtain

$$F_\bullet : \begin{bmatrix} \bigwedge^0 \\ S^0 \\ S^2 \\ S^6 \\ S^7 \end{bmatrix} \leftarrow \begin{bmatrix} \bigwedge^2 \\ \widetilde{D}^0 \\ S^0 \\ S^4 \\ S^5 \end{bmatrix}(-2) \leftarrow \begin{bmatrix} \bigwedge^4 \\ \widetilde{D}^2 \\ \widetilde{D}^0 \\ S^2 \\ S^3 \end{bmatrix}(-4) \leftarrow \begin{bmatrix} \bigwedge^5 \\ \widetilde{D}^3 \\ \widetilde{D}^1 \\ S^1 \\ S^2 \end{bmatrix}(-5) \leftarrow \begin{bmatrix} \bigwedge^6 \\ \widetilde{D}^4 \\ \widetilde{D}^2 \\ S^0 \\ S^1 \end{bmatrix}(-6) \leftarrow 0$$

and

$$F'_\bullet: \quad \begin{bmatrix} \bigwedge^1 \\ \widetilde{\mathrm{D}}^0 \\ \mathrm{S}^1 \\ \mathrm{S}^3 \\ \mathrm{S}^4 \end{bmatrix}(-1) \leftarrow \begin{bmatrix} \bigwedge^2 \\ \widetilde{\mathrm{D}}^1 \\ \mathrm{S}^0 \\ \mathrm{S}^2 \\ \mathrm{S}^3 \end{bmatrix}(-2) \leftarrow \begin{bmatrix} \bigwedge^4 \\ \widetilde{\mathrm{D}}^3 \\ \widetilde{\mathrm{D}}^0 \\ \mathrm{S}^0 \\ \mathrm{S}^1 \end{bmatrix}(-4) \leftarrow \begin{bmatrix} \bigwedge^7 \\ \widetilde{\mathrm{D}}^6 \\ \widetilde{\mathrm{D}}^3 \\ \widetilde{\mathrm{D}}^1 \\ \widetilde{\mathrm{D}}^0 \end{bmatrix}(-7) \leftarrow 0.$$

The nonzero map between these resolutions is induced by the natural inclusion $A' \subseteq A$ whose cokernel is the final summand of $\mathbb{Z}$ in A. See also Remark 7.4.

12. Detailed example of a tensor complex

Example 12.1. Let ϕ be the universal $4 \times (2,2)$ tensor, and $w = (0,0,2)$. We consider the complex $F(\phi, w)_\bullet$. This is one of the simplest examples of a tensor complex which is not a matrix complex. The resulting complex $F(\phi^{4\times(2,2)}, (0,0,2))_\bullet$ is

$$\begin{bmatrix} \bigwedge^0 \\ \mathrm{S}^0 \\ \mathrm{S}^2 \end{bmatrix} \xleftarrow{\partial_1} \begin{bmatrix} \bigwedge^2 \\ \widetilde{\mathrm{D}}^0 \\ \mathrm{S}^0 \end{bmatrix}(-2) \xleftarrow{\partial_2} \begin{bmatrix} \bigwedge^4 \\ \widetilde{\mathrm{D}}^2 \\ \widetilde{\mathrm{D}}^0 \end{bmatrix}(-4) \leftarrow 0,$$

which has the Betti diagram

$$\begin{pmatrix} 3 & - & - \\ - & 6 & - \\ - & - & 3 \end{pmatrix}.$$

To describe the differentials ∂_1 and ∂_2, we first write the flattening $\phi^\flat\colon A^* \to B_1^* \otimes B_2^*$:

$$\phi^\flat = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} x_{1,(1,1)} & x_{2,(1,1)} & x_{3,(1,1)} & x_{4,(1,1)} \\ x_{1,(1,2)} & x_{2,(1,2)} & x_{3,(1,2)} & x_{4,(1,2)} \\ x_{1,(2,1)} & x_{2,(2,1)} & x_{3,(2,1)} & x_{4,(2,1)} \\ x_{1,(2,2)} & x_{2,(2,2)} & x_{3,(2,2)} & x_{4,(2,2)} \end{pmatrix} \end{array}.$$

For $I \subseteq \{a,b,c,d\}$ and $J \subseteq \{1,2,3,4\}$ with $|I| = |J|$, we denote the corresponding minor of $\phi^\flat$ by $\Delta_{I;J}$. For instance, $\Delta_{ab;12}$ is the 2×2 minor from the upper left corner of $\phi^\flat$.

We set $a_1, \ldots, a_4$ as a basis of A, u_1, u_2 a basis of B_1, and v_1, v_2 a basis of B_2. Following the notation and the method of §4.1, we then obtain

$$\partial_1^T = \begin{array}{c} \\ f_{\{1,2\},\{1,2\},\emptyset} \\ f_{\{1,3\},\{1,2\},\emptyset} \\ f_{\{1,4\},\{1,2\},\emptyset} \\ f_{\{2,3\},\{1,2\},\emptyset} \\ f_{\{2,4\},\{1,2\},\emptyset} \\ f_{\{3,4\},\{1,2\},\emptyset} \end{array} \begin{array}{c} \begin{array}{ccc} g_{\emptyset,\emptyset,(2,0)} & g_{\emptyset,\emptyset,(1,1)} & g_{\emptyset,\emptyset,(0,2)} \end{array} \\ \begin{pmatrix} \Delta_{ac;12} & \Delta_{ad;12} + \Delta_{bc;12} & \Delta_{bd;12} \\ \Delta_{ac;13} & \Delta_{ad;13} + \Delta_{bc;13} & \Delta_{bd;13} \\ \Delta_{ac;14} & \Delta_{ad;14} + \Delta_{bc;14} & \Delta_{bd;14} \\ \Delta_{ac;23} & \Delta_{ad;23} + \Delta_{bc;23} & \Delta_{bd;23} \\ \Delta_{ac;24} & \Delta_{ad;24} + \Delta_{bc;24} & \Delta_{bd;24} \\ \Delta_{ac;34} & \Delta_{ad;34} + \Delta_{bc;34} & \Delta_{bd;34} \end{pmatrix} \end{array}$$

and

$$\partial_2 = \begin{array}{c} \\ f_{\{1,2\},\{1,2\},\emptyset} \\ f_{\{1,3\},\{1,2\},\emptyset} \\ f_{\{1,4\},\{1,2\},\emptyset} \\ f_{\{2,3\},\{1,2\},\emptyset} \\ f_{\{2,4\},\{1,2\},\emptyset} \\ f_{\{3,4\},\{1,2\},\emptyset} \end{array} \begin{array}{c} \begin{array}{ccc} e_{\{1234\},(2,0),\emptyset} & e_{\{1234\},(1,1),\emptyset} & e_{\{1234\},(0,2),\emptyset} \end{array} \\ \begin{pmatrix} \Delta_{ab;34} & (\Delta_{ad;34} - \Delta_{bc;34}) & \Delta_{cd;34} \\ -\Delta_{ab;24} & -(\Delta_{ad;24} - \Delta_{bc;24}) & -\Delta_{cd;24} \\ \Delta_{ab;23} & (\Delta_{ad;23} - \Delta_{bc;23}) & \Delta_{cd;23} \\ \Delta_{ab;14} & (\Delta_{ad;14} - \Delta_{bc;14}) & \Delta_{cd;14} \\ -\Delta_{ab;13} & -(\Delta_{ad;13} - \Delta_{bc;13}) & -\Delta_{cd;13} \\ \Delta_{ab;12} & (\Delta_{ad;12} - \Delta_{bc;12}) & \Delta_{cd;12} \end{pmatrix} \end{array}.$$

The fact that each entry of $\partial_1\partial_2$ equals zero follows from a generalized Laplace expansion of a singular matrix. For instance, let us consider the $(1, 1)$ entry of $\partial_1\partial_2$, which is given by

$$\begin{aligned}(\partial_1\partial_2)_{1,1} = \Delta_{ac;12}\Delta_{ab;34} - \Delta_{ac;13}\Delta_{ab;24} + \Delta_{ac;14}\Delta_{ab;23} + \Delta_{ac;23}\Delta_{ab;14} \\ - \Delta_{ac;24}\Delta_{ab;13} + \Delta_{ac;34}\Delta_{ab;12}.\end{aligned}$$

By the generalized Laplace expansion formula [Nor, §1.6], this equals the determinant of

$$\begin{array}{c} \\ a \\ c \\ a \\ b \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} x_{1,(1,1)} & x_{2,(1,1)} & x_{3,(1,1)} & x_{4,(1,1)} \\ x_{1,(2,1)} & x_{2,(2,1)} & x_{3,(2,1)} & x_{4,(2,1)} \\ x_{1,(1,1)} & x_{2,(1,1)} & x_{3,(1,1)} & x_{4,(1,1)} \\ x_{1,(1,2)} & x_{2,(1,2)} & x_{3,(1,2)} & x_{4,(1,2)} \end{pmatrix} \end{array}.$$

But the above matrix has a repeated row, and hence this determinant is zero. Similar arguments show that all entries of $(\partial_1\partial_2)$ equal 0.

Example 12.2. Continuing with the $4 \times 2 \times 2$ example above, we compute the defining ideal of $Y(\phi)$. In order to use representation theory and computations from `Macaulay2` [M2], we work over $\mathbb{Q}$ instead of $\mathbb{Z}$. From the presentation matrix ∂_1 for $M(\phi, w)$, we compute directly in `Macaulay2` that $Y(\phi)$ is defined by one quartic and ten sextic equations. The quartic equation arises as the determinant of $\phi^\flat$, which corresponds to the subrepresentation

$$\mathbf{S}_{1,1,1,1}(A) \otimes \mathbf{S}_{2,2}(B_1^*) \otimes \mathbf{S}_{2,2}(B_2^*) \subseteq \mathrm{S}^4(A \otimes B_1^* \otimes B_2^*).$$

The sextic equations correspond to the hyperdeterminants of all $3 \times 2 \times 2$ subtensors of ϕ and arise as the subrepresentation

$$\mathbf{S}_{2,2,2}(A) \otimes \mathbf{S}_{3,3}(B_1^*) \otimes \mathbf{S}_{3,3}(B_2^*) \subseteq \mathrm{S}^6(A \otimes B_1^* \otimes B_2^*).$$

These equations all have geometric significance. Namely, as discussed in Remark 8.2, $Y(\phi)$ parametrizes quadruples $(f_1, \ldots, f_4)$ of multilinear forms on $\mathbb{P}^1 \times \mathbb{P}^1$ with $V(f_1, \ldots, f_4) \neq \emptyset$. Since $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1, 1))$ is 4-dimensional and base-point-free, the vector space $\langle f_1, \ldots, f_4\rangle$ has dimension at most 3. This explains the presence of the quartic $\det(\phi^\flat)$.

In addition, if $V(f_1, \ldots, f_4) \neq \emptyset$, then $V(g_1, g_2, g_3) \neq \emptyset$ for every triplet $g_1, g_2, g_3 \in \langle f_1, \ldots, f_4\rangle$. For such a triplet, $V(g_1, g_2, g_3) \neq \emptyset$ if and only if its corresponding $3 \times 2 \times 2$ hyperdeterminant vanishes. Applying this to all $3 \times 2 \times 2$ subtensors yields the 10-dimensional space of sextic equations.

Appendix A. Characteristic-free multilinear algebra

We review some characteristic-free multilinear algebra. See [Wey, §1.1] and [ABW][2] for more details.

Let E be a finitely generated $\mathbb{Z}$-module and d a positive integer. Let Σ_d denote the symmetric group on d letters. The *symmetric power* $\mathrm{S}^d(E)$ is the quotient of $E^{\otimes d}$ by the submodule generated by elements of the form $e_1 \otimes \cdots \otimes e_d - e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(d)}$ for $\sigma \in \Sigma_d$. The *divided power* $\mathrm{D}^d(E)$ is the submodule of Σ_d-invariants of $E^{\otimes d}$. We have a canonical isomorphism $\mathrm{D}^d(E^*) = \mathrm{S}^d(E)^*$. The *exterior power* $\bigwedge^d E$ is the quotient of $E^{\otimes d}$ by the submodule generated by elements of the form $e_1 \otimes \cdots \otimes e_d - \mathrm{sgn}(\sigma) e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(d)}$ for $\sigma \in \Sigma_d$, where $\mathrm{sgn}(\sigma)$ is the determinant of σ when written as a permutation matrix. One could also define the exterior power as a submodule of $E^{\otimes d}$, but in accordance with Remark 2.7, one must make a distinction between the two when E is a $\mathbb{Z}/2$-graded module. If E is a free $\mathbb{Z}$-module, then each module defined is also a free $\mathbb{Z}$-module.

For each of the three definitions above, one can take direct sums over all $d \geq 0$, and the resulting modules can be given the structure of a Hopf algebra. In particular, they are equipped with a multiplication m and comultiplication Δ, which we will make use of.

Now for E and F free $\mathbb{Z}$-modules of finite rank, we define the following inclusions.

(i) $\bigwedge^d E \otimes \bigwedge^d F \to \mathrm{S}^d(E \otimes F)$ is defined by mapping $e_1 \wedge \cdots \wedge e_d \otimes f_1 \wedge \cdots \wedge f_d$ to the determinant of the matrix $(e_i \otimes f_j)_{i,j=1,\ldots,d}$.

(ii) $\Phi_d \colon \bigwedge^d E \otimes \mathrm{D}^d F \to \bigwedge^d (E \otimes F)$ will be defined by induction on d. For the base case, set Φ_1 to be the identity. For $d > 1$, extend linearly the map on elements of the form $x = e_1 \wedge \cdots \wedge e_d \otimes f_1^{(\alpha_1)} \cdots f_r^{(\alpha_r)}$, where $\alpha_1 + \cdots + \alpha_r = d$, given by

$$\Phi_d(x) := \sum_{i=1}^{r} (e_1 \otimes f_i) \wedge \Phi_{d-1}(e_2 \wedge \cdots \wedge e_d \otimes f_1^{(\alpha_1)} \cdots f_i^{(\alpha_i - 1)} \cdots f_r^{(\alpha_r)}).$$

(iii) $\mathrm{D}^d E \otimes \mathrm{D}^d F \to \mathrm{D}^d(E \otimes F)$ is the dual of the map $\mathrm{S}^d(E^* \otimes F^*) \to \mathrm{S}^d(E^*) \otimes \mathrm{S}^d(F^*)$, which is given by $(e_{i_1} \otimes f_{j_1}) \cdots (e_{i_d} \otimes f_{j_d}) \mapsto (e_{i_1} \cdots e_{i_d}) \otimes (f_{j_1} \cdots f_{j_d})$.

Appendix B. Schur functors in characteristic zero

We review some representation theory of $G = \mathbf{GL}_n(\mathbb{Q})$ and Schur–Weyl duality. See [Wey, §2] and [KP, §5] for general background.

A sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ of nonnegative integers is a *partition* if $\lambda_1 \geq \cdots \geq \lambda_n$. If $\lambda_n \neq 0$, then n is the length of λ. We set $|\lambda| := \lambda_1 + \cdots + \lambda_n$, and write $\lambda \vdash |\lambda|$. Write 1^d for the partition consisting of d 1's. Given two partitions λ and μ, we write $\lambda \subseteq \mu$ if $\lambda_i \leq \mu_i$ for all i. If $\lambda \subseteq \mu$, we say that μ/λ is a *horizontal strip* if $\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \geq \mu_n \geq \lambda_n$, denoted $\mu/\lambda \in \mathrm{HS}$. Define λ' to be the partition such that

[2] The first formula in [ABW, p. 247] is a multiple of the second formula there and does not have desirable characteristic-free properties.

$\lambda'_i = \#\{j \mid \lambda_j \geq i\}$. Given $\lambda \subseteq \mu$, we say that μ/λ is a *vertical strip* if $\mu'/\lambda' \in \mathrm{HS}$, denoted $\mu/\lambda \in \mathrm{VS}$.

Let E be the n-dimensional vector representation of G. The finite-dimensional irreducible polynomial representations of G are indexed by partitions λ of length at most n, and a general finite-dimensional polynomial representation of G is a direct sum of irreducible representations. Let $\mathbf{S}_\lambda E$ denote the irreducible representation corresponding to λ, using the convention that $\mathbf{S}_\lambda E = 0$ if $\lambda_{n+1} > 0$. In particular, $\mathrm{S}^d E = \mathbf{S}_{(d)} E$ and $\bigwedge^d E = \mathbf{S}_{1^d} E$.

Pieri's rule gives tensor product decompositions

$$\mathbf{S}_\lambda E \otimes \mathrm{S}^d E \cong \bigoplus_{\substack{\mu \vdash |\lambda|+d \\ \mu/\lambda \in \mathrm{HS}}} \mathbf{S}_\mu E \quad \text{and} \quad \mathbf{S}_\lambda E \otimes \bigwedge^d E \cong \bigoplus_{\substack{\mu \vdash |\lambda|+d \\ \mu/\lambda \in \mathrm{VS}}} \mathbf{S}_\mu E. \tag{B.1}$$

See [Wey, Corollary 2.3.5] (there, $L_\lambda E$ is isomorphic to our $\mathbf{S}_{\lambda'} E$). These formulas remain valid if we replace E by its dual E^*.

Let Σ_k be the symmetric group on k letters. There are commuting actions of G and Σ_k on $E^{\otimes k}$. *Schur–Weyl duality* [KP, Proposition 5.9] is the $G \times \Sigma_k$-equivariant decomposition

$$E^{\otimes k} \cong \bigoplus_{\substack{\lambda \vdash k \\ \lambda_{n+1}=0}} \mathbf{S}_\lambda E \otimes \chi_\lambda,$$

where χ_λ are irreducible representations of Σ_k. We use the facts that $\chi_{(k)}$ is the trivial representation of Σ_k, $\chi_{(1^k)}$ is the one-dimensional sign representation, and more generally, $\chi_\lambda \otimes \chi_{(1^k)} = \chi_{\lambda'}$.

We use the following consequence of Schur–Weyl duality. Let $E_1, \dots, E_r$ be vector spaces and consider $\mathbf{S}_\lambda(E_1 \otimes \cdots \otimes E_r)$ as a representation of $\mathbf{GL}(E_1) \times \cdots \times \mathbf{GL}(E_r)$. The irreducible representations of $\mathbf{GL}(E_1) \times \cdots \times \mathbf{GL}(E_r)$ are indexed by r-tuples of partitions, so

$$\mathbf{S}_\lambda(E_1 \otimes \cdots \otimes E_r) \cong \bigoplus_{\mu^1,\dots,\mu^r} (\mathbf{S}_{\mu^1} E_1 \otimes \cdots \otimes \mathbf{S}_{\mu^r} E_r)^{\oplus g_{\lambda,\mu^1,\dots,\mu^r}} \tag{B.2}$$

for some nonnegative integers $g_{\lambda,\mu^1,\dots,\mu^r}$ (the *Kronecker coefficients*). We now apply Schur–Weyl duality to $(E_1 \otimes \cdots \otimes E_r)^{\otimes k}$ in two different ways, where $k = |\lambda|$. First, we have

$$(E_1 \otimes \cdots \otimes E_r)^{\otimes k} \cong \bigoplus_{\nu \vdash k} \mathbf{S}_\nu(E_1 \otimes \cdots \otimes E_r) \otimes \chi_\nu$$

as $\mathbf{GL}(E_1 \otimes \cdots \otimes E_r) \times \Sigma_k$-representations. Second, we have

$$E_1^{\otimes k} \otimes \cdots \otimes E_r^{\otimes k} \cong \Big(\bigoplus_{\mu^1 \vdash k} \mathbf{S}_{\mu^1} E_1 \otimes \chi_{\mu^1}\Big) \otimes \cdots \otimes \Big(\bigoplus_{\mu^r \vdash k} \mathbf{S}_{\mu^r} E_r \otimes \chi_{\mu^r}\Big)$$

as $\mathbf{GL}(E_1) \times \cdots \times \mathbf{GL}(E_r) \times \Sigma_k$-representations. Restricting to the action of Σ_k and comparing the χ_λ-isotypic component of both expressions, we see that $g_{\lambda,\mu^1,\dots,\mu^r}$ is the multiplicity of χ_λ in the product $\chi_{\mu^1} \otimes \cdots \otimes \chi_{\mu^r}$. Since all representations of Σ_k are self-dual, this yields

$$g_{\lambda,\mu^1,\dots,\mu^r} = \dim(\chi_\lambda \otimes \chi_{\mu^1} \otimes \cdots \otimes \chi_{\mu^r})^{\Sigma_k}, \tag{B.3}$$

where the superscript indicates that invariants are taken. In light of (B.3), $g_{\lambda,\mu^1,\dots,\mu^r}$ is invariant under permutation of all of its indices. In particular, we deduce the Cauchy identities

$$\mathrm{S}^d(E_1 \otimes E_2) \cong \bigoplus_{\lambda \vdash d} \mathbf{S}_\lambda E_1 \otimes \mathbf{S}_\lambda E_2, \qquad \bigwedge^d(E_1 \otimes E_2) \cong \bigoplus_{\lambda \vdash d} \mathbf{S}_\lambda E_1 \otimes \mathbf{S}_{\lambda'} E_2. \tag{B.4}$$

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