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Probabilistic well-posedness for the cubic wave equation

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Abstract. The purpose of this article is to introduce for dispersive partial differential equations with random initial data, the notion of well-posedness (in the Hadamard-probabilistic sense). We restrict the study to one of the simplest examples of such equations: the periodic cubic semi-linear wave equation. Our contributions in this work are twofold: first, we break the algebraic rigidity involved in our previous works and allow much more general randomizations (general infinite product measures vs. Gibbs measures), and second, we show that the flow that we are able to construct enjoys very nice dynamical properties, including a new notion of probabilistic continuity.

1. Introduction

In [10], we developed a general method for obtaining local existence and uniqueness of semi-linear wave equations with data of supercritical regularity. In addition, in [11] we gave a very particular example (based on invariant measures considerations) of global existence with data of supercritical regularity. Our goal here is to make a significant extension of [10, 11] by presenting a quite general scheme to get global well-posedness for semi-linear dispersive equations with random data. More importantly, we propose a natural notion of probabilistic Hadamard well-posedness in this setting. We decided to restrict our attention to a very simple example. A further development of the ideas we introduce here will appear in a forthcoming work.

Let (M, g) be a $3d$ boundaryless Riemannian manifold with associated Laplace–Beltrami operator Δ_g . Consider the cubic defocusing wave equation

$$\begin{aligned} (\partial_t^2 - \Delta_g)u + u^3 &= 0, & u : \mathbb{R} \times M &\rightarrow \mathbb{R}, \\ u|_{t=0} = u_0, \partial_t u|_{t=0} &= u_1, & (u_0, u_1) \in H^s(M) \times H^{s-1}(M) &\equiv \mathcal{H}^s(M), \end{aligned} \quad (1.1)$$

where $H^s(M)$ denotes the classical Sobolev spaces on M . By using simple scaling considerations one finds that $s = 1/2$ is the critical Sobolev regularity associated to (1.1). It turns out that this regularity is the border line of the deterministic theory, in the sense of local well-posedness in the Hadamard sense (existence, uniqueness and continuous dependence on the data). More precisely, we have the following statement.

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Theorem 1. *The Cauchy problem (1.1) is locally well-posed for data in \mathcal{H}^s , $s \geq 1/2$ (and even globally for $s \geq 3/4$). In the opposite direction, for $s \in (0, 1/2)$, the Cauchy problem (1.1) is not locally well-posed in \mathcal{H}^s . For instance one can contradict the continuous dependence by showing that there exists a sequence (u_n) of global smooth solutions of (1.1) such that*

$$\lim_{n \rightarrow \infty} \|(u_n(0), \partial_t u_n(0))\|_{\mathcal{H}^s} = 0$$

and

$$\lim_{n \rightarrow \infty} \|(u_n(t), \partial_t u_n(t))\|_{L^\infty([0, T]; \mathcal{H}^s)} = \infty, \quad \forall T > 0.$$

Moreover, one can also contradict the existence by showing that there exists an initial datum $(u_0, u_1) \in \mathcal{H}^s$ such that for every $T > 0$ the problem (1.1) has no solution in $L^\infty([0, T]; \mathcal{H}^s)$, if we suppose that in addition the flow satisfies a suitable finite speed of propagation. Such a property is natural in the context of wave equations.

One may wish to compare the result of Theorem 1 with the classical Hadamard counterexample in the context of the Laplace equation

$$(\partial_t^2 + \partial_x^2)v = 0, \quad v : \mathbb{R}_t \times S_x^1 \rightarrow \mathbb{R}. \quad (1.2)$$

The equation (1.2) has the explicit solution

$$v_n(t, x) = e^{-\sqrt{n}} \operatorname{sh}(nt) \cos(nx).$$

Then for every s , v_n satisfies

$$\|(v_n(0), \partial_t v_n(0))\|_{\mathcal{H}^s(S^1)} \lesssim e^{-\sqrt{n}} n^s \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

but for $t \neq 0$,

$$\|(v_n(t), \partial_t v_n(t))\|_{\mathcal{H}^s(S^1)} \gtrsim e^{n|t|} e^{-\sqrt{n}} n^s \rightarrow +\infty \quad \text{as } n \rightarrow \infty.$$

Consequently, (1.2) is not well-posed in \mathcal{H}^s for every $s \in \mathbb{R}$, which is the analogue of the \mathcal{H}^s , $s < 1/2$, result of Theorem 1. On the other hand, (1.2) is well-posed in analytic spaces, which is the analogue of the $s > 1/2$ result in Theorem 1. Let us also observe that one may show the ill-posedness of (1.2) in Sobolev spaces by an indirect argument based on elliptic regularity. We are not aware of a similar indirect argument in the context of the wave equation (1.1) for $s < 1/2$.

The well-posedness part of Theorem 1 can be proved as in the works by Ginibre–Velo [16] and Lindblad–Sogge [23], by invoking the Strichartz estimates for the wave equation on a Riemannian manifold due to Kapitanskiĭ [18]. For $s > 1/2$ the well-posedness holds in a stronger sense since the time existence can be chosen the same for all data in a fixed bounded set of \mathcal{H}^s and moreover the flow map is uniformly continuous on bounded sets of \mathcal{H}^s . In the case $s = 1/2$ the situation is more delicate since the existence time depends in a more subtle way on the data. The global well-posedness part of Theorem 1 can be obtained (following ideas by Bourgain [4]) by adapting the proofs of Kenig–Ponce–Vega [19], Gallagher–Planchon [15] and Bahouri–Chemin [1] to the compact setting. We also refer to the works by Roy [24] for further investigations in

the direction of deterministic global well-posedness for (1.1) with rough data. The ill-posedness statement of Theorem 1 is proved in our previous article [10], by using the approaches of Christ–Colliander–Tao [12] and Lebeau [22].

One may however ask whether some sort of well-posedness for (1.1) survives for $s < 1/2$. In [10] we have shown that the answer is positive, at least locally in time, if one accepts to randomize the initial data. Moreover, the method of [10] works for a quite general class of randomizations. As already mentioned, the approach of [11] to get global in time results is restricted only to very particular randomizations. More precisely, it is based on a global control on the flow given by an invariant Gibbs measure (see also [4]). In [10, Remark 1.5] we asked whether the globalization argument can be performed by using other global controls on the flow such as conservation laws. In the present work we give a positive answer to this question. We also refer to [14] for a recent related work.

Let us now describe the initial data randomization we use. We suppose that $M = \mathbb{T}^3$ with the flat metric. Starting from $(u_0, u_1) \in \mathcal{H}^s$ given by their Fourier series

$$u_j(x) = a_j + \sum_{n \in \mathbb{Z}_\star^3} (b_{n,j} \cos(n \cdot x) + c_{n,j} \sin(n \cdot x)), \quad j = 0, 1, \mathbb{Z}_\star^3 = \mathbb{Z}^3 \setminus \{0\}$$

we define u_j^ω by

$$u_j^\omega(x) = \alpha_j(\omega)a_j + \sum_{n \in \mathbb{Z}_\star^3} (\beta_{n,j}(\omega)b_{n,j} \cos(n \cdot x) + \gamma_{n,j}(\omega)c_{n,j} \sin(n \cdot x)), \quad (1.3)$$

where $(\alpha_j(\omega), \beta_{n,j}(\omega), \gamma_{n,j}(\omega))$, $n \in \mathbb{Z}_\star^3$, $j = 0, 1$, is a sequence of real random variables on a probability space (Ω, p, \mathcal{F}) . We assume that the random variables $(\alpha_j, \beta_{n,j}, \gamma_{n,j})_{n \in \mathbb{Z}_\star^3, j=0,1}$ are independent identically distributed real random variables with a joint distribution θ satisfying

$$\exists c > 0, \forall \gamma \in \mathbb{R}, \int_{-\infty}^{\infty} e^{\gamma x} d\theta(x) \leq e^{c\gamma^2} \quad (1.4)$$

(notice that under the assumption (1.4) the random variables are necessarily of mean zero). Typical examples of random variables satisfying (1.4) are the standard Gaussians, i.e. $d\theta(x) = (2\pi)^{-1/2} \exp(-x^2/2)dx$ (with equality in (1.4)) or any family of random variables having compactly supported distribution function θ , e.g. the Bernoulli variables $d\theta(x) = \frac{1}{2}(\delta_{-1} + \delta_1)$. An advantage of the Bernoulli randomization is that it keeps the \mathcal{H}^s norm of the original function. The Gaussian randomization has the advantage to “generate” a dense set in \mathcal{H}^s via the map

$$\omega \in \Omega \mapsto (u_0^\omega, u_1^\omega) \in \mathcal{H}^s \quad (1.5)$$

for many $(u_0, u_1) \in \mathcal{H}^s$ (see Proposition 1.2). Notice finally that we could relax the “identical distribution” assumption provided (1.4) is uniformly satisfied by the family of random variables.

Definition 1.1. For fixed $(u_0, u_1) \in \mathcal{H}^s$, the map (1.5) is a measurable map from (Ω, \mathcal{F}) to \mathcal{H}^0 endowed with the Borel σ -algebra since the partial sums from a Cauchy sequence in $L^2(\Omega; \mathcal{H}^0)$. Thus (1.5) endows the space $\mathcal{H}^0(\mathbb{T}^3)$ with a probability measure which is the direct image of p . Denote this measure by $\mu_{(u_0, u_1)}$. Then

$$\forall A \subset \mathcal{H}^0, \quad \mu_{(u_0, u_1)}(A) = p(\{\omega \in \Omega : (u_0^\omega, u_1^\omega) \in A\}).$$

Denote by \mathcal{M}^s the set of measures obtained by means of this construction:

$$\mathcal{M}^s = \bigcup_{(u_0, u_1) \in \mathcal{H}^s} \{\mu_{(u_0, u_1)}\}.$$

Let us recall some basic properties of these measures.

Proposition 1.2. *For any $(u_0, u_1) \in \mathcal{H}^s$, the measure $\mu_{(u_0, u_1)}$ is supported by \mathcal{H}^s . Furthermore, for any $s' > s$, if $(u_0, u_1) \notin \mathcal{H}^{s'}$, then $\mu_{(u_0, u_1)}(\mathcal{H}^{s'}) = 0$. In other words, the randomization (1.5) does not regularize in the scale of the L^2 -based Sobolev spaces (this fact is obvious for the Bernoulli randomization). Finally, if (u_0, u_1) have all their Fourier coefficients different from zero and if the measure θ charges all open sets of \mathbb{R} then the support of $\mu_{(u_0, u_1)}$ is \mathcal{H}^s (recall that the support of $\mu_{(u_0, u_1)}$ is the complement of the largest open set $U \subset \mathcal{H}^s$ such that $\mu_{(u_0, u_1)}(U) = 0$). In other words, under these assumptions, for any $(w_0, w_1) \in \mathcal{H}^s$ and any $\epsilon > 0$,*

$$\mu_{(u_0, u_1)}(\{(v_0, v_1) \in \mathcal{H}^s : \|(w_0, w_1) - (v_0, v_1)\|_{\mathcal{H}^s} < \epsilon\}) > 0, \quad (1.6)$$

or in yet other words, “the intersection of any set of $\mu_{(u_0, u_1)}$ -measure 1 with any (non-trivial) ball in \mathcal{H}^s is non-empty”.

We refer to [10] for the proof of the first part of this result and will only prove in Appendix B.2 the density of the support (1.6).

As mentioned above, for fixed (u_0, u_1) the measure $\mu_{(u_0, u_1)}$ depends heavily on the choice of the random variables $(\alpha_j(\omega), \beta_{n,j}(\omega), \gamma_{n,j}(\omega))$. On the other hand for a fixed randomization $(\alpha_j(\omega), \beta_{n,j}(\omega), \gamma_{n,j}(\omega))$ the measure $\mu_{(u_0, u_1)}$ depends largely on the choice of (u_0, u_1) . For instance, let us consider a Gaussian randomization, i.e. we suppose that $(\alpha_j(\omega), \beta_{n,j}(\omega), \gamma_{n,j}(\omega))$ are independent centered Gaussian random variables. Then if (u_0, u_1) and $(\tilde{u}_0, \tilde{u}_1)$ are given by the Fourier expansions

$$\begin{aligned} u_j(x) &= a_j + \sum_{n \in \mathbb{Z}_*^3} (b_{n,j} \cos(n \cdot x) + c_{n,j} \sin(n \cdot x)), \quad j = 0, 1, \\ \tilde{u}_j(x) &= \tilde{a}_j + \sum_{n \in \mathbb{Z}_*^3} (\tilde{b}_{n,j} \cos(n \cdot x) + \tilde{c}_{n,j} \sin(n \cdot x)), \quad j = 0, 1, \end{aligned}$$

then, following Kakutani [17], when the random variables are normalized Gaussians, it is possible (see Appendix B.1) to prove that the associated measures $\mu_{(u_0, u_1)}$ and $\mu_{(\tilde{u}_0, \tilde{u}_1)}$ are mutually singular if

$$\sum_n \left(\left| \frac{\tilde{b}_{n,j}}{b_{n,j}} - 1 \right|^2 + \left| \frac{\tilde{c}_{n,j}}{c_{n,j}} - 1 \right|^2 \right) = +\infty. \quad (1.7)$$

In other words, if (1.7) is satisfied then there exists a set A such that $\mu_{(u_0, u_1)}(A) = 1$ and $\mu_{(\tilde{u}_0, \tilde{u}_1)}(A) = 0$. On the other hand, if (1.7) is not satisfied and all the coefficients are non-zero (or vanish pairwise simultaneously) then $\mu_{(u_0, u_1)} \ll \mu_{(\tilde{u}_0, \tilde{u}_1)} \ll \mu_{(u_0, u_1)}$. We refer to the Appendix for more general statements concerning the dependence of $\mu_{(u_0, u_1)}$ on (u_0, u_1) and the randomization α, β, γ .

We can now state our first result.

Theorem 2. *Let $M = \mathbb{T}^3$ with the flat metric and fix $\mu \in \mathcal{M}^s$, $0 \leq s < 1$. Then there exists a full μ -measure set $\Sigma \subset \mathcal{H}^s(\mathbb{T}^3)$ such that for every $(v_0, v_1) \in \Sigma$, there exists a unique global solution v of the non-linear wave equation*

$$(\partial_t^2 - \Delta_{\mathbb{T}^3})v + v^3 = 0, \quad (v(0), \partial_t v(0)) = (v_0, v_1), \quad (1.8)$$

satisfying

$$(v(t), \partial_t v(t)) \in (S(t)(v_0, v_1), \partial_t S(t)(v_0, v_1)) + C(\mathbb{R}_t; H^1(\mathbb{T}^3) \times L^2(\mathbb{T}^3))$$

($S(t)$ denotes the free evolution defined by (2.2) below). Furthermore, if we denote by

$$\Phi(t)(v_0, v_1) \equiv (v(t), \partial_t v(t))$$

the flow thus defined, the set Σ is invariant under the map $\Phi(t)$, that is,

$$\Phi(t)(\Sigma) = \Sigma, \quad \forall t \in \mathbb{R}.$$

Finally, for any $\varepsilon > 0$ there exist $C, \delta > 0$ such that for μ -almost every $(v_0, v_1) \in \mathcal{H}^s(\mathbb{T}^3)$, there exists $M > 0$ such that the global solution to (1.8) previously constructed satisfies

$$v(t) = S(t)\Pi^0(v_0, v_1) + w(t)$$

(Π_0 is the orthogonal projector on constants and $\Pi^0 = \text{Id} - \Pi_0$) with

$$\|(w(t), \partial_t w(t))\|_{\mathcal{H}^1(\mathbb{T}^3)} \leq C(M + |t|)^{(1-s)/s+\varepsilon} \quad \text{if } s > 0, \quad (1.9)$$

$$\|(w(t), \partial_t w(t))\|_{\mathcal{H}^1(\mathbb{T}^3)} \leq C e^{C(t+M)^2} \quad \text{if } s = 0, \quad (1.10)$$

and

$$\mu(\{(v_0, v_1) \in \mathcal{H}^s : M > \lambda\}) \leq C e^{-\lambda^\delta}.$$

Remark 1.3. In the context of the one-dimensional Schrödinger equation (or rather the Wick-ordered renormalization), an analog of the first part of the result (global existence) was previously proved, in the particular case of Gaussian random variables and for very special choices of sequences (b_n, c_n) (i.e. some particular powers of $|n|$), by Colliander–Oh [14] who implemented in this context Bourgain’s High/Low decomposition method [6] which requires a careful control with respect to the truncation parameter. Notice still that in the range studied by Colliander–Oh, the (Wick-ordered) NLS appears not to be ill-posed in the sense of Theorem 1 (see the estimates by Koch–Tataru [20] on the line and the work announced by Christ–Holmer–Tataru [13] on the torus) and it might very well happen that a deterministic approach applies in this context, suppressing the

need for any probabilistic tool. On this particular point, taking benefit from our different model, our approach is much simpler and more direct than Bourgain's [5] (for local existence) and Colliander–Oh's (for global existence) in the sense that the local Cauchy theory does not require delicate $X^{s,b}$ space analysis while the globalization argument does not require any High/Low decomposition and relies on an elementary Gronwall type inequality (see Section 2). However, we do follow Bourgain's High/Low decomposition approach (or rather the converse of his approach [15, 2]) when proving the large time estimates (1.9) in Section 4. Finally, let us notice that the analysis of the critical case, $s = 0$ (including (1.10)) contains a completely different ingredient with respect to these previous works and relies on an implementation in this probabilistic setting of the ideas developed by Yudovich [26] to prove the global existence of solutions of the two-dimensional Euler equations (see Section 5).

Beyond this very simple global existence result, we want to emphasize that what we think is our main contribution in this work is the understanding of the notion of a *probabilistic global well-posedness*, i.e. with a form of continuity of the flow, as opposed to the mere *global existence* results as developed in our previous works [11, 9] or in [5, 14] (notice however that the existence of Gibbs measures [5, 11, 9] in itself is an interesting dynamical property of the flow). Indeed, having established a large time (unique) dynamics on an invariant set of full measure on $\mathcal{H}^s(\mathbb{T}^3)$, the very first crucial and central question to address is precisely the continuity of the flow (a discontinuous dynamics would clearly lead to suspicions about the relevance of any model). Let us recall that for any event B (of non-null probability) the conditioned probability $\mathcal{P}(\cdot|B)$ is the natural probability measure supported by B , defined by

$$\mathcal{P}(A|B) = \frac{\mathcal{P}(A \cap B)}{\mathcal{P}(B)}.$$

Notice (see the last part in Theorem 3) that the sequences constructed following the approach by Lebeau and Christ–Colliander–Tao give an obstruction to the (deterministic) continuity of our flow. However, we are able to prove that it is still continuous *in probability* and consequently the supercritical Cauchy problem (1.1) is globally well-posed in the following *Hadamard-probabilistic* sense:

Theorem 3. *Fix $s \in (0, 1)$, let $A > 0$, let $B_A \equiv \{V \in \mathcal{H}^s : \|V\|_{\mathcal{H}^s} \leq A\}$ be the closed ball of radius A centered at the origin of \mathcal{H}^s and let $T > 0$. Let $\mu \in \mathcal{M}^s$ and suppose that θ (the law of our random variables) is symmetric. Let $\Phi(t)$ be the flow of the cubic wave equations defined μ -almost everywhere in Theorem 2. Then for $\varepsilon, \eta > 0$, we have the bound*

$$\mu \otimes \mu \left(\left\{ (V, V') \in \mathcal{H}^s \times \mathcal{H}^s : \|\Phi(t)(V) - \Phi(t)(V')\|_{X_T} > \varepsilon \mid \|V - V'\|_{\mathcal{H}^s} < \eta \text{ and } (V, V') \in B_A \times B_A \right\} \right) \leq g(\varepsilon, \eta), \quad (1.11)$$

where $X_T \equiv (C([0, T]; \mathcal{H}^s) \cap L^4([0, T] \times \mathbb{T}^3)) \times C([0, T]; H^{s-1})$ and $g(\varepsilon, \eta)$ is such that

$$\lim_{\eta \rightarrow 0} g(\varepsilon, \eta) = 0, \quad \forall \varepsilon > 0.$$

Moreover, if in addition we assume that the support of μ is the whole \mathcal{H}^s (which is true iff in the definition of the measure μ , we have $a_i, b_{n,j}, c_{n,j} \neq 0$ for all $n \in \mathbb{Z}^d$ and the support of the distribution function of the random variables is \mathbb{R}), then there exists $\varepsilon > 0$ such that for every $\eta > 0$ the left-hand side in (1.11) is positive.

In other words, as soon as $\eta \ll \varepsilon$, among the initial data which are η -close to each other, the probability of finding two for which the corresponding solutions to (1.1) do not remain ε -close to each other, is very small! The last part of the statement is saying that the deterministic version of the uniform continuity property (1.11) does not hold and somehow that one cannot get rid of a probabilistic approach on this problem. A crucial element in the proof is the ill-posedness result displayed in Theorem 1. It is likely that Theorem 3 also holds for $s = 0$, modulo some additional technicalities.

In a forthcoming work, we show that similar results can be obtained for general manifolds by suitably modifying the randomization.

As mentioned in [10] it would be interesting to develop similar results in the case of the non-linear Schrödinger equation (NLS). In this case there are at least two difficulties compared to the wave equation. The first one is that the smoothing in the non-linearity is no longer present in the case of NLS. The second one is that the deterministic Cauchy theory in the case of NLS, posed on a compact domain, is much more intricate compared to the non-linear wave equation or the NLS in the Euclidean space (see e.g. [3, 8]). One can however show that in some cases one may at least control almost surely the first iteration at a supercritical regularity (see the appendix of [25]). Finally, let us mention that we obtained with Thomann a first step toward similar results for the non-linear Schrödinger equation in [9].

The remaining part of this paper is organized as follows. In the following section, we give the global existence part of the proof of Theorem 2 for $s > 0$. Next, we construct an invariant set of full measure. Section 4 is devoted to the possible growth of Sobolev norms for $s > 0$. We then consider the case $s = 0$ in Section 5. Section 6 is devoted to the proof of Theorem 3. Finally in an appendix, we collect the results on large deviation estimates for random series used in the previous sections, including conditioned versions which do not seem to appear in the literature. We also give a criterion for the orthogonality of two measures of \mathcal{M}^s .

Notation. A probability measure θ on \mathbb{R} is called *symmetric* if $\int_{\mathbb{R}} f(x) d\theta(x) = \int_{\mathbb{R}} f(-x) d\theta(x)$ for every $f \in L^1(d\theta)$. A real random variable is called symmetric if its distribution is a symmetric measure on \mathbb{R} . Finally, we will use the bracket $\langle x \rangle = (1 + |x|^2)^{1/2}$.

2. Almost sure global well-posedness for $s > 0$

Let us first recall the following local existence result.

Proposition 2.1. *Consider the problem*

$$(\partial_t^2 - \Delta_{\mathbb{T}^3})v + (f + v)^3 = 0. \quad (2.1)$$

There exists a constant C such that for every time interval $I = [a, b]$ of size 1, every $\Lambda \geq 1$, and every $(v_0, v_1, f) \in H^1 \times L^2 \times L^3(I; L^6)$ satisfying $\|v_0\|_{H^1} + \|v_1\|_{L^2} + \|f\|_{L^3(I; L^6)}^3 \leq \Lambda$ there exists a unique solution of (2.1) on the time interval $[a, a + C^{-1}\Lambda^{-2}]$ with initial data

$$v(a, x) = v_0(x), \quad \partial_t v(a, x) = v_1(x).$$

Moreover the solution satisfies $\|(v, \partial_t v)\|_{L^\infty([a, a + C\Lambda^{-2}]; H^1 \times L^2)} \leq C\Lambda$, $(v, \partial_t v)$ is unique in the class $L^\infty([a, a + C\Lambda^{-2}]; H^1 \times L^2)$ and the dependence in time is continuous.

Proof. By translation invariance in time, we can suppose that $I = [0, 1]$. Define the free evolution $S(t)$ by

$$S(t)(v_0, v_1) \equiv \cos(t\sqrt{-\Delta})(v_0) + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}(v_1), \quad (2.2)$$

with the natural convention concerning the zero Fourier mode. Then we can rewrite the problem as

$$v(t) = S(t)(v_0, v_1) - \int_0^t \frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} ((f(\tau) + v(\tau))^3) d\tau. \quad (2.3)$$

Set

$$\Phi_{v_0, v_1, f}(v) \equiv S(t)(v_0, v_1) - \int_0^t \frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} ((f(\tau) + v(\tau))^3) d\tau.$$

Then for $T \in (0, 1]$, using the Sobolev embedding $H^1(\mathbb{T}^3) \subset L^6(\mathbb{T}^3)$, we get

$$\begin{aligned} \|\Phi_{v_0, v_1, f}(v)\|_{L^\infty([0, T], H^1)} &\leq C \left(\|v_0\|_{H^1} + \|v_1\|_{L^2} + T \sup_{\tau \in [0, T]} \|f(\tau) + v(\tau)\|_{L^6}^3 \right) \\ &\leq C \left(\|v_0\|_{H^1} + \|v_1\|_{L^2} + \sup_{\tau \in [0, T]} \|f(\tau)\|_{L^6}^3 \right) + CT \|v\|_{L^\infty([0, T], H^1)}^3. \end{aligned}$$

It is now clear that for $T \approx \Lambda^{-2}$ the map $\Phi_{v_0, v_1, f}$ sends the ball $\{v : \|v\|_{L^\infty([0, T], H^1)} \leq C\Lambda\}$ onto itself. Moreover by a similar argument, this map is a contraction on the same ball. Thus we obtain the existence part and the bound on v in H^1 . The estimate of $\|\partial_t v\|_{L^2}$ follows by differentiating in t the Duhamel formula (2.3). This completes the proof of Proposition 2.1. \square

We can now deduce the global existence and uniqueness part in Theorem 2 in the case $s > 0$.

Proposition 2.2. *Assume that $s > 0$ and fix $\mu \in \mathcal{M}^s$. Then, for μ -almost every $(v_0, v_1) \in \mathcal{H}^s(\mathbb{T}^3)$, there exists a unique global solution*

$$(v(t), \partial_t v(t)) \in (S(t)(v_0, v_1), \partial_t S(t)(v_0, v_1)) + C(\mathbb{R}; H^1(\mathbb{T}^3) \times L^2(\mathbb{T}^3))$$

of the non-linear wave equation

$$(\partial_t^2 - \Delta_{\mathbb{T}^3})v + v^3 = 0$$

with initial data

$$v(0, x) = v_0(x), \quad \partial_t v(0, x) = v_1(x).$$

Proof. We look for v in the form $v(t) = S(t)(v_0, v_1) + w(t)$. Then w solves

$$(\partial_t^2 - \Delta_{\mathbb{T}^3})w + (S(t)(v_0, v_1) + w)^3 = 0, \quad w|_{t=0} = 0, \quad \partial_t w|_{t=0} = 0. \quad (2.4)$$

From Corollary A.4, if $\delta > 1 + 1/p$, we know that μ -almost surely

$$\|\langle t \rangle^{-\delta} S(t)(v_0, v_1)\|_{L^p(\mathbb{R}_t; W^{s,p}(\mathbb{T}^3))} < \infty.$$

Taking p large enough so that $3/p < s$ and consequently $W^{s,p}(\mathbb{T}^3) \subset L^\infty(\mathbb{T}^3)$, we deduce that μ -almost surely,

$$g(t) = \|S(t)(v_0, v_1)\|_{L^6(\mathbb{T}^3)}^3 \in L_{\text{loc}}^1(\mathbb{R}_t), \quad f(t) = \|S(t)(v_0, v_1)\|_{L^\infty(\mathbb{T}^3)} \in L_{\text{loc}}^1(\mathbb{R}_t). \quad (2.5)$$

The local existence now follows from Proposition 2.1 and the first estimate in (2.5). We also deduce from Proposition 2.1 that as long as the $H^1 \times L^2$ norm of $(w, \partial_t w)$ remains bounded, the solution w of (2.4) exists. Set

$$\mathcal{E}(w(t)) = \frac{1}{2} \int_{\mathbb{T}^3} \left((\partial_t w)^2 + |\nabla_x w|^2 + \frac{1}{2} w^4 \right) dx.$$

Using the equation solved by w , we now compute

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(w(t)) &= \int_{\mathbb{T}^3} (\partial_t w \partial_t^2 w + \nabla_x \partial_t w \cdot \nabla_x w + \partial_t w w^3) dx \\ &= \int_{\mathbb{T}^3} \partial_t w (\partial_t^2 w - \Delta w + w^3) dx \\ &= \int_{\mathbb{T}^3} \partial_t w (w^3 - (S(t)(v_0, v_1) + w)^3) dx. \end{aligned}$$

Now, using the Cauchy–Schwarz and Hölder inequalities, we can write

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(w(t)) &\leq C(\mathcal{E}(w(t)))^{1/2} \|w^3 - (S(t)(v_0, v_1) + w)^3\|_{L^2(\mathbb{T}^3)} \\ &\leq C(\mathcal{E}(w(t)))^{1/2} (\|S(t)(v_0, v_1)\|_{L^6(\mathbb{T}^3)}^3 + \|S(t)(v_0, v_1)\|_{L^\infty(\mathbb{T}^3)} \|w^2\|_{L^2(\mathbb{T}^3)}) \\ &\leq C(\mathcal{E}(w(t)))^{1/2} (g(t) + f(t)(\mathcal{E}(w(t)))^{1/2}) \end{aligned}$$

and consequently, according to the Gronwall inequality and (2.5), w exists globally in time. This completes the proof of Proposition 1. \square

3. Construction of an invariant set, $s > 0$

The construction of the previous section yields the global existence on a set of full μ -measure but it does not exclude the possibility to have a dynamics sending for some $t \neq 0$ the set of full measure where the global existence holds to a set of small measure. Notice that in a similar discussion in [11] the set of full measure where the global existence holds is invariant under the dynamics by construction. Our purpose in this section is to establish a global dynamics on an invariant set of full measure in the context of the argument of the previous section.

Define

$$\Theta \equiv \{(v_0, v_1) \in \mathcal{H}^s : \|S(t)(v_0, v_1)\|_{L^6(\mathbb{T}^3)}^3 \in L^1_{\text{loc}}(\mathbb{R}_t), \\ \|S(t)(v_0, v_1)\|_{L^\infty(\mathbb{T}^3)} \in L^1_{\text{loc}}(\mathbb{R}_t)\}$$

and $\Sigma \equiv \Theta + \mathcal{H}^1$. Then Σ is of full μ -measure for every $\mu \in \mathcal{H}^s$, since so is Θ . We have the following proposition.

Proposition 3.1. *Assume that $s > 0$ and fix $\mu \in \mathcal{M}^s$. Then, for every $(v_0, v_1) \in \Sigma$, there exists a unique global solution*

$$(v(t), \partial_t v(t)) \in (S(t)(v_0, v_1), \partial_t S(t)(v_0, v_1)) + C(\mathbb{R}; H^1(\mathbb{T}^3) \times L^2(\mathbb{T}^3))$$

of the non-linear wave equation

$$(\partial_t^2 - \Delta_{\mathbb{T}^3})v + v^3 = 0, \quad (v(0, x), \partial_t v(0, x)) = (v_0(x), v_1(x)). \quad (3.1)$$

Moreover, $(v(t), \partial_t v(t)) \in \Sigma$ for every $t \in \mathbb{R}$, and thus by time reversibility Σ is invariant under the flow of (3.1).

Proof. By assumption, we can write $(v_0, v_1) = (\tilde{v}_0, \tilde{v}_1) + (w_0, w_1)$ with $(\tilde{v}_0, \tilde{v}_1) \in \Theta$ and $(w_0, w_1) \in \mathcal{H}^1$. We look for v in the form $v(t) = S(t)(\tilde{v}_0, \tilde{v}_1) + w(t)$. Then w solves

$$(\partial_t^2 - \Delta_{\mathbb{T}^3})w + (S(t)(\tilde{v}_0, \tilde{v}_1) + w)^3 = 0, \quad w|_{t=0} = w_0, \quad \partial_t w|_{t=0} = w_1.$$

Now, exactly as in the proof of Proposition 2.2, we obtain

$$\frac{d}{dt} \mathcal{E}(w(t)) \leq C(\mathcal{E}(w(t)))^{1/2} (g(t) + f(t)(\mathcal{E}(w(t)))^{1/2}),$$

where

$$g(t) = \|S(t)(\tilde{v}_0, \tilde{v}_1)\|_{L^6(\mathbb{T}^3)}^3, \quad f(t) = \|S(t)(\tilde{v}_0, \tilde{v}_1)\|_{L^\infty(\mathbb{T}^3)}.$$

Therefore thanks to the Gronwall lemma, using that $\mathcal{E}(w(0))$ is well-defined, we obtain the global existence for w . Thus the solution of (3.1) can be written as

$$v(t) = S(t)(\tilde{v}_0, \tilde{v}_1) + w(t), \quad (w, \partial_t w) \in C(\mathbb{R}; \mathcal{H}^1).$$

Coming back to the definition of Θ , we observe that

$$S(t)(\Theta) = \Theta.$$

Thus $(v(t), \partial_t v(t)) \in \Sigma$. This completes the proof of Proposition 3.1. \square

4. Bounds on the possible growth of the Sobolev norms, the case $s > 0$

In this section, we are going to follow the high-low decomposition method of Bourgain [6], or more precisely the reverse of Bourgain's method, as developed for instance in the work by Gallagher and Planchon [15] (see also [19]) or Bona and the second author [2] and refine the global well-posedness results obtained in the previous sections, to prove (1.9). Notice that this strategy has recently been used by Colliander and Oh [14] in the context of the well-posedness of the cubic one-dimensional non-linear Schrödinger equation below L^2 . In the context of randomly forced parabolic-type equations, such sub-linear estimates appear rather naturally (see the work by Kuksin and Shirikyan [21]).

Let us introduce the Dirichlet projectors to low/high frequencies. If a function u on the torus is given by its Fourier series

$$u(x) = a + \sum_{n \in \mathbb{Z}_*^2} (b_n \cos(n \cdot x) + c_n \sin(n \cdot x)),$$

for $N \geq 0$ an integer we set

$$\Pi_N(u) \equiv a + \sum_{|n| \leq N} (b_n \cos(n \cdot x) + c_n \sin(n \cdot x)), \quad \Pi^N(u) \equiv (1 - \Pi_N)(u). \quad (4.1)$$

We also set $\Pi_0(u) \equiv a$. The goal of this section is to prove the following statement.

Proposition 4.1. *Let $1 > s > 0$ and $\mu \in \mathcal{M}^s$. Consider the flow of (3.1) established in the previous section. Then for any $\varepsilon > 0$ there exist $C, \delta > 0$ such that for every $(v_0, v_1) \in \Sigma$, there exists $M > 0$ such that the global solution to (3.1) constructed in the previous section satisfies*

$$v(t) = S(t)\Pi^0(v_0, v_1) + w(t), \quad \|(w(t), \partial_t w(t))\|_{\mathcal{H}^1(\mathbb{T}^3)} \leq C(M + |t|)^{(1-s)/s+\varepsilon},$$

with $\mu(M > \lambda) \leq C e^{-\lambda^\delta}$.

Proof. We only give the proof for positive times, the analysis for negative times being analogous. For $\varepsilon > 0$, $\delta > 1/2$ and $\tilde{\delta} > 1/3$, we introduce the sets

$$\begin{aligned} F_N &= \{(v_0, v_1) \in \Sigma : \|\Pi_N(v_0, v_1)\|_{\mathcal{H}^1} \leq N^{1-s+\varepsilon}\}, \\ G_N &= \{(v_0, v_1) \in \Sigma : \|\Pi_N(v_0)\|_{L^4(\mathbb{T}^3)} \leq N^\varepsilon\}, \\ H_N &= \{(v_0, v_1) \in \Sigma : \|\langle t \rangle^{-\delta} S(t)(\Pi^N(v_0, v_1))\|_{L^2(\mathbb{R}_t; L^\infty(\mathbb{T}^3))} \leq N^{\varepsilon-s}\}, \\ K_N &= \{(v_0, v_1) \in \Sigma : \|\langle t \rangle^{-\tilde{\delta}} \tilde{S}(t)(\Pi^N(v_0, v_1))\|_{L^3(\mathbb{R}_t; L^6(\mathbb{T}^3))} \leq N^{\varepsilon-s}\}. \end{aligned}$$

Lemma 4.2. *Let $\delta > 1$ and $\tilde{\delta} > 1/3$, There exist $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$, there exists $C, c > 0$ such that for every $N \geq 1$,*

$$\mu(F_N^c) \leq C e^{-cN^{2\varepsilon}}, \quad \mu(G_N^c) \leq C e^{-cN^{2\varepsilon}}, \quad \mu(H_N^c) \leq C e^{-cN^{2\varepsilon}}, \quad \mu(K_N^c) \leq C e^{-cN^{2\varepsilon}}.$$

Proof. For $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3)$, we have

$$\|\Pi_N(u_0, u_1)\|_{\mathcal{H}^1(\mathbb{T}^3)} \leq CN^{1-s} \|(u_0, u_1)\|_{\mathcal{H}^s}.$$

Therefore according to (A.5),

$$\mu(F_N^c) \leq Ce^{-cN^{2\varepsilon}}.$$

Next, using (A.4), we infer that

$$\mu(G_N^c) \leq Ce^{-cN^{2\varepsilon}}.$$

On the other hand, we have

$$\|\Pi^N(u_0, u_1)\|_{\mathcal{H}^0(\mathbb{T}^3)} \leq CN^{-s} \|(u_0, u_1)\|_{\mathcal{H}^s(\mathbb{T}^3)}.$$

Therefore using Remark A.3 and Corollary A.5, we deduce that

$$\mu(K_N^c) \leq Ce^{-cN^{2\varepsilon}}, \quad \mu(H_N^c) \leq Ce^{-cN^{2\varepsilon}}.$$

This completes the proof of Lemma 4.2. \square

Let us now define for $N \geq 1$ an integer,

$$E_N \equiv F_N \cap G_N \cap H_N \cap K_N.$$

According to Lemma 4.2, we have

$$\mu(E_N^c) \leq Ce^{-cN^\kappa}, \quad \kappa > 0.$$

Fix $\varepsilon_1 > 0$. Then we fix $\varepsilon > 0$ small enough such that

$$\frac{1-s+\varepsilon}{s-2\varepsilon} \leq \frac{1-s}{s} + \varepsilon_1, \quad \varepsilon < \frac{s}{2}. \quad (4.2)$$

Let us finally fix $\delta > 1/2$ and $\tilde{\delta} > 1/3$ such that

$$(\delta - 1/2)s < 2\delta\varepsilon, \quad \tilde{\delta} < 1. \quad (4.3)$$

We have the following statement.

Lemma 4.3. *For every $c > 0$ there exists $C > 0$ such that for every $t \geq 1$ and every integer $N \geq 1$ such that $t \leq cN^{s-2\varepsilon}$, and every $(v_0, v_1) \in E_N$, the solution of (3.1) with data (v_0, v_1) satisfies*

$$\|v(t) - S(t)\Pi^0(v_0, v_1)\|_{\mathcal{H}^1(\mathbb{T}^3)} \leq CN^{1-s+\varepsilon}.$$

In particular, thanks to (4.2), if $t \approx N^{s-2\varepsilon}$ then

$$\|v(t) - S(t)\Pi^0(v_0, v_1)\|_{\mathcal{H}^1(\mathbb{T}^3)} \lesssim t^{(1-s)/s+\varepsilon_1}.$$

Proof. For $(v_0, v_1) \in E_N$ and we decompose the solution of (3.1) with data (v_0, v_1) as

$$v(t) = S(t)\Pi^N(v_0, v_1) + w_N,$$

where w_N solves the problem

$$(\partial_t^2 - \Delta_{\mathbb{T}^3})w_N + (w_N + S(t)\Pi^N(v_0, v_1))^3 = 0, \quad (w(0), \partial_t w(0)) = \Pi_N(v_0, v_1). \quad (4.4)$$

Using the energy estimates applied in the previous sections, we get the bound

$$\frac{d}{dt}\mathcal{E}(w_N(t)) \leq C(\mathcal{E}(w_N(t)))^{1/2}(g_N(t) + f_N(t)(\mathcal{E}(w_N(t)))^{1/2}), \quad (4.5)$$

where

$$g_N(t) = \|S(t)\Pi^N(v_0, v_1)\|_{L^6(\mathbb{T}^3)}^3, \quad f_N(t) = \|S(t)\Pi^N(v_0, v_1)\|_{L^\infty(\mathbb{T}^3)}.$$

Integrating (4.5), we get

$$(\mathcal{E}(w_N(t)))^{1/2} \leq C e^{C \int_0^t f_N(\tau) d\tau} \left((\mathcal{E}(w_N(0)))^{1/2} + \int_0^t g_N(\tau) d\tau \right). \quad (4.6)$$

We now observe that for $(v_0, v_1) \in E_N$,

$$\left| \int_0^t g_N(\tau) d\tau \right| \leq CN^{3(-s+\varepsilon)} \langle t \rangle^{3\tilde{\delta}} \leq CN^{3(-s+\varepsilon)+3\tilde{\delta}(s-2\varepsilon)} \leq C,$$

provided

$$-s + \varepsilon + \tilde{\delta}(s - 2\varepsilon) \leq 0.$$

The last condition can be readily satisfied according to (4.3).

Next, we have (using the Cauchy–Schwarz inequality in time) that for $(v_0, v_1) \in E_N$,

$$\left| \int_0^t f_N(\tau) d\tau \right| \leq \|\langle \tau \rangle^{-\delta} f_N\|_{L^2(\mathbb{R})} \langle t \rangle^{\delta+1/2} \leq CN^{-s+\varepsilon} \langle t \rangle^{\delta+1/2} \leq CN^{-s+\varepsilon+(\delta+1/2)(s-2\varepsilon)} \leq C,$$

provided $-s + \varepsilon + (\delta + 1/2)(s - 2\varepsilon) \leq 0$, a condition which is satisfied thanks to (4.3).

For $(v_0, v_1) \in E_N$, we have

$$(\mathcal{E}(w_N(0)))^{1/2} \leq C(\|\Pi_N(v_0, v_1)\|_{\mathcal{H}^1} + \|\Pi_N(v_0)\|_{L^4}^2) \leq CN^{1-s+\varepsilon},$$

and coming back to (4.6), we get

$$(\mathcal{E}(w_N(t)))^{1/2} \leq CN^{1-s+\varepsilon}.$$

Recall that

$$v(t) = w_N(t) + S(t)\Pi^N(v_0, v_1) = S(t)\Pi^0(v_0, v_1) + w_N(t) - S(t)\Pi_N\Pi^0(v_0, v_1).$$

We find that for a solution to the linear wave equation, the linear energy

$$\|\nabla_x u\|_{L^2(\mathbb{T}^3)}^2 + \|\partial_t u\|_{L^2(\mathbb{T}^3)}^2$$

is independent of time, and if $(u, \partial_t u)$ is orthogonal to constants ($(u, \partial_t u) = \Pi^0(u, \partial_t u)$), then this energy controls the $\mathcal{H}^1(\mathbb{T}^3)$ norm; we deduce for $(v_0, v_1) \in E_N \subset F_N$ that

$$\|S(t)\Pi_N\Pi^0(v_0, v_1)\|_{\mathcal{H}^1(\mathbb{T}^3)} \leq CN^{1-s+\varepsilon}$$

and therefore

$$\|v(t) - S(t)\Pi^0(v_0, v_1)\|_{\mathcal{H}^1(\mathbb{T}^3)} \leq CN^{1-s+\varepsilon}.$$

This completes the proof of Lemma 4.3. \square

Next we set

$$E^N = \bigcap_{M \geq N} E_M,$$

where the intersection is taken over the dyadic values of M , i.e. $M = 2^j$ with j an integer. Thus $\mu(E^N)$ tends to 1 as $N \rightarrow \infty$. Using Lemma 4.3, we conclude that there exists $C > 0$ such that for every $t \geq 1$, every N , and every $(v_0, v_1) \in E^N$,

$$\|v(t) - S(t)\Pi^0(v_0, v_1)\|_{\mathcal{H}^1(\mathbb{T}^3)} \leq C(N^{1-s+\varepsilon} + t^{(1-s)/s+\varepsilon_1}).$$

Finally, we set

$$E = \bigcup_{N=1}^{\infty} E^N.$$

We have thus shown μ -almost sure bounds on the possible growths of the Sobolev norms of the solutions established in the previous section for data in E , which is of full μ -measure. This completes the proof of Proposition 4.1. \square

5. Global existence, the case $s = 0$

In the previous sections, to prove global existence for $s > 0$ we used at some points the Sobolev embeddings $W^{\varepsilon,p} \subset L^\infty$ for $\varepsilon > 0$ and p large enough. This argument implied an ε loss in the estimates. Here, such a loss is forbidden and we are going to follow instead a probabilistic version of the strategy initiated by Yudovich [26] to prove global existence for the two-dimensional Euler equation (see also Brezis–Gallouët [7] and Burq–Gérard–Tzvetkov [8] for similar ideas in the context of dispersive equations). Our goal in this section is to prove the following statement.

Proposition 5.1. *Let $\mu \in \mathcal{M}^0$. Then there exists $C > 0$ such that for μ -almost every $(v_0, v_1) \in L^2(\mathbb{T}^3) \times H^{-1}(\mathbb{T}^3)$, there exists a unique global solution*

$$v(t) \in S(t)\Pi^0(v_0, v_1) + C(\mathbb{R}; H^1(\mathbb{T}^3) \times L^2(\mathbb{T}^3))$$

of the non-linear wave equation (3.1). Moreover there exists $M = M(v_0, v_1) > 0$ such that the solution satisfies

$$v(t) = S(t)\Pi^0(v_0, v_1) + w(t), \quad \|(w(t), \partial_t w(t))\|_{\mathcal{H}^1(\mathbb{T}^3)} \leq Ce^{C(t+M)^2},$$

with $\mu(M > \lambda) \leq Ce^{-\lambda^\delta}$ for some $\delta > 0$.

Proof. Again, we shall only consider positive times. Let us notice that according to Proposition A.1 (with $p_1 = p_2 = j$, $\delta_j = 2/j$), there exists $C > 0$ such that for any $j \geq 2$, we have

$$\begin{aligned} \mu(\{(v_0, v_1) \in \mathcal{H}^0 : \|\langle t \rangle^{-\delta_j} \Pi^0 S(t)(v_0, v_1)\|_{L^j(\mathbb{R}_t \times \mathbb{T}^3)} > \lambda\}) &\leq \left(C \frac{\sqrt{j}(\delta_j j - 1)^{-1/j}}{\lambda} \right)^j \\ &= \left(\frac{C\sqrt{j}}{\lambda} \right)^j. \end{aligned}$$

On the other hand, according to Corollary A.2 (with $p_1 = 3$, $p_2 = 6$, $\delta = 1 > 1/3$),

$$\mu(\{(v_0, v_1) \in \mathcal{H}^0 : \|\langle t \rangle^{-1} \Pi^0 S(t)(v_0, v_1)\|_{L^3(\mathbb{R}_t; L^6(\mathbb{T}^3))} > \lambda\}) \leq C e^{-c\lambda^2}.$$

For any integer $k \geq 1$, we set (notice that for the global existence part in the case $s = 0$ in Theorem 2, we only use the case $k = 1$, the other cases will only be used to prove the invariance of our set under the flow)

$$F_{j,k} = \{(v_0, v_1) \in \mathcal{H}^0 : \|\langle t \rangle^{-\delta_j} \Pi^0 S(t)(v_0, v_1)\|_{L^j(\mathbb{R}_t \times \mathbb{T}^3)} \leq 2kC\sqrt{j}\}$$

and

$$\begin{aligned} G_{j,k} &= \{(v_0, v_1) \in \mathcal{H}^0 : \\ &\|\langle t \rangle^{-1} \Pi^0 S(t)(v_0, v_1)\|_{L^3(\mathbb{R}_t; L^6(\mathbb{T}^3))}^3 + \|\Pi_0(v_0, v_1)\|_{\mathcal{H}^1} + \|\Pi_0 v_0\|_{L^4(\mathbb{T}^3)}^2 \leq kj\}. \end{aligned}$$

Therefore

$$\mu(F_{j,k}^c) + \mu(G_{j,k}^c) \lesssim e^{-cj^\delta}, \quad \delta > 0. \quad (5.1)$$

Next we set $E_{j,k} = G_{j,k} \cap F_{j,k}$. For any $(v_0, v_1) \in E_{j,k}$, we write

$$v(t) = S(t)\Pi^0(v_0, v_1) + w(t), \quad (w(0), \partial_t w(0)) = \Pi_0(v_0, v_1).$$

Using the energy estimate, already applied several times in the previous sections, we get

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(w(t)) &\leq C(\mathcal{E}(w(t)))^{1/2} \|w^3 - (\Pi^0 S(t)(v_0, v_1) + w)^3\|_{L^2(\mathbb{T}^3)} \\ &\leq C(\mathcal{E}(w(t)))^{1/2} (\|\Pi^0 S(t)(v_0, v_1)\|_{L^6(\mathbb{T}^3)}^3 + \|\Pi^0 S(t)(v_0, v_1)\|_{L^j(\mathbb{T}^3)} \|w^2\|_{L^{\tilde{j}}(\mathbb{T}^3)}), \end{aligned}$$

provided $1/j + 1/\tilde{j} = 1/2$. On the other hand

$$\|v\|_{L^{2\tilde{j}}} \leq \|v\|_{L^4}^{1-\theta} \|v\|_{L^6}^\theta, \quad \frac{1-\theta}{4} + \frac{\theta}{6} = \frac{1}{2\tilde{j}} \Rightarrow \theta = \frac{6}{j},$$

and thus we obtain

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(w(t)) &\leq C(\mathcal{E}(w(t)))^{1/2} \\ &\times (\|\Pi^0 S(t)(v_0, v_1)\|_{L^6(\mathbb{T}^3)}^3 + \|\Pi^0 S(t)(v_0, v_1)\|_{L^j(\mathbb{T}^3)} (\mathcal{E}(w(t)))^{\frac{1}{2}(1+6/j)}). \end{aligned}$$

Consequently, as long as $(\mathcal{E}(w(t)))^{1/2}$ remains smaller than 2^j , we have $(\mathcal{E}(w(t)))^{3/j} \leq C$ and consequently by the Gronwall inequality, for $(v_0, v_1) \in E_{j,k}$ and $t \lesssim \sqrt{j}$,

$$\begin{aligned} & (\mathcal{E}(w(t)))^{1/2} \\ & \leq C e^{\int_0^t \|\Pi^0 S(\tau)(v_0, v_1)\|_{L^j(\mathbb{T}^3)} d\tau} \left(\int_0^t \|\Pi^0 S(\tau)(v_0, v_1)\|_{L^6(\mathbb{T}^3)}^3 d\tau + (\mathcal{E}(w(0)))^{1/2} \right) \\ & \leq C j^{3/2} e^{c \|(1-\Pi_0)S(t)(v_0, v_1)\|_{L^j(0,t) \times \mathbb{T}^3} \times t} t^{1-1/j}. \end{aligned}$$

Next, we can write, for $t \lesssim \sqrt{j}$ and $(v_0, v_1) \in E_{j,k}$,

$$\|(1 - \Pi_0)S(t)(v_0, v_1)\|_{L^j(0,t) \times \mathbb{T}^3} \leq C j^{\delta_j/2} \| \langle t \rangle^{-\delta_j} (1 - \Pi_0)S(t)(v_0, v_1) \|_{L^j(\mathbb{R} \times \mathbb{T}^3)}$$

and consequently there exists a small $\alpha > 0$ and a large j_0 depending only on k such that for $j \geq j_0$, $t \leq \alpha\sqrt{j}$ and $(v_0, v_1) \in E_{j,k}$, as long as $(\mathcal{E}(w(t)))^{1/2}$ remains bounded by 2^j and $t \leq \alpha\sqrt{j}$, we have

$$(\mathcal{E}(w(t)))^{1/2} \leq C j^{3/2} e^{2C\sqrt{j} j^{\delta_j/2} \times |t|^{1-1/j}} \leq C j^{3/2} e^{2C'\alpha j} \leq 2^{j/2}.$$

From the usual bootstrap argument, we deduce that the bootstrap assumption $(\mathcal{E}(w(t)))^{1/2} \leq 2^j$ remains satisfied for $t \leq \alpha\sqrt{j}$. Thus,

$$\forall k \geq 1, \exists \alpha > 0, \exists j_0 > 0, \forall j \geq j_0, \forall (v_0, v_1) \in E_{j,k}, \quad (\mathcal{E}(w(t)))^{1/2} \leq 2^j, \quad t \leq \alpha\sqrt{j}.$$

Next, we set

$$E_k^j = \bigcap_{N \geq \max(j, j_0)} E_{N,k}.$$

Thus, according to (5.1),

$$\mu((E_k^j)^c) \leq \sum_{n \geq \max(j, j_0)} C e^{-cn^\delta} \leq C e^{-j^\delta}$$

and consequently the set

$$\Sigma_k = \bigcup_{j=1}^{\infty} E_k^j$$

is of full μ -measure.

Let $(v_0, v_1) \in E_k^j$. The solution is already defined up to time $\alpha \max(\sqrt{j_0}, \sqrt{j})$. For $t \geq \alpha \max(\sqrt{j_0}, \sqrt{j})$, there exists a dyadic $N \geq \max(j_0, j)$ such that $\alpha\sqrt{N} \leq t \leq \alpha\sqrt{2N}$ and we deduce that the solution is defined up to time $\alpha\sqrt{2N}$ with bounds

$$(\mathcal{E}(w(t)))^{1/2} \leq 2^{2N} \leq C e^{c(1+t^2)}.$$

Hence, the solution is globally defined. The above discussion also gives the claimed bound on the possible growth of the Sobolev norms. Namely, for every $t \geq 1$, and every $(v_0, v_1) \in E_k^j$,

$$(\mathcal{E}(w(t)))^{1/2} \leq C e^{c(j+j_0+t^2)}.$$

Thus we have the needed bound for every $(v_0, v_1) \in \Sigma_k$. This ends the proof of Proposition 5.1. \square

Let us now define a set of full measure invariant under the dynamics established in Proposition 5.1. Set

$$\Sigma = \bigcap_{k=1}^{\infty} \Sigma_k.$$

Then Σ is of full measure. Coming back to the definition of Σ_k , we infer that for every t there exists k_0 such that $S(t)(\Sigma_k) \subset \Sigma_{k+k_0}$ for every k , which in turn implies that $S(t)(\Sigma) = \Sigma$. It remains to observe that the argument of the proof of Proposition 5.1 implies that the set $\Sigma + \mathcal{H}^1$ is invariant under the dynamics (and of full measure). This completes the proof of Theorem 2.

6. Probabilistic continuity of the flow

The purpose of this section is to prove Theorem 3. According to Proposition A.7, for any $2 \leq p_1 < \infty$, $2 \leq p_2 \leq \infty$, $\delta > 1 + 1/p_1$, $\eta \in (0, 1)$, $\alpha \in (0, 1)$ and $\beta > 0$,

$$\begin{aligned} \mu \otimes \mu \{ (V_0, V_1) \in \mathcal{H}^s \times \mathcal{H}^s : \langle t \rangle^{-\delta} S(t)(V_0 - V_1) \|_{L^{p_1}(\mathbb{R}_t; L^{p_2}(\mathbb{T}^3))} > \eta^{1-\alpha} \\ \text{or } \langle t \rangle^{-\delta} S(t)(V_j) \|_{L^{p_1}(\mathbb{R}_t; L^{p_2}(\mathbb{T}^3))} > \beta \log \log(\eta^{-1}), j = 0, 1 \mid \\ \|V_0 - V_1\|_{\mathcal{H}^s(\mathbb{T}^3)} < \eta \text{ and } \|V_j\|_{\mathcal{H}^s(\mathbb{T}^3)} \leq A, j = 0, 1 \} \rightarrow 0 \end{aligned}$$

as $\eta \rightarrow 0$. Therefore, we can also suppose that

$$\langle t \rangle^{-\delta} S(t)(V_0 - V_1) \|_{L^{p_1}(\mathbb{R}_t; L^{p_2}(\mathbb{T}^3))} \leq \eta^{1-\alpha} \quad (6.1)$$

and

$$\langle t \rangle^{-\delta} S(t)(V_j) \|_{L^{p_1}(\mathbb{R}_t; L^{p_2}(\mathbb{T}^3))} \leq \beta \log \log(\eta^{-1}), \quad j = 0, 1, \quad (6.2)$$

when we estimate the needed conditional probability.

We therefore need to estimate the difference of two solutions under the assumptions (6.1) and (6.2), in the regime $\eta \ll 1$. Let

$$v_j(t) = S(t)(V_j) + w_j(t), \quad j = 0, 1,$$

be two solutions of the cubic wave equation with data V_j (and thus $(w_j(0), \partial_t w_j(0)) = (0, 0)$). Applying the energy estimate, performed several times in this paper, we get the bound

$$\frac{d}{dt} (\mathcal{E}(w_j(t)))^{1/2} \leq C (\|S(t)(V_j)\|_{L^6(\mathbb{T}^3)}^3 + \|S(t)(V_j)\|_{L^\infty(\mathbb{T}^3)} (\mathcal{E}(w_j(t)))^{1/2}), \quad j = 0, 1,$$

and therefore, under the assumptions (6.1) and (6.2),

$$(\mathcal{E}(w_j(t)))^{1/2} \leq C_T e^{C_T \beta \log \log(\eta^{-1})} \log \log(\eta^{-1}) \leq C_T [\log(\eta^{-1})]^{C_T \beta}, \quad t \in [0, T]; \quad (6.3)$$

here and in what follows we denote by C_T different constants depending only on T (but independent of η).

We next estimate the difference $w_0 - w_1$. Using the equations solved by w_0, w_1 , we infer that

$$\begin{aligned} \frac{d}{dt} \|w_0(t, \cdot) - w_1(t, \cdot)\|_{\mathcal{H}^1(\mathbb{T}^3)}^2 &\leq 2 \left| \int_{\mathbb{T}^3} \partial_t (w_0(t, x) - w_1(t, x)) (\partial_t^2 - \Delta) (w_0(t, x) - w_1(t, x)) dx \right| \\ &\leq C \|w_0(t, \cdot) - w_1(t, \cdot)\|_{\mathcal{H}^1(\mathbb{T}^3)} \| (w_0 + S(t)(V_0))^3 - (w_1 + S(t)(V_1))^3 \|_{L^2(\mathbb{T}^3)}. \end{aligned}$$

Therefore using the Sobolev embedding $H^1(\mathbb{T}^3) \subset L^6(\mathbb{T}^3)$, we get

$$\begin{aligned} \frac{d}{dt} \|w_0(t, \cdot) - w_1(t, \cdot)\|_{\mathcal{H}^1(\mathbb{T}^3)} &\leq C (\|w_0(t, \cdot) - w_1(t, \cdot)\|_{\mathcal{H}^1(\mathbb{T}^3)} + \|S(t)(V_0 - V_1)\|_{L^6(\mathbb{T}^3)}) \\ &\quad \times (\|w_0(t, \cdot)\|_{L^6(\mathbb{T}^3)}^2 + \|w_1(t, \cdot)\|_{L^6(\mathbb{T}^3)}^2 + \|S(t)(V_0)\|_{L^6(\mathbb{T}^3)}^2 + \|S(t)(V_1)\|_{L^6(\mathbb{T}^3)}^2). \end{aligned}$$

Therefore, using (6.3) and the Gronwall lemma, under the assumptions (6.1) and (6.2), for $t \in [0, T]$,

$$\begin{aligned} \|w_0(t, \cdot) - w_1(t, \cdot)\|_{\mathcal{H}^1(\mathbb{T}^3)} &\leq C_T \eta^{1-\alpha} [\log(\eta^{-1})]^{C_T \beta} e^{C_T [\log(\eta^{-1})]^{C_T \beta}} \leq C_T \eta^{1-\alpha - C_T \beta} \\ &\leq C_T \eta^{1/2}, \end{aligned}$$

provided $\alpha, \beta \ll 1$. In particular by the Sobolev embedding

$$\|w_0 - w_1\|_{L^4([0, T] \times \mathbb{T}^3)} \leq C_T \eta^{1/2},$$

and therefore under the assumption (6.1),

$$\|v_0 - v_1\|_{L^4([0, T] \times \mathbb{T}^3)} \leq C_T \eta^{1/2}.$$

In summary, we have deduced that for a fixed $\varepsilon > 0$, the $\mu \otimes \mu$ -measure of the set of (V_0, V_1) such that

$$\|\Phi(t)(V_0) - \Phi(t)(V_1)\|_{X_T} > \varepsilon$$

under the conditions (6.1), (6.2) and $\|V_0 - V_1\|_{\mathcal{H}^s} < \eta$ is zero, as far as $\eta > 0$ is sufficiently small. Therefore, the left-hand side of (1.11) tends to zero as $\eta \rightarrow 0$. This ends the proof of the first part of Theorem 3.

For the second part, we argue by contradiction. Suppose that for every $\varepsilon > 0$ there exist $\eta > 0$ and Σ of full $\mu \otimes \mu$ -measure such that

$$\forall (V, V') \in \Sigma \cap (B_A \times B_A), \|V - V'\|_{\mathcal{H}^s} < \eta \Rightarrow \|\Phi(t)(V) - \Phi(t)(V')\|_{X_T} < \varepsilon. \quad (6.4)$$

Let us apply this with $\varepsilon = 1/n, n = 1, 2, \dots$, which produces full measure sets $\Sigma(n)$. Set

$$\Sigma_1 \equiv \bigcap_{n=1}^{\infty} \Sigma(n).$$

Then Σ_1 is of full $\mu \otimes \mu$ -measure and

$$\forall \varepsilon > 0, \exists \eta > 0, \forall (V, V') \in \Sigma_1 \cap (B_A \times B_A), \\ \|V - V'\|_{\mathcal{H}^s} < \eta \Rightarrow \|\Phi(t)(V) - \Phi(t)(V')\|_{X_T} < \varepsilon.$$

Next for $V \in \mathcal{H}^s$ we define $\mathcal{A}(V) \subset \mathcal{H}^s$ by $\mathcal{A}(V) \equiv \{V' \in \mathcal{H}^s : (V, V') \in \Sigma_1\}$. According to the Fubini Theorem, there exists a set $\mathcal{E} \subset \mathcal{H}^s$ of full μ -measure such that for every $V \in \mathcal{E}$ the set $\mathcal{A}(V)$ is of full μ -measure.

We are going to extend $\Phi(t)$ to a uniformly continuous map on B_A . For that purpose, we first extend $\Phi(t)$ to a uniformly continuous map on dense set of B_A . Let $\{(V_j)_{j \in \mathbb{N}}\}$ be a dense set of B_A for the \mathcal{H}^s topology. For $j \in \mathbb{N}$, we can construct by induction a sequence $(V_{j,n})$ such that

$$V_{j,n} \in B_A \cap \mathcal{E} \cap \bigcap_{m < n} \mathcal{A}(V_{j,m}) \cap \bigcap_{l < j, q \in \mathbb{N}} \mathcal{A}(V_{l,q}), \quad \|V_{j,n} - V_j\|_{\mathcal{H}^s} < 1/n.$$

Indeed, the induction assumption guarantees that the set $\mathcal{E} \cap \bigcap_{m < n} \mathcal{A}(V_{j,m}) \cap \bigcap_{l < j, q \in \mathbb{N}} \mathcal{A}(V_{l,q})$ has measure 1 (as an intersection of sets of measure 1) and consequently is dense. Notice that by construction, we have

$$(V_{k,n}, V_{l,m}) \in \Sigma, \quad \forall k < l, \forall n, m \in \mathbb{N}, \text{ and } \forall k = l, n < m. \quad (6.5)$$

Using (6.5) for $k = l$, we find from (6.4) that for any fixed k , the sequence $(\Phi(t)(V_{k,n}))_{n \in \mathbb{N}}$ is a Cauchy sequence in X_T and we can define $\overline{\Phi(t)}(V_j)$ as its limit. Using again (6.5) for $k \neq l$, we see from (6.4) that the map $\overline{\Phi(t)}$ is uniformly continuous on the set $\{(V_j)_{j \in \mathbb{N}}\}$. Therefore $\overline{\Phi(t)}$ can be extended by density to a uniformly continuous map on the whole B_A ; denote the extension by $\overline{\Phi(t)}$. We therefore have

$$\forall \varepsilon > 0, \exists \eta > 0, \forall V, V' \in B_A, \quad \|V - V'\|_{\mathcal{H}^s} < \eta \Rightarrow \|\overline{\Phi(t)}(V) - \overline{\Phi(t)}(V')\|_{X_T} < \varepsilon. \quad (6.6)$$

We have the following lemma.

Lemma 6.1. *For $V \in (C^\infty(\mathbb{T}^3) \times C^\infty(\mathbb{T}^3)) \cap B_A$, we have $\overline{\Phi(t)}(V) = (u, u_t)$, where u is the unique classical solution on $[0, T]$ of*

$$(\partial_t^2 - \Delta)u + u^3 = 0, \quad (u(0), \partial_t u(0)) = V.$$

Proof. Let us first show that that first component of $\overline{\Phi(t)}(V) \equiv (\overline{\Phi_1(t)}(V), \overline{\Phi_2(t)}(V))$ is a solution of the cubic wave equation. Observe that by construction, necessarily $\overline{\Phi_2(t)}(V) = \partial_t \overline{\Phi_1(t)}(V)$ in the distributional sense (in $\mathcal{D}'((0, T) \times \mathbb{T}^3)$).

Again by construction, we have

$$V = \lim_{n \rightarrow \infty} V_n$$

in \mathcal{H}^s where V_n are such that

$$(\partial_t^2 - \Delta)(\Phi_1(t)(V_n)) + (\Phi_1(t)(V_n))^3 = 0, \quad (6.7)$$

with the notation $\Phi(t) = (\Phi_1(t), \Phi_2(t))$. In addition,

$$\overline{\Phi(t)}(V) = \lim_{n \rightarrow \infty} \Phi(t)(V_n)$$

in X_T . We therefore have

$$(\partial_t^2 - \Delta)(\overline{\Phi_1(t)}(V)) = \lim_{n \rightarrow \infty} (\partial_t^2 - \Delta)(\Phi_1(t)(V_n))$$

in the distributional sense. Moreover, coming back to the definition of X_T , we also obtain

$$(\overline{\Phi_1(t)}(V))^3 = \lim_{n \rightarrow \infty} (\Phi_1(t)(V_n))^3,$$

in $L^{4/3}([0, T] \times \mathbb{T}^3)$. Therefore, letting $n \rightarrow \infty$ in (6.7), we see that $\overline{\Phi_1(t)}(V)$ solves the cubic wave equation (with data V). Moreover, since $(\overline{\Phi_1(t)}(V))^3 \in L^{4/3}([0, T] \times \mathbb{T}^3)$, it also satisfies the Duhamel formulation of the equation.

Denote by $u(t)$, $t \in [0, T]$, the classical solution of

$$(\partial_t^2 - \Delta)u + u^3 = 0, \quad (u(0), \partial_t u(0)) = V,$$

defined by Theorem 1. Set $v \equiv \overline{\Phi_1(t)}(V)$. Since our previous analysis has shown that v is a solution of the cubic wave equation, we have

$$(\partial_t^2 - \Delta)(u - v) + u^3 - v^3 = 0, \quad (u(0), \partial_t u(0)) = (0, 0). \quad (6.8)$$

We now invoke the L^4 - $L^{4/3}$ non-homogeneous estimates for the three-dimensional wave equation. Namely, there exists a constant (depending on T) such that for every interval $I \subset [0, T]$, the solution of the wave equation

$$(\partial_t^2 - \Delta)w = F, \quad (w(0), \partial_t w(0)) = (0, 0),$$

satisfies

$$\|w\|_{L^4(I \times \mathbb{T}^3)} \leq C \|F\|_{L^{4/3}(I \times \mathbb{T}^3)}. \quad (6.9)$$

Applying (6.9) in the context of (6.8) together with the Hölder inequality yields the bound

$$\|u - v\|_{L^4(I \times \mathbb{T}^3)} \leq C(\|u\|_{L^4(I \times \mathbb{T}^3)}^2 + \|v\|_{L^4(I \times \mathbb{T}^3)}^2)\|u - v\|_{L^4(I \times \mathbb{T}^3)}. \quad (6.10)$$

Since $u, v \in L^4(I \times \mathbb{T}^3)$, we can find a partition of $[0, T]$ into intervals I_1, \dots, I_l such that

$$C(\|u\|_{L^4(I_j \times \mathbb{T}^3)}^2 + \|v\|_{L^4(I_j \times \mathbb{T}^3)}^2) < 1/2, \quad j = 1, \dots, l.$$

We now apply (6.10) with $I = I_j$, $j = 1, \dots, l$, to conclude that $u = v$ on I_1 , then on I_2 and so on up to I_l , which shows that $u = v$ on $[0, T]$. Thus $u = \overline{\Phi_1(t)}(V)$ and therefore also $\partial_t u = \overline{\Phi_2(t)}(V)$. This completes the proof of Lemma 6.1. \square

It remains now to apply Lemma 6.1 to the sequence of smooth data in the statement of Theorem 1 to get a contradiction with (6.6). More precisely, if (U_n) is the sequence involved in the statement of Theorem 1, the result of Theorem 1 is that $\overline{\Phi(t)}(U_n)$ tends to infinity in $L^\infty([0, T]; \mathcal{H}^s)$, while (6.6) states that the same sequence tends to zero in the same space.

Appendix A. Random series

In this appendix, we collect the various results we need about random series. Most of them are well known in slightly different contexts, and the proofs we give are essentially adaptations of the classical proofs. The conditioned versions of our estimates (see Section A.2), though very natural, do not seem to appear in the literature.

A.1. Basic large deviation estimates

Proposition A.1. *Fix $\mu \in \mathcal{M}^s$, $s \in [0, 1)$ and suppose that μ is induced via the map (1.5) from the couple $(u_0, u_1) \in \mathcal{H}^s$. Then there exists a positive constant C such that for every $2 \leq p_1, p_2 \leq q < \infty$ and every $\delta > 1/p_1$,*

$$\begin{aligned} \mu(\{(v_0, v_1) \in \mathcal{H}^s : \|\langle t \rangle^{-\delta} (1 - \Pi_0) S(t)(v_0, v_1)\|_{L^{p_1}(\mathbb{R}_t; L^{p_2}(\mathbb{T}^3))} > \lambda\}) \\ \leq \left(C \frac{\sqrt{q} \|(u_0, u_1)\|_{\mathcal{H}^0(\mathbb{T}^3)} (\delta p_1 - 1)^{-1/p_1}}{\lambda} \right)^q. \end{aligned} \quad (\text{A.1})$$

Proof. By definition, the left-hand side of (A.1) equals

$$p(\{\omega \in \Omega : \|\langle t \rangle^{-\delta} (1 - \Pi_0) S(t)(u_0^\omega, u_1^\omega)\|_{L^{p_1}(\mathbb{R}_t; L^{p_2}(\mathbb{T}^3))} > \lambda\}).$$

We decompose

$$\begin{aligned} \Pi^0 S(t)(u_0^\omega, u_1^\omega) = \sum_{n \in \mathbb{Z}_*^3} \left(\left(\beta_{n,0}(\omega) b_{n,0} \cos(t|n|) + \beta_{n,1}(\omega) b_{n,1} \frac{\sin(t|n|)}{|n|} \right) \cos(n \cdot x) \right. \\ \left. + \left(\gamma_{n,0}(\omega) c_{n,0} \cos(t|n|) + \gamma_{n,1}(\omega) c_{n,1} \frac{\sin(t|n|)}{|n|} \right) \sin(n \cdot x) \right), \end{aligned}$$

with

$$\sum_{n \in \mathbb{Z}_*^3} (|b_{n,0}|^2 + |c_{n,0}|^2 + |n|^{-2} (|b_{n,1}|^2 + |c_{n,1}|^2)) \leq C \|(u_0, u_1)\|_{\mathcal{H}^0(\mathbb{T}^3)}^2.$$

Now, using the triangle inequality, by writing $\cos(n \cdot x)$ and $\sin(n \cdot x)$ as linear combinations of $\exp(\pm i(n \cdot x))$, we observe that it suffices to get the bound

$$\begin{aligned} p\left(\left\{\omega : \left\| \langle t \rangle^{-\delta} \sum_n d_n(t) c_n g_n^\omega e^{in \cdot x} \right\|_{L^{p_1}(\mathbb{R}_t; L^{p_2}(\mathbb{T}^3))} > \lambda \right\}\right) \\ \leq \left(\frac{C \sqrt{q} \|(c_n)\|_{l^2} (\delta p_1 - 1)^{-1/p_1}}{\lambda} \right)^q, \end{aligned}$$

where (g_n^ω) are independent real random variables with joint distribution satisfying (1.4) and $|d_n(t)| \leq 1$. Using the Minkowski inequality, we can write for $q \geq p$,

$$\begin{aligned} & \left\| \langle t \rangle^{-\delta} \sum_n d_n(t) c_n g_n^\omega e^{in \cdot x} \right\|_{L^q(\Omega; L^{p_1}(\mathbb{R}_t; L^{p_2}(\mathbb{T}^3)))} \\ & \leq \left\| \langle t \rangle^{-\delta} \sum_n d_n(t) c_n g_n^\omega e^{in \cdot x} \right\|_{L^{p_1}(\mathbb{R}_t; L^{p_2}(\mathbb{T}^3); L^q(\Omega))} \\ & = \left\| \left\| \langle t \rangle^{-\delta} \sum_n d_n(t) c_n g_n^\omega e^{in \cdot x} \right\|_{L^q(\Omega)} \right\|_{L^{p_1}(\mathbb{R}_t; L^{p_2}(\mathbb{T}^3))}. \end{aligned}$$

By using [10, Lemma 3.1], we get

$$\begin{aligned} & \left\| \langle t \rangle^{-\delta} \sum_n d_n(t) c_n g_n^\omega e^{in \cdot x} \right\|_{L^q(\Omega; L^{p_1}(\mathbb{R}_t; L^{p_2}(\mathbb{T}^3)))} \\ & \leq \left\| C \sqrt{q} \left(\sum_n |\langle t \rangle^{-\delta} d_n(t) c_n e^{in \cdot x}|^2 \right)^{1/2} \right\|_{L^{p_1}(\mathbb{R}_t; L^{p_2}(\mathbb{T}^3))} \\ & \leq C \sqrt{q} \left\| \sum_n \langle t \rangle^{-2\delta} |c_n|^2 \right\|_{L^{p_1/2}(\mathbb{R}_t; L^{p_2/2}(\mathbb{T}^3))}^{1/2} \leq C \sqrt{q} \left(\sum_n \|\langle t \rangle^{-2\delta}\|_{L^{p_1/2}(\mathbb{R}_t)} |\alpha_n|^2 \right)^{1/2} \\ & \leq C \sqrt{q} (\delta p_1 - 1)^{-1/p_1} \left(\sum_n |\alpha_n|^2 \right)^{1/2}, \end{aligned}$$

and we conclude the proof of Proposition A.1 by using the Chebyshev inequality

$$p(\{\omega : |A(\omega)| > \lambda\}) \leq \lambda^{-q} \|A\|_{L^q(\Omega)}^q. \quad \square$$

For fixed p_1, p_2 , we can optimize the estimate by taking

$$C \frac{\sqrt{q} \|(u_0, u_1)\|_{\mathcal{H}^0(\mathbb{T}^3)}}{\lambda} = \frac{1}{2} \Leftrightarrow q = \frac{\lambda^2 \|(u_0, u_1)\|_{\mathcal{H}^0(\mathbb{T}^3)}^{-2}}{4C^2}, \quad (\text{A.2})$$

and we deduce the following statement.

Corollary A.2. *There exist $C, c > 0$ such that under the assumptions of Proposition A.1, for every $\lambda > 0$,*

$$\begin{aligned} & \mu(\{(v_0, v_1) \in \mathcal{H}^s : \|\langle t \rangle^{-\delta} \Pi^0 S(t)(v_0, v_1)\|_{L^{p_1}(\mathbb{R}_t; L^{p_2}(\mathbb{T}^3))} > \lambda\}) \\ & \leq C \exp\left(-\frac{c\lambda^2}{\|(u_0, u_1)\|_{\mathcal{H}^0(\mathbb{T}^3)}^2}\right). \end{aligned}$$

Remark A.3. Notice that the measure μ is the tensor product of two probability measures μ_N and μ^N defined on the images of the projectors Π_N and Π^N respectively. As a

consequence, applying Corollary A.2 to the measure μ^N , we find that under the assumptions of Proposition A.1, for every $\lambda > 0$,

$$\begin{aligned} & \mu(\{(v_0, v_1) \in \mathcal{H}^s : \|\langle t \rangle^{-\delta} \Pi^N S(t)(v_0, v_1)\|_{L^{p_1}(\mathbb{R}_t; L^{p_2}(\mathbb{T}^3))} > \lambda\}) \\ &= \mu^N(\{(v_0^N, v_1^N) \in \Pi^N(\mathcal{H}^s) : \|\langle t \rangle^{-\delta} S(t)(v_0^N, v_1^N)\|_{L^{p_1}(\mathbb{R}_t; L^{p_2}(\mathbb{T}^3))} > \lambda\}) \\ &\leq C \exp\left(-\frac{c\lambda^2}{\|\Pi^N(u_0, u_1)\|_{\mathcal{H}^0(\mathbb{T}^3)}^2}\right) \leq C \exp\left(-\frac{c\lambda^2}{N^{-2s} \|\Pi^N(u_0, u_1)\|_{\mathcal{H}^s(\mathbb{T}^3)}^2}\right). \end{aligned} \quad (\text{A.3})$$

Notice now that if u_0 and u_1 are constant, the free evolution is

$$S(t)(u_0, u_1) = u_0 + u_1 t.$$

Therefore we deduce the following statement.

Corollary A.4. *Fix $\mu \in \mathcal{M}^s$, $s \in [0, 1)$, and suppose that μ is induced via the map (1.5) from the couple $(u_0, u_1) \in \mathcal{H}^s$. Fix also $2 \leq p_1, p_2 < \infty$ and $\delta > 1 + 1/p_1$. Then there exists a positive constant C such that for every $\lambda > 0$,*

$$\begin{aligned} & \mu(\{(v_0, v_1) \in \mathcal{H}^s : \|\langle t \rangle^{-\delta} S(t)(v_0, v_1)\|_{L^{p_1}(\mathbb{R}_t; L^{p_2}(\mathbb{T}^3))} > \lambda\}) \\ & \leq C \exp\left(-\frac{c\lambda^2}{\|(u_0, u_1)\|_{\mathcal{H}^0(\mathbb{T}^3)}^2}\right). \end{aligned}$$

Notice finally that using the Sobolev embeddings $W^{\sigma, p}(\mathbb{T}^3) \subset L^\infty(\mathbb{T}^3)$, $\sigma > 3/p$, where the $W^{\sigma, p}$ norm is defined by

$$\|u\|_{W^{\sigma, p}(\mathbb{T}^3)} = \|(1 - \Delta)^{\sigma/2} u\|_{L^p(\mathbb{T}^3)},$$

we also obtain

Corollary A.5. *Fix $s > 0$ and $\mu \in \mathcal{M}^s$. Let $0 < \sigma \leq s$ and suppose that μ is induced via the map (1.5) from the couple $(u_0, u_1) \in \mathcal{H}^s$. Fix also $2 \leq p_1 < \infty$, $2 \leq p_2 \leq \infty$ and $\delta > 1 + 1/p_1$. Then there exists a positive constant C such that for every $\lambda > 0$,*

$$\begin{aligned} & \mu(\{(v_0, v_1) \in \mathcal{H}^s : \|\langle t \rangle^{-\delta} S(t)(v_0, v_1)\|_{L^{p_1}(\mathbb{R}_t; L^{p_2}(\mathbb{T}^3))} > \lambda\}) \\ & \leq C \exp\left(-\frac{c\lambda^2}{\|(u_0, u_1)\|_{\mathcal{H}^\sigma(\mathbb{T}^3)}^2}\right). \end{aligned}$$

Remark A.6. The same argument as in the proof of Proposition A.1 shows that for every $2 \leq p < \infty$ and $s \geq 0$ there exist $C, c > 0$ such that under the assumptions of Proposition A.1 defining μ , for every $\lambda > 0$ and every integer $N \geq 0$,

$$\mu(\{(v_0, v_1) \in \mathcal{H}^s : \|\Pi_N v_0\|_{L^p(\mathbb{T}^3)} > \lambda\}) \leq C \exp\left(-\frac{c\lambda^2}{\|(u_0, u_1)\|_{\mathcal{H}^0(\mathbb{T}^3)}^2}\right), \quad (\text{A.4})$$

$$\mu(\{(v_0, v_1) \in \mathcal{H}^s : \|(v_0, v_1)\|_{\mathcal{H}^s(\mathbb{T}^3)} > \lambda\}) \leq C \exp\left(-\frac{c\lambda^2}{\|(u_0, u_1)\|_{\mathcal{H}^s(\mathbb{T}^3)}^2}\right). \quad (\text{A.5})$$

A.2. Conditioned large deviation estimates

The purpose of this section is to deduce the following conditioned versions of our previous large deviation estimates:

Proposition A.7. *Let $\mu \in \mathcal{M}^s$, $s \in (0, 1)$ and suppose that the real random variable with distribution θ , involved in the definition of μ , is symmetric. Then for $\delta > 1 + 1/p_1$, $2 \leq p_1 < \infty$ and $2 \leq p_2 \leq \infty$ there exist positive constants c, C such that for every positive $\varepsilon, \lambda, \Lambda$ and A ,*

$$\begin{aligned} \mu \otimes \mu \left(\left\{ ((v_0, v_1), (v'_0, v'_1)) \in \mathcal{H}^s \times \mathcal{H}^s : \|\langle t \rangle^{-\delta} S(t)(v_0 - v'_0, v_1 - v'_1)\|_{L^{p_1}(\mathbb{R}_t; L^{p_2}(\mathbb{T}^3))} > \lambda \right. \right. \\ \left. \left. \text{or } \|\langle t \rangle^{-\delta} S(t)(v_0 + v'_0, v_1 + v'_1)\|_{L^{p_1}(\mathbb{R}_t; L^{p_2}(\mathbb{T}^3))} > \Lambda \mid \|(v_0 - v'_0, u_1 - u'_1)\|_{\mathcal{H}^s(\mathbb{T}^3)} \leq \varepsilon \right. \right. \\ \left. \left. \text{and } \|(v_0 + v'_0, u_1 + u'_1)\|_{\mathcal{H}^s(\mathbb{T}^3)} \leq A \right\} \right) \leq C(e^{-c\lambda^2/\varepsilon^2} + e^{-c\Lambda^2/A^2}). \end{aligned} \quad (\text{A.6})$$

Proof. The proof of this result can be obtained by coming back to the original proof of Paley and Zygmund's of the L^p boundedness of random series on the torus. However, we will follow a suggestion by J.-P. Kahane and show that in fact, we can deduce it directly from the large deviation estimates proved in the previous section. The basic result which will allow this procedure is the following lemma.

Lemma A.8. *For $j = 1, 2$, let E_j be two Banach spaces endowed with measures μ_j . Let $f : E_1 \times E_2 \rightarrow \mathbb{C}$ and $g_1, g_2 : E_2 \rightarrow \mathbb{C}$ be three measurable functions. Then*

$$\begin{aligned} \mu_1 \otimes \mu_2 \left(\left\{ (x_1, x_2) \in E_1 \times E_2 : |f(x_1, x_2)| > \lambda \mid |g_1(x_2)| \leq \varepsilon, |g_2(x_2)| \leq A \right\} \right) \\ \leq \sup_{x_2 \in E_2, |g_1(x_2)| \leq \varepsilon, |g_2(x_2)| \leq A} \mu_1(\{x_1 \in E_1 : |f(x_1, x_2)| > \lambda\}), \end{aligned}$$

where by sup we mean the essential supremum.

Proof. We may write

$$\begin{aligned} \int_{E_1} \chi(|f(x_1, x_2)| > \lambda) \chi(|g_1(x_2)| \leq \varepsilon) \chi(|g_2(x_2)| \leq A) d\mu_1(x_1) \\ \leq \left(\sup_{\substack{x_2 \in E_2, |g_1(x_2)| \leq \varepsilon \\ |g_2(x_2)| \leq A}} \mu_1(\{x_1 \in E_1 : |f(x_1, x_2)| > \lambda\}) \right) \chi(|g_1(x_2)| \leq \varepsilon) \chi(|g_2(x_2)| \leq A) \end{aligned} \quad (\text{A.7})$$

for a.e. $x_2 \in E_2$. Here by $\chi(\cdot)$ we denote the characteristic function of the corresponding set. Now, we integrate the inequality (A.7) over $x_2 \in E_2$ with respect to μ_2 , to achieve the claimed bound. \square

We shall also use the following lemma.

Lemma A.9. *Let g_1 and g_2 be two independent identically distributed real random variables with symmetric distribution. Then $g_1 \pm g_2$ have symmetric distributions. Moreover if h is a Bernoulli random variable independent of g_1 then hg_1 has the same distribution as g_1 .*

The proof is straightforward (in the analysis of $g_1 - g_2$ we do not need the symmetry assumption on g_1, g_2).

Proof of Proposition A.7. Define

$$\mathcal{E} \equiv \mathbb{R} \times \mathbb{R}^{\mathbb{Z}_*^3} \times \mathbb{R}^{\mathbb{Z}_*^3},$$

equipped with the natural Banach space structure coming from the l^∞ norm. We endow \mathcal{E} with a probability measure μ_0 defined via the map

$$\omega \mapsto (k_0(\omega), (l_n(\omega))_{n \in \mathbb{Z}_*^3}, (h_n(\omega))_{n \in \mathbb{Z}_*^3}),$$

where (k_0, l_n, h_n) is a system of independent Bernoulli variables.

For $h = (x, (y_n)_{n \in \mathbb{Z}_*^3}, (z_n)_{n \in \mathbb{Z}_*^3}) \in \mathcal{E}$ and

$$u(x) = a + \sum_{n \in \mathbb{Z}_*^3} (b_n \cos(n \cdot x) + c_n \sin(n \cdot x)),$$

we define the operation \odot by

$$h \odot u \equiv ax + \sum_{n \in \mathbb{Z}_*^3} (b_n y_n \cos(n \cdot x) + c_n z_n \sin(n \cdot x)).$$

Let us first evaluate the quantity

$$\mu \otimes \mu \left(\left\{ ((v_0, v_1), (v'_0, v'_1)) \in \mathcal{H}^s \times \mathcal{H}^s : \|\langle t \rangle^{-\delta} S(t)(v_0 - v'_0, v_1 - v'_1)\|_{L^{p_1}(\mathbb{R}_t; L^{p_2}(\mathbb{T}^3))} > \lambda \right. \right. \\ \left. \left. \mid \|(v_0 - v'_0, v_1 - v'_1)\|_{\mathcal{H}^s(\mathbb{T}^3)} \leq \varepsilon \text{ and } \|(v_0 + v'_0, v_1 + v'_1)\|_{\mathcal{H}^s(\mathbb{T}^3)} \leq A \right\} \right). \quad (\text{A.8})$$

Observe that, thanks to Lemma A.9, (A.8) equals

$$\mu \otimes \mu \otimes \mu_0 \otimes \mu_0 \left(\left\{ ((v_0, v_1), (v'_0, v'_1), (h_0, h_1)) \in \mathcal{H}^s \times \mathcal{H}^s \times \mathcal{E} \times \mathcal{E} : \right. \right. \\ \|\langle t \rangle^{-\delta} S(t)(h_0 \odot (v_0 - v'_0), h_1 \odot (v_1 - v'_1))\|_{L^{p_1}(\mathbb{R}_t; L^{p_2}(\mathbb{T}^3))} > \lambda \mid \\ \|(h_0 \odot (v_0 - v'_0), h_1 \odot (v_1 - v'_1))\|_{\mathcal{H}^s(\mathbb{T}^3)} \leq \varepsilon \\ \left. \left. \text{and } \|(h_0 \odot (v_0 + v'_0), h_1 \odot (v_1 + v'_1))\|_{\mathcal{H}^s(\mathbb{T}^3)} \leq A \right\} \right). \quad (\text{A.9})$$

Since the $H^s(\mathbb{T}^3)$ norm of a function f depends only on the absolute value of its Fourier coefficients, we deduce that (A.9) equals

$$\mu \otimes \mu \otimes \mu_0 \otimes \mu_0 \left(\left\{ ((v_0, v_1), (v'_0, v'_1), (h_0, h_1)) \in \mathcal{H}^s \times \mathcal{H}^s \times \mathcal{E} \times \mathcal{E} : \right. \right. \\ \|\langle t \rangle^{-\delta} S(t)(h_0 \odot (v_0 - v'_0), h_1 \odot (v_1 - v'_1))\|_{L^{p_1}(\mathbb{R}_t; L^{p_2}(\mathbb{T}^3))} > \lambda \mid \\ \|(v_0 - v'_0, v_1 - v'_1)\|_{\mathcal{H}^s(\mathbb{T}^3)} \leq \varepsilon \text{ and } \|(v_0 + v'_0, v_1 + v'_1)\|_{\mathcal{H}^s(\mathbb{T}^3)} \leq A \left. \right\} \right). \quad (\text{A.10})$$

We now apply Lemma A.8 with $\mu_1 = \mu_0 \otimes \mu_0$ and $\mu_2 = \mu \otimes \mu$ to find that (A.10) is bounded by

$$\sup_{\|(v_0 - v'_0, v_1 - v'_1)\|_{\mathcal{H}^s(\mathbb{T}^3)} \leq \varepsilon} \mu_0 \otimes \mu_0(\{(h_0, h_1) \in \mathcal{E} \times \mathcal{E} : \|\langle t \rangle^{-\delta} S(t)(h_0 \odot (v_0 - v'_0), h_1 \odot (v_1 - v'_1))\|_{L^{p_1}(\mathbb{R}_t; L^{p_2}(\mathbb{T}^3))} > \lambda\}). \quad (\text{A.11})$$

We next apply Corollary A.5 (with Bernoulli variables) to deduce that (A.8) is bounded by $C \exp(-c\lambda^2/\varepsilon^2)$. A very similar argument gives

$$\mu \otimes \mu(\{(v_0, v_1), (v'_0, v'_1) \in \mathcal{H}^s \times \mathcal{H}^s : \|\langle t \rangle^{-\delta} S(t)(v_0 + v'_0, v_1 + v'_1)\|_{L^{p_1}(\mathbb{R}_t; L^{p_2}(\mathbb{T}^3))} > \Lambda \mid \|(v_0 - v'_0, v_1 - v'_1)\|_{\mathcal{H}^s(\mathbb{T}^3)} \leq \varepsilon \text{ and } \|(v_0 + v'_0, v_1 + v'_1)\|_{\mathcal{H}^s(\mathbb{T}^3)} \leq A\})$$

is bounded by $C \exp(-c\Lambda^2/A^2)$. This completes the proof of Proposition A.7. \square

B. Properties of the measures $\mu_{(u_0, u_1)}$

B.1. Conditions for orthogonality of the measures

Via the choice of coordinates induced by the decomposition (1.3)

$$(u_0, u_1) \in \mathcal{H}^s \mapsto (a_0, (b_{n,0}, c_{n,0})_{n \in \mathbb{Z}_*^3}, a_1, (b_{n,1}, c_{n,1})_{n \in \mathbb{Z}_*^3}) \in (\mathbb{R} \times \mathbb{R}^{\mathbb{Z}_*^3} \times \mathbb{R}^{\mathbb{Z}_*^3})^2,$$

the measure $\mu_{(u_0, u_1)}$ can be seen as an infinite tensor product of probability measures on

$$(\mathbb{R} \times \mathbb{R}^{\mathbb{Z}_*^3} \times \mathbb{R}^{\mathbb{Z}_*^3})^2, \quad \mu \sim \mu_{0,0} \otimes \bigotimes_{n \in \mathbb{Z}_*^3} \mu_{n,0,b} \otimes \bigotimes_{n \in \mathbb{Z}_*^3} \mu_{n,0,c} \otimes \mu_{0,1} \otimes \bigotimes_{n \in \mathbb{Z}_*^3} \mu_{n,1,b} \otimes \bigotimes_{n \in \mathbb{Z}_*^3} \mu_{n,1,c},$$

where

$$\mu_{0,0}, \mu_{n,0,b}, \mu_{n,0,c}, \mu_{0,1}, \mu_{n,1,b}, \mu_{n,1,c}$$

are the distributions of the random variables

$$a_0 \alpha_0, b_{n,0} \beta_{n,0}, c_{n,0} \gamma_{n,0}, a_1 \alpha_1, b_{n,1} \beta_{n,1}, c_{n,1} \gamma_{n,1}$$

respectively. As a consequence, we will be able to apply the following result by Kakutani [17].

Theorem (Kakutani). *Consider the infinite tensor products of probability measures on $\mathbb{R}^{\mathbb{N}}$,*

$$\mu_i = \bigotimes_{n \in \mathbb{N}} \mu_{n,i}, \quad i = 1, 2.$$

Then the measures μ_1 and μ_2 on $\mathbb{R}^{\mathbb{N}}$ endowed with its cylindrical Borel σ -algebra are absolutely continuous with respect to each other, $\mu_1 \ll \mu_2$ and $\mu_2 \ll \mu_1$, if and only if the following holds:

- (1) The measures $\mu_{n,1}$ and $\mu_{n,2}$ are for each n absolutely continuous with respect to each other: there exist functions $g_n \in L^1(\mathbb{R}, d\mu_{n,2})$ and $k_n \in L^1(\mathbb{R}, d\mu_{n,1})$ such that

$$d\mu_{n,1} = g_n d\mu_{n,2}, \quad d\mu_{n,2} = k_n d\mu_{n,1}.$$

- (2) The functions g_n are such that the infinite product

$$\prod_{n \in \mathbb{N}} \int_{\mathbb{R}} g_n^{1/2} d\mu_{n,2} = \prod_{n \in \mathbb{N}} \int_{\mathbb{R}} \sqrt{d\mu_{n,1}} \sqrt{d\mu_{n,2}} \quad (\text{B.1})$$

is convergent (i.e. positive).

Furthermore, if any of the conditions above is not satisfied (i.e. if the two measures μ_1 and μ_2 are not absolutely continuous with respect to each other), then the two measures are mutually singular: there exists a set $A \subset \mathbb{R}^{\mathbb{N}}$ such that

$$\mu_1(A) = 1, \quad \mu_2(A) = 0.$$

The theorem above implies the following statement concerning the measures we studied in this paper in the context of the cubic wave equation (1.1).

Proposition B.1. Assume that the random variables $(\alpha_j(\omega), \beta_{n,j}(\omega), \gamma_{n,j}(\omega))$, $j = 0, 1$, $n \in \mathbb{Z}_*^3$, used to obtain the randomization as explained in the introduction are independent centered Gaussian random variables. Let

$$u_j(x) = a_{0,j} + \sum_{n \in \mathbb{Z}_*^3} (b_{n,j} \cos(n \cdot x) + c_{n,0} \sin(n \cdot x)), \quad j = 0, 1,$$

$$\tilde{u}_j(x) = \tilde{a}_{0,j} + \sum_{n \in \mathbb{Z}_*^3} (\tilde{b}_{n,j} \cos(n \cdot x) + \tilde{c}_{n,0} \sin(n \cdot x)), \quad j = 0, 1.$$

Then the measures $\mu_{(u_0, u_1)}$ and $\mu_{(\tilde{u}_0, \tilde{u}_1)}$ are absolutely continuous with respect to each other if and only if neither of the coefficients $a, b, c, \tilde{a}, \tilde{b}, \tilde{c}$ above vanishes (or then they must vanish simultaneously, i.e. if $a_{0,j} = 0$, then $\tilde{a}_{0,j} = 0$, etc.) and

$$\sum_{j=0}^1 \left(\left| \frac{\tilde{a}_{0,j}}{a_{0,j}} \right| - 1 \right)^2 + \sum_{n \in \mathbb{Z}_*^3} \left[\left(\left| \frac{\tilde{b}_{n,j}}{b_{n,j}} \right| - 1 \right)^2 + \left(\left| \frac{\tilde{c}_{n,j}}{c_{n,j}} \right| - 1 \right)^2 \right] < \infty.$$

Furthermore, if this condition is not satisfied, then the two measures $\mu_{(u_0, u_1)}$ and $\mu_{(\tilde{u}_0, \tilde{u}_1)}$ are mutually singular.

Proof. Indeed, if g is a normalized Gaussian random variable, the random variable αg is a Gaussian random variable centered and with variance α^2 , and eliminating the trivial contributions when the coefficients vanish simultaneously, the result amounts to proving that if $\mu_i = \bigotimes_{n \in \mathbb{N}} \mu_{n,i}$, where $\mu_{n,i}$ are Gaussian distributions of variance $x_{n,i}^2$, $x_{n,i} > 0$, then the measures μ_1 and μ_2 are absolutely continuous with respect to each other if and only if

$$\sum_n \left(\frac{x_{n,1}}{x_{n,2}} - 1 \right)^2 < \infty,$$

and in this case we have

$$d\mu_{n,i} = \frac{1}{x_{n,i}\sqrt{2\pi}} e^{-t^2/2x_{n,i}^2} dt \quad \text{and} \quad g_n = \frac{x_{n,2}}{x_{n,1}} e^{t^2/2x_{n,2}^2 - t^2/2x_{n,1}^2}.$$

Consequently (taking $x_{n,i} > 0$),

$$\begin{aligned} \int_{\mathbb{R}} g_n^{1/2} d\mu_{n,2} &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi x_{n,1}x_{n,2}}} e^{-t^2/4x_{n,2}^2 - t^2/4x_{n,1}^2} dt = \left(\frac{2x_{n,1}x_{n,2}}{x_{n,1}^2 + x_{n,2}^2} \right)^{1/2} \\ &= \left(\frac{x_{n,1}/x_{n,2} + x_{n,2}/x_{n,1}}{2} \right)^{-1/2} \end{aligned} \quad (\text{B.2})$$

and we deduce that if the infinite product (B.1) is convergent then necessarily the quotients $x_{n,1}/x_{n,2}$ tend to 1. Now, writing $x_{n,2}/x_{n,1} = 1 + \varepsilon_n$, we have

$$\left(\frac{2x_{n,1}x_{n,2}}{x_{n,1}^2 + x_{n,2}^2} \right)^{1/2} = 1 - \frac{1}{4}\varepsilon_n^2 + \mathcal{O}(\varepsilon_n^3).$$

Finally, by taking the logarithm, we conclude that the infinite product (B.1) is convergent if and only if

$$\sum_n \varepsilon_n^2 < \infty. \quad \square$$

B.2. Proof of Proposition 1.2

Recall that Π_N is the orthogonal projection to the space of functions having Fourier modes $n \in \mathbb{Z}^3$ such that $|n| \leq N$ and $\Pi^N = \text{Id} - \Pi_N$ (see (4.1)). Then for fixed $\epsilon > 0$, we have

$$\begin{aligned} &\mu_{(u_0, u_1)}(\{(v_0, v_1) \in \mathcal{H}^s : \|(v_0, v_1) - (w_0, w_1)\|_{\mathcal{H}^s} < \epsilon\}) \\ &\geq \mu_{(u_0, u_1)}(\{(v_0, v_1) \in \mathcal{H}^s : \|\Pi_N((v_0, v_1) - (w_0, w_1))\|_{\mathcal{H}^s} < \epsilon/2 \text{ and} \\ &\quad \|\Pi^N((v_0, v_1) - (w_0, w_1))\|_{\mathcal{H}^s} < \epsilon/2\}) \end{aligned} \quad (\text{B.3})$$

Since the two events are independent (due to the independence of the random variables), we deduce

$$\begin{aligned} &\mu_{(u_0, u_1)}(\{(v_0, v_1) \in \mathcal{H}^s : \|(v_0, v_1) - (w_0, w_1)\|_{\mathcal{H}^s} < \epsilon\}) \\ &\geq \mu_{(u_0, u_1)}(\{(v_0, v_1) \in \mathcal{H}^s : \|\Pi_N(v_0, v_1) - \Pi_N(w_0, w_1)\|_{\mathcal{H}^s} < \epsilon/2\}) \\ &\quad \times \mu_{(u_0, u_1)}(\{(v_0, v_1) \in \mathcal{H}^s : \|\Pi^N(v_0, v_1) - \Pi^N(w_0, w_1)\|_{\mathcal{H}^s} < \epsilon/2\}) \\ &\geq \mu_{(u_0, u_1)}(\{(v_0, v_1) \in \mathcal{H}^s : \|\Pi_N(v_0, v_1) - \Pi_N(w_0, w_1)\|_{\mathcal{H}^s} < \epsilon/2\}) \\ &\quad \times \mu_{(u_0, u_1)}(\{(v_0, v_1) \in \mathcal{H}^s : \|\Pi^N(w_0, w_1)\|_{\mathcal{H}^s} < \epsilon/4 \text{ and } \|\Pi^N(v_0, v_1)\|_{\mathcal{H}^s} < \epsilon/4\}) \end{aligned} \quad (\text{B.4})$$

Since

$$\lim_{N \rightarrow \infty} \|\Pi^N(u_0, u_1)\|_{\mathcal{H}^s} = \lim_{N \rightarrow \infty} \|\Pi^N(w_0, w_1)\|_{\mathcal{H}^s} = 0$$

and according to (A.5) applied to $\Pi^N(v_0, v_1)$,

$$\begin{aligned} \mu_{(u_0, u_1)}(\{(v_0, v_1) \in \mathcal{H}^s : \|\Pi^N(v_0, v_1)\|_{\mathcal{H}^s} \geq \epsilon/4\}) \\ \leq C e^{-c\epsilon^2/\|\Pi^N(u_0, u_1)\|_{\mathcal{H}^s}^2} \rightarrow 0 \quad \text{as } N \rightarrow \infty, \end{aligned}$$

we deduce that for N large enough (depending on $\epsilon > 0$ and $(u_0, u_1), (w_0, w_1)$),

$$\mu_{(u_0, u_1)}(\{(v_0, v_1) \in \mathcal{H}^s : \|\Pi^N(w_0, w_1)\|_{\mathcal{H}^s} < \epsilon/4 \text{ and } \|\Pi^N(v_0, v_1)\|_{\mathcal{H}^s} < \epsilon/4\}) \geq 1/2.$$

and consequently we obtain for such N ,

$$\begin{aligned} \mu_{(u_0, u_1)}(\{(v_0, v_1) \in \mathcal{H}^s : \|(v_0, v_1) - (w_0, w_1)\|_{\mathcal{H}^s} < \epsilon\}) \\ \geq \frac{1}{2} \mu_{(u_0, u_1)}(\{(v_0, v_1) \in \mathcal{H}^s : \|\Pi_N(v_0, v_1) - \Pi_N(w_0, w_1)\|_{\mathcal{H}^s} < \epsilon/2\}), \quad (\text{B.5}) \end{aligned}$$

and since now the study is reduced to a finite-dimensional problem, (1.6) follows easily from the assumptions that all Fourier modes of (u_0, u_1) are non-vanishing and the distribution function θ of our random variables charges all open sets of \mathbb{R} .

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