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Gilles Francfort · Alessandro Giacomini

On periodic homogenization in perfect elasto-plasticity

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Abstract. The limit behavior of a periodic assembly of a finite number of elasto-plastic phases is investigated as the period becomes vanishingly small. A limit quasi-static evolution is derived through two-scale convergence techniques. It can be thermodynamically viewed as an elasto-plastic model, albeit with an infinite number of internal variables.

Keywords. Elasticity, plasticity, space of bounded deformations, lower semicontinuity, Radon measures, periodic homogenization, evolution problems

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1. Introduction

1.1. Introductory remarks

In a previous paper [11], we undertook what we believe to be a thorough revamping of heterogeneous, small strain elasto-plastic evolutions, so as to account for multi-phase composites with arbitrary yield surfaces and elasticities, provided only that the interfaces between the phases be piecewise C^1. This laid the ground work for the present investigation in which we propose to (re)visit periodic homogenization in the same context.

Elasto-plastic composites belong to the familiar of many engineering fields, and their behavior has been meticulously investigated in a plethoric literature. When focussing on

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G. Francfort: LAGA, Université Paris-Nord & Institut Universitaire de France;
e-mail: gilles.francfort@univ-paris13.fr

A. Giacomini: DICATAM, Sezione di Matematica, Università degli Studi di Brescia,
Via Valotti 9, 25133 Brescia, Italy; e-mail: alessandro.giacomini@ing.unibs.it

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limit analysis, that is, on the prediction of the ultimate load that a composite elasto-plastic structure can withstand, the engineering literature is extensive, while the mathematical analysis of the underlying variational problem has been successfully undertaken in various works of G. Bouchitté and/or P.-M. Suquet (see e.g. [5], [20], [6], [7]). However, when elasto-plastic evolutions are envisioned, both engineering and mathematical literature fall short of any bona fide discussion of the interaction between the evolution and the elasto-plastic microstructure. Rather, the default position is to rely on strain hardening as a regularizing mechanism under which the homogenization procedure becomes much simpler (see e.g. [22], [23], [18], [16] as far as the mathematical literature is concerned).

In this paper, we propose to confront the homogenization of the evolution of a periodic multi-phase elasto-plastic composite without any regularizing effect. The periodicity restriction is unfortunate, but, in all fairness, we are clueless if departing from the periodic framework, although we suspect that ergodicity could easily replace periodicity. In turn, the periodicity assumption will allow us to resort to the very efficient method of two-scale convergence first proposed by [17], [1] in a classical elliptic setting, then refined by many authors. As in our previous contribution [11], we pay close attention to the issue of the duality between the stress fields which are essentially square-integrable functions and the plastic strains which are bounded measures; we attempt to clearly circumscribe those steps where duality is truly needed.

The paper is organized as follows.

In Section 2, we detail the structure of the envisioned periodic microstructures and apply the existence results for a quasi-static evolution that were derived in [11] to the specific setting at hand. It proves most convenient to view the periodic structure as that which is given on an N-dimensional torus, denoted henceforth by $\mathcal{Y}$. In Section 3, we state the various consequences of the existence result (maximal dissipation, flow rule, …) for an evolution that takes place exclusively on $\mathcal{Y}$. We do so because those results will then serve as the building block for the interpretation of the resulting "homogenized evolution" (an evolution in both the macroscopic variable x and its microscopic counterpart y), provided that the macroscopic dependence of all fields can be properly localized.

Elasto-plasticity gives rise to plastic strains that are merely bounded measures, so that the tools that will be used in the homogenization process have to account for weak* convergences in measure spaces. Since we have specialized the microstructures to the periodic setting, two-scale convergence is the usual tool that we extend to our specific setting. Of course, two-scale convergence of bounded measures has already been extensively discussed, starting with [2] in a BV setting. However, our measures are born out of the complex kinematics of elasto-plasticity, which is why we revisit the two-scale convergence process in this specific framework in Section 4. In the first subsection, we reframe the general existence result for two-scale limits of sequences of bounded measures, so as to prove in Lemma 4.6 a two-scale version of Reshetnyak's lower semicontinuity theorem (see e.g. [19, Theorem 1.7]); of course, we do not contend that Lemma 4.6 is new in and of itself. In Subsection 4.2, we characterize more specifically those measures that arise out of symmetrized gradients of BD functions (see Propositions 4.7 and 4.10), which in turn allows us to define the proper two-scale kinematics in Definition 5.1. Even when restricted to BV functions, our characterization is more elementary than that proposed in [2] because we avoid the use of Banach-space-valued measures (more specifically, of measures with values in periodic BV functions).

In Subsection 6.1, we address the homogenization process for the elasto-plastic evolution. To this end, we first have to prove a lower semicontinuity result for the dissipation in a two-scale setting (see Theorem 5.7) which is reminiscent of an analogous result in the heterogeneous setting [11, Proposition 2.3]. We then prove an inequality between two-scale dissipation and two-scale plastic work (Remark 5.13) which heavily relies on the results of Section 3. Finally, we prove that the heterogeneous elasto-plastic evolution of Section 2 two-scale converges at each time to a two-scale evolution (Theorem 6.2). That evolution is an evolution on the two-scale limits at each time, $u(t,x)$, $E(t,x,y)$, $P(t,x,y)$, of the various kinematic fields, i.e., the displacement field $u^\varepsilon(t)$, the elastic strain $e^\varepsilon(t)$, and the plastic strain $p^\varepsilon(t)$. In the resulting evolution, the y-dependence—that is, the dependence upon the micro-structural variable—cannot be integrated out, which results in a thermodynamical model with an infinite number of internal variables (essentially the plastic strains at each point y of the torus $\mathcal{Y}$).

In which sense is this still an elasto-plastic evolution? That is the question we address in the final subsection of this paper (Subsection 6.2). The goal is to recover some kind of flow rule, a harbinger of plasticity. This is achieved in Theorem 6.6 which demonstrates that, at almost every macroscopic point x, the two-scale plastic flow follows the rules of normality—that is, it is oriented along the normal to the yield surface, a y-dependent hypersurface—and this at all points of the torus $\mathcal{Y}$. The proof of Theorem 6.6 heavily relies upon Theorem 5.12 which is in turn a localized version of the previously mentioned Remark 5.13.

To achieve the results of Section 6 and in the spirit of e.g. [21], [14], [10], [11], we need to use the duality between plastic strain and its counterpart, the deviatoric stress. But those are not defined on the same set of macroscopic points x because the plastic strain is a measure in both x and y, which can thus concentrate in both variables, while the deviatoric stress is only defined $\mathcal{L}^N_x \otimes \mathcal{L}^N_y$-a.e. Consequently, to even make sense of the duality for a fixed x, we need to resort to the concept of disintegration of measures, Specifically, we need to disintegrate the two-scale kinematically admissible fields and to define the accompanying duality results. This is performed in the technical Section 5 which also includes the already mentioned lower semicontinuity result (Theorem 5.7) and the inequality between dissipation and the global stress-plastic strain duality product (Remark 5.13).

Because of that flow rule, we are seemingly at liberty to incorporate the resulting two-scale evolution into the framework of standard generalized materials advocated in [13]. To do so, however, we do need an infinite number of internal variables. Those are the plastic strains $P_x(t,y) := P(t,x,y)$, where $y \in \mathcal{Y}$. See Remark 6.7 for more details on the extent to which the previous statement is justified.

Finally, the reader will undoubtedly note that force loads are not considered in this work. As explained in [11, Remark 2.9], this is no restriction, provided that a uniform safe load condition with a smooth enough associated deviatoric stress is satisfied; for details refer to that remark in [11]. If that is not the case, then one should be very careful because, drawing a parallel with the discussion in [6], one should expect that, besides the bulk-type homogenization detailed in this work, a boundary-type homogenization also occurs.

1.2. Notation

The following notation will be adopted throughout.

General notation. For $A \subseteq \mathbb{R}^N$, 1_A denotes the characteristic function of A, i.e., $1_A(x) = 1$ for $x \in A$ and $1_A(x) = 0$ for $x \notin A$. The indicator function of A, denoted by $\mathbb{I}_A$, is defined as $\mathbb{I}_A(x) = 0$ for $x \in A$, and $\mathbb{I}_A(x) = +\infty$ for $x \notin A$. The symbol $\lfloor_A$ stands for "restricted to A". Finally $\mathcal{L}^N$ stands for the usual Lebesgue measure, while $\mathcal{H}^{N-1}$ denotes the $(N-1)$ dimensional Hausdorff measure.

Matrices. We denote by $\mathrm{M}^N_{\mathrm{sym}}$ the set of $(N \times N)$-symmetric matrices and by M^N_D the set of trace-free elements of $\mathrm{M}^N_{\mathrm{sym}}$. If M is an element of $\mathrm{M}^N_{\mathrm{sym}}$, then M_D denotes its deviatoric part, i.e., its projection onto the subspace M^N_D of $\mathrm{M}^N_{\mathrm{sym}}$ orthogonal to the identity mapping $\mathfrak{i}$ for the Frobenius inner product. The symbol $\cdot$ denotes that inner product. We denote by $\mathcal{L}_s(\mathrm{M}^N_D)$ the set of symmetric endomorphisms on M^N_D. For $a, b \in \mathbb{R}^N$, $a \odot b$ stands for the symmetric matrix such that $(a \odot b)_{ij} := (a_i b_j + a_j b_i)/2$.

Measures. If E is a locally compact separable metric space, and X a finite-dimensional normed space, $\mathcal{M}_b(E; X)$ will denote the space of finite Radon measures on E with values in X. For $\mu \in \mathcal{M}_b(E; X)$, we denote by $|\mu|$ its total variation. The space $\mathcal{M}_b(E; X)$ is the topological dual of $C^0_0(E; X^*)$, the set of continuous functions u from E to the vector dual X^* of X which "vanish at the boundary", i.e., for every $\varepsilon > 0$ there exists a compact set $K \subseteq E$ with $|u(x)| < \varepsilon$ for $x \notin K$. We will denote by $\mathcal{M}^+_b(E)$ the space of positive bounded Radon measures on E.

If $\mu \in \mathcal{M}^+_b(\mathbb{R}^N)$ we will denote by μ^s the singular part of μ with respect to the N-dimensional Lebesgue measure.

We will make extensive use of the technique of generalized products and disintegration of measures, for which we refer the reader to [4, Section 2.5]. Given E, F locally compact separable metric spaces, and $\eta \in \mathcal{M}^+_b(E)$, a map $x \mapsto \mu_x \in \mathcal{M}_b(F)$ is said to be *η-measurable* if the map $x \mapsto \mu_x(B)$ is η-measurable for every Borel set $B \subseteq F$. Assuming moreover that the map $x \mapsto |\mu_x|(F)$ is η-summable, the generalized product $\eta \overset{\text{gen.}}{\otimes} \mu_x \in \mathcal{M}_b(E \times F)$ is defined through the equality

$$\langle \eta \overset{\text{gen.}}{\otimes} \mu_x, f \rangle := \int_E \left(\int_F f(x, y)\, d\mu_x(y) \right) d\eta(x), \qquad f \in C^0_0(E \times F).$$

Moreover (see [4, Theorem 2.28]), every $\mu \in \mathcal{M}_b(E \times F)$ can be *disintegrated*, i.e., it can be written as a generalized product $\eta \overset{\text{gen.}}{\otimes} \mu_x$. Here η is the push forward of $|\mu|$ along the projection on E, i.e., for every Borel set $B \subseteq E$,

$$\eta(B) := |\mu|(B \times F),$$

while $x \mapsto \mu_x \in \mathcal{M}_b(F)$ is a suitable η-measurable map.

Further (see [4, Corollary 2.29]), $|\mu| = \eta \overset{\text{gen.}}{\otimes} |\mu_x|$.

The generalized product technique, and the associated disintegration result, are easily extended to the case of vector-valued finite Radon measure.

By contrast, if μ and ν are in $\mathcal{M}_b(E)$ and $\mathcal{M}_b(F)$, respectively, we will denote by $\mu \otimes \nu$ the classical product measure in $\mathcal{M}_b(E \times F)$. Let us emphasize that, if $\pi \in \mathcal{M}_b(E \times F)$ disintegrates as $\pi = \mu \overset{\text{gen.}}{\otimes} [a(x, y)\nu]$, then we cannot assert *a priori* that a is $\mu \otimes \nu$-measurable. However, π is then absolutely continuous with respect to $\mu \otimes \nu$, so that there exists a Borel map $h : E \times F \to \mathbb{R}$ such that $\pi = h(x, y)(\mu \otimes \nu)$. The relation between h and a will have to be established on a case-by-case basis and this will be a source of difficulties in the proof of Proposition 4.7 and in Lemma 5.4. In the case where $E = F = \mathbb{R}$ and $\mu = \nu = \mathcal{L}^1$, an example due to W. Sierpiński provides the existence of a non-measurable set $A \subset \mathbb{R}^2$ such that all its sections $A_x := \{y \in \mathbb{R} : (x, y) \in A\}$ are reduced to a point (see [12]). Then

$$\mathcal{L}^1_x \overset{\text{gen.}}{\otimes} 1_A(x, y)\mathcal{L}^1_y \equiv 0,$$

so that adding $1_A(x, y)$ to $a(x, y)$ will clearly prevent any possible identification of h to a.

The (kinematic) space *BD*. Let $\Omega \subseteq \mathbb{R}^N$ be an open set. In this paper as in previous works on elasto-plasticity the displacement field u lies in $BD(\Omega)$, the space of functions with bounded deformations. We refer the reader to e.g. [21, Chapter 2] and [3] for background material. Besides elementary properties of $BD(\Omega)$, we will only appeal to two "finer" results. Firstly, the measure Eu does not charge $\mathcal{H}^{N-1}$-negligible sets (see [3, Remark 3.3]). Secondly, assuming that Ω is bounded with Lipschitz boundary and given $\Gamma_d \subseteq \partial\Omega$ with $\mathcal{H}^{N-1}(\Gamma_d) > 0$, Poincaré–Korn's inequality states that there exists $C > 0$ such that

$$\|u\|_{BD(\Omega)} \le C\left(\int_{\Gamma_d} |u|\, d\mathcal{H}^{N-1} + \|Eu\|_{\mathcal{M}_b(\Omega;\mathbb{M}^N_{\text{sym}})}\right),$$

where Eu denotes the symmetrized gradient of u, and the integral on Γ_d involves the trace of u on $\partial\Omega$ which is well defined as an element of $L^1(\partial\Omega; \mathbb{R}^N)$; see [21, Chapter 2, Remark 2.5(ii)].

We say that

$$u_n \overset{*}{\rightharpoonup} u \quad \text{weakly* in } BD(\Omega)$$

iff

$$u_n \to u \quad \text{strongly in } L^1(\Omega; \mathbb{R}^N) \quad \text{and} \quad Eu_n \overset{*}{\rightharpoonup} Eu \quad \text{weakly* in } \mathcal{M}_b(\Omega; \mathbb{M}^N_{\text{sym}}).$$

If Ω is bounded and Lipschitz, bounded sequences in $BD(\Omega)$ always admit a weakly* converging subsequence.

Functional spaces. Given $E \subseteq \mathbb{R}^N$ measurable, $1 \le p < \infty$, and M a finite-dimensional normed space, $L^p(E; M)$ stands for the space of p-summable functions on E with values in M, with associated norm denoted by $\|\cdot\|_p$. Given $A \subseteq \mathbb{R}^N$ open, $H^1(A; M)$ is the Sobolev space of functions in $L^2(A; M)$ with distributional derivatives in L^2.

Finally, let X be a normed space. We denote by $BV(a, b; X)$ and $AC(a, b; X)$ the space of functions with bounded variation and the space of absolutely continuous functions from $[a, b]$ to X, respectively. We recall that the total variation of $f \in BV(a, b; X)$

is defined as

$$\mathcal{V}_X(f; a, b) := \sup\Big\{\sum_{j=1}^{k} \|f(t_j) - f(t_{j-1})\|_X \,:\, a = t_0 < t_1 < \cdots < t_k = b\Big\}.$$

Periodicity. Our analysis of the homogenization problem relies on an extensive use of two-scale convergence (see Section 4). We thus need to consider the space of $[0, 1]^N$-periodic continuous (or C^1) functions on $\mathbb{R}^N$, and its dual, a space of measures that enjoys suitable periodicity properties. These spaces are most conveniently viewed as acting on a torus.

Let $\mathcal{Y} := \mathbb{R}^N/\mathbb{Z}^N$ be the N-dimensional torus, $Y := [0, 1)^N$, and let $\mathcal{I} : \mathcal{Y} \to Y$ denote the corresponding canonical identification. For future reference, we set

$$\mathcal{C} := \mathcal{I}^{-1}(\partial Y). \tag{1.1}$$

For any $\mathcal{Z} \subset \mathcal{Y}$, we define

$$\mathcal{Z}_\varepsilon := \{x \in \mathbb{R}^N : x/\varepsilon \in \mathbb{Z}^N + \mathcal{I}(\mathcal{Z})\}, \tag{1.2}$$

while for any function $F : \mathcal{Y} \to X$, where X is some set, the ε-periodic function $F_\varepsilon : \mathbb{R}^N \to X$ is defined as

$$F_\varepsilon(x) := F(y_\varepsilon) \quad \text{with } x/\varepsilon - [x/\varepsilon] = \mathcal{I}(y_\varepsilon) \in Y. \tag{1.3}$$

The ε-periodic function F_ε will be abbreviated as $F(x/\varepsilon)$ unless confusion might ensue.

Remark 1.1. Note that, if $\mathcal{D}$ is a Lipschitz hypersurface in $\mathcal{Y}$, then the normal $\nu_\varepsilon(x)$ at a given point $x \in \mathcal{D}_\varepsilon$ is actually of the form $\nu(y)$ for some $y \in \mathcal{Y}$.

Throughout the paper, if X a finite-dimensional vector space, we will identify the space of $[0, 1]^N$-periodic and continuous (resp. C^1) functions with values in X with $C^0(\mathcal{Y}; X)$ (resp. $C^1(\mathcal{Y}; X)$). The dual space is then naturally identified with $\mathcal{M}_b(\mathcal{Y}; X)$.

For our applications to plasticity, we need to consider *BD* functions on $\mathcal{Y}$, i.e., those functions $u \in L^1(\mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}})$ whose symmetrized gradient $E_y u$—defined by means of a local coordinates system associated with the very definition of $\mathcal{Y}$ as a quotient space—is a finite Radon measure on $\mathcal{Y}$ with values in $\mathrm{M}^N_{\mathrm{sym}}$. These can be identified with those functions $u : \mathbb{R}^N \to \mathbb{R}^N$ which are locally *BD* and Y-periodic. In other words, besides Y-periodicity, there exists $C > 0$ such that

$$\left|\int_Y u \cdot \operatorname{div} \psi \, dx\right| \le C\|\psi\|_\infty$$

for every $\psi \in C^1_{\mathrm{per}}([0, 1]^N; \mathrm{M}^N_{\mathrm{sym}})$. Thanks to periodicity, if $u \in BD(\mathcal{Y})$ is such that $E_y u = 0$, that is, if u is a periodic “infinitesimal rigid body motion”, then u is a constant vector on $\mathcal{Y}$. In particular, we will use the following form of the Poincaré–Korn inequality on $BD(\mathcal{Y})$: there exists $C > 0$ such that for every $u \in BD(\mathcal{Y})$ with $\int_{\mathcal{Y}} u \, dy = 0$,

$$\int_{\mathcal{Y}} |u| \, dy \le C|E_y u|(\mathcal{Y}).$$

2. Quasi-static evolutions in periodic heterogeneous materials

In this section we detail the structure of *periodic* heterogeneous materials and of elasto-plastic evolutions for such materials.

The reference configuration. In all that follows, $\Omega \subset \mathbb{R}^N$ is an open, bounded set with (at least) Lipschitz boundary and exterior normal ν. Further, the Dirichlet part Γ_d of $\partial\Omega$ is a non-empty open set in the relative topology of $\partial\Omega$ with boundary $\partial\lfloor_{\partial\Omega}\Gamma_d$ in $\partial\Omega$ and we set $\Gamma_t := \partial\Omega \setminus \bar{\Gamma}_d$. Reproducing the setting of [11, Section 6], we introduce the following

Definition 2.1. We will say that $\partial\lfloor_{\partial\Omega}\Gamma_d$ is *admissible* iff, for any $\sigma \in L^2(\Omega; \mathrm{M}^N_{\mathrm{sym}})$ with

$$\operatorname{div}\sigma = f \text{ in } \Omega, \qquad \sigma\nu = g \text{ on } \Gamma_t, \qquad \sigma_D \in L^\infty(\Omega; \mathrm{M}^N_D) \tag{2.1}$$

where $f \in L^N(\Omega; \mathbb{R}^N)$ and $g \in L^\infty(\Gamma_t; \mathbb{R}^N)$, and every $p \in \mathcal{M}_b(\Omega \cup \Gamma_d; \mathrm{M}^N_D)$ such that there exists an associated pair $(u, e) \in BD(\Omega) \times L^{N/(N-1)}(\Omega; \mathrm{M}^N_{\mathrm{sym}})$ with

$$Eu = e + p \quad \text{in } \Omega, \qquad p = (w - u) \odot \nu\mathcal{H}^{N-1}\lfloor_{\Gamma_d} \quad \text{on } \Gamma_d,$$

the distribution, defined for all $\varphi \in C^\infty_c(\mathbb{R}^N)$ by

$$\begin{aligned}\langle\sigma_D, p\rangle(\varphi) := &-\int_\Omega \varphi\sigma \cdot (e - Ew)\,dx - \int_\Omega \varphi f \cdot (u - w)\,dx \\ &-\int_\Omega \sigma \cdot [(u - w) \odot \nabla\varphi]\,dx + \int_{\Gamma_t} \varphi g \cdot (u - w)\,d\mathcal{H}^{N-1}\end{aligned} \tag{2.2}$$

is a bounded Radon measure on $\mathbb{R}^N$ with $|\langle\sigma_D, p\rangle| \le \|\sigma_D\|_\infty |p|$.

Definition 2.1 covers many "practical" settings like those of a hypercube with one of its faces being the Dirichlet part Γ_d of the boundary; see [11, Section 6] for that and other such settings.

Remark 2.2. Expression (2.2) defines a meaningful distribution on $\mathbb{R}^N$. Indeed, according to [11, Proposition 6.1], if $\sigma \in L^2(\Omega; \mathrm{M}^N_{\mathrm{sym}})$ is such that $\operatorname{div}\sigma \in L^N(\Omega; \mathbb{R}^N)$ and $\sigma_D \in L^\infty(\Omega; \mathrm{M}^N_D)$, then $\sigma \in L^r(\Omega; \mathrm{M}^N_{\mathrm{sym}})$ for every $1 \le r < \infty$ with

$$\|\sigma\|_r \le C_r(\|\sigma\|_2 + \|\operatorname{div}\sigma\|_N + \|\sigma_D\|_\infty)$$

for some $C_r > 0$. On the other hand, $u \in L^{N/(N-1)}(\Omega; \mathbb{R}^N)$ in view of the embedding of $BD(\Omega)$ into $L^{N/(N-1)}(\Omega; \mathbb{R}^N)$. Further, u has a trace on $\partial\Omega$ which belongs to $L^1(\partial\Omega; \mathbb{R}^N)$. Finally note that, if σ is the restriction to Ω of a C^1-function and if $\mathcal{H}^{N-1}(\partial\lfloor_{\partial\Omega}\Gamma_d) = 0$, then, performing an integration by parts in BD (see [21, Chapter 2, Theorem 2.1]), the right hand side of (2.2) coincides with the integral of φ with respect to the (well defined) measure $\sigma_D p$.

Geometry. Let $Y := [0,1)^N$ be the unit cell in $\mathbb{R}^N$, while $\mathcal{Y}$ is the associated N-dimensional torus. We view $\mathcal{Y}$ as being made of finitely many phases $\mathcal{Y}_i$, together with their interfaces, i.e., $\mathcal{Y} = \bigcup \bar{\mathcal{Y}}_i$. We assume that those phases are pairwise disjoint open sets with Lipschitz boundary. Moreover it is not restrictive to assume that the transversality condition

$$\mathcal{H}^{N-1}(\partial\mathcal{Y}_i \cap \mathcal{C}) = 0 \tag{2.3}$$

holds true ($\mathcal{C}$ was defined in (1.1)). This can be achieved by a translation of the unit cell Y, and a suitable redefining of the associated identification map $\mathcal{I} : \mathcal{Y} \to Y$.

Denoting by Γ the interfaces, i.e.,

$$\Gamma := \bigcup_{i,j} \partial\mathcal{Y}_i \cap \partial\mathcal{Y}_j,$$

we assume that there exists a compact set $\mathcal{S} \subset \Gamma$ with $\mathcal{H}^{N-1}(\mathcal{S}) = 0$ and

$$\Gamma \setminus \mathcal{S} \text{ is a } C^1\text{-hypersurface.}$$

We will write

$$\Gamma = \bigcup_{i \neq j} \Gamma_{ij},$$

where Γ_{ij} stands for the interface between $\mathcal{Y}_i$ and $\mathcal{Y}_j$.

A torus $\mathcal{Y}$ that satisfies the collection of those (minimal) assumptions will be referred to henceforth as a *geometrically admissible multiphase torus*.

Throughout the rest of this paper it will be assumed that $\mathcal{Y}$ is *a geometrically admissible multiphase torus*. If, further, $\Gamma \setminus \mathcal{S}$ is a C^2-hypersurface, then $\mathcal{Y}$ will be referred to as a *C^2-geometrically admissible multiphase torus*.

Given $\varepsilon > 0$, we assume that our domain Ω is made up of the various phases $(\mathcal{Y}_i)_\varepsilon$ (see (1.2)). Note that, provided that ε is chosen such that $\mathcal{H}^{N-1}((\bigcup_i (\partial\mathcal{Y}_i)_\varepsilon) \cap \Gamma_d) = 0$, then each point of Γ_d outside an $\mathcal{H}^{N-1}$-negligible set belongs to a well defined phase. Therefore, $\Omega \cup \Gamma_d$ is a geometrically admissible multiphase domain in the sense of [11, Subsection 1.2]. Only those ε's will be considered from this point on.

Kinematic admissibility. Given the boundary displacement $w \in H^1(\Omega; \mathbb{R}^N)$, we adopt the following

Definition 2.3 (Admissible configurations). $\mathcal{A}(w)$, the family of *admissible configurations* relative to w, is the set of triplets (u, e, p) with

$$u \in BD(\Omega), \qquad e \in L^2(\Omega; \mathrm{M}^N_{\mathrm{sym}}), \qquad p \in \mathcal{M}_b(\Omega \cup \Gamma_d; \mathrm{M}^N_D),$$

and such that

$$Eu = e + p \quad \text{in } \Omega, \qquad p = (w - u) \odot \nu \mathcal{H}^{N-1} \lfloor_{\Gamma_d} \quad \text{on } \Gamma_d, \tag{2.4}$$

where ν denotes the outer normal to $\partial\Omega$ and $w - u$ is to be understood in the sense of traces.

The function u denotes the displacement field on Ω, while e and p are the associated elastic and plastic strains. In view of the additive decomposition (2.4) of Eu and of the general properties of BD functions recalled earlier, p does not charge $\mathcal{H}^{N-1}$-negligible sets. Moreover, given a Lipschitz hypersurface $D \subset \Omega$ dividing Ω locally into the subdomains Ω^+ and Ω^-, we have

$$p\lfloor_D = (u^+ - u^-) \odot \nu \mathcal{H}^{N-1}\lfloor_D,$$

where ν is the normal to D pointing from Ω^- to Ω^+, and $u^\pm$ are the traces on D of the restrictions of u to $\Omega^\pm$. Since p is assumed to take values in the space of deviatoric matrices M^N_D, $u^+ - u^-$ is perpendicular to ν, so that only particular plastic strains are activated along D.

These properties will be used below when defining the plastic properties of the multiphase material Ω.

Elastic and plastic properties. The elasto-plastic properties of Ω are given in terms of a periodic elastic tensor and a periodic dissipation potential.

The elasticity tensor. We consider elasticity tensors (Hooke's law) of the form

$$\mathbb{C}(y)M := \mathbb{C}_D(y)M_D + k(y)\,\mathrm{tr}(M)\mathrm{i}, \qquad y \in \mathcal{Y}, \tag{2.5}$$

with $\mathbb{C}_D := (\mathbb{C}_D)_i \in \mathcal{L}_s(\mathrm{M}^N_D)$ and $k := k_i > 0$ on every $\mathcal{Y}_i$, with $(\mathbb{C}_D)_i$ such that

$$(\mathbb{C}_D)_i M \cdot M \geq c_1 |M|^2, \quad \forall M \in \mathrm{M}^N_D; \quad k_i \geq c_1, \tag{2.6}$$

for some $c_1 > 0$.

For every $\varepsilon > 0$ and $e \in L^2(\Omega; \mathrm{M}^N_{\mathrm{sym}})$ we consider the elastic energy

$$\mathcal{Q}_\varepsilon(e) := \frac{1}{2}\int_\Omega \mathbb{C}_\varepsilon e \cdot e\, dx, \tag{2.7}$$

where $\mathbb{C}_\varepsilon(x) := \mathbb{C}(x/\varepsilon)$ for every $x \in \Omega$ (see (1.3)).

The set of admissible stresses. In elasto-plasticity, the deviatoric part of the stress σ is restricted by the yield condition. Thus, here, we are led to assuming the existence of a convex compact set $K_i \subset \mathrm{M}^N_D$ for each phase $\mathcal{Y}_i$. We further assume that those sets cannot be too small or too large, i.e., there exist $c_3, c_4 > 0$ such that for every i,

$$B(0, c_3) \subset K_i \subset B(0, c_4). \tag{2.8}$$

We define

$$K(y) := K_i \quad \text{for } y \in \mathcal{Y}_i, \tag{2.9}$$

and $K_\varepsilon(x) = K(x/\varepsilon)$ for $x \in \Omega$.

Our formulation of the problem uses the Legendre transform of $\mathbb{I}_{K_i}$, which is often referred to as the dissipation potential.

The dissipation potential. For all $y \in \mathcal{Y}_i$ and $\xi \in \mathbb{M}^N_D$, we define the *dissipation potential* to be

$$H(y, \xi) = H_i(\xi) := \sup\{\tau \cdot \xi : \tau \in K_i\}. \tag{2.10}$$

This defines, for a.e. $y \in \mathcal{Y}$, a convex, one-homogeneous function in ξ which further satisfies

$$c_3|\xi| \le H(y, \xi) \le c_4|\xi| \quad \text{for a.e. } y \in \mathcal{Y}.$$

This is not however sufficient for our purpose because we need the dissipation potential to act upon the plastic strain (or plastic strain rate) which, being a measure, may concentrate on sets of zero Lebesgue measure. Moreover, plastic strains can concentrate on the inner interfaces where they will only activate particular strain directions, as previously mentioned. We thus have to extend H to every point in $\mathcal{Y} \times \mathbb{M}^N_D$.

The dissipation potential $H : \mathcal{Y} \times \mathbb{M}^N_D \to [0, +\infty]$ of a *geometrically admissible multiphase torus* is constructed as follows.

(a) In each phase $\mathcal{Y}_i$, we take

$$H(y, \xi) = H_i(\xi) \quad \text{for } y \in \mathcal{Y}_i$$

with $H_i : \mathbb{M}^N_D \to \mathbb{R}$ such that

$$\xi \mapsto H_i(\xi) \text{ is convex and positively one-homogeneous in } \xi \tag{2.11}$$

with

$$c_3|\xi| \le H_i(\xi) \le c_4|\xi|, \tag{2.12}$$

where $c_3, c_4 > 0$ are independent of the phase i.

(b) At a point $y \in \Gamma \setminus \mathcal{S}$ on the interface between $\mathcal{Y}_i$ and $\mathcal{Y}_j$ such that the associated normal $\nu(y)$ points from $\mathcal{Y}_j$ to $\mathcal{Y}_i$, we set

$$\begin{cases} H(y, \xi) := H_{ij}(a, \nu(y)) & \text{for every } \xi = a \odot \nu(y) \in \mathbb{M}^N_D, \\ H(y, \xi) = +\infty & \text{otherwise on } \mathbb{M}^N_D, \end{cases} \tag{2.13}$$

where for every $a \in \mathbb{R}^N$ and $\nu \perp a \in S^{N-1}$,

$$H_{ij}(a, \nu) := \inf\{H_i(a_i \odot \nu) + H_j(a_j \odot \nu) : a = a_i + a_j,\ a_i, a_j \in \mathbb{R}^N,\ a_i \perp \nu,\ a_j \perp \nu\}.$$

Note that $\xi \mapsto H(y, \xi)$ is convex and positively one-homogeneous and, for every $a \odot \nu(y) \in \mathbb{M}^N_D$,

$$c_3|a \odot \nu(y)| \le H(y, a \odot \nu(y)) \le c_4|a \odot \nu(y)|. \tag{2.14}$$

Also observe that, since H_i, H_j are continuous functions of ξ, ν is a continuous function of $y \in \Gamma \setminus S$, while, by coercivity, the infimum in the inf-convolution is actually a minimum, $H(y, \xi)$ is lower semicontinuous on $(\Gamma \setminus \mathcal{S}) \times \mathbb{M}^N_D$. Thus $(y, \xi) \mapsto H(y, \xi)$ is a Borel function.

(c) Finally, we define $H(y, \xi)$ arbitrarily for $y \in \mathcal{S}$, for example as $c_3|\xi|$, since those points will not be relevant for the admissible plastic strains because $\mathcal{H}^{N-1}(\mathcal{S}) = 0$.

It is readily seen that the resulting dissipation potential $H : \mathcal{Y} \times \mathrm{M}^N_D \to [0, +\infty]$ is a Borel function.

Remark 2.4. By convex conjugation, we can associate with the dissipation at $y \in \varGamma_{ij} \setminus \mathcal{S}$ the set

$$K(y) = \{\Sigma_D \in \mathrm{M}^N_D : (\Sigma_D \nu(y))_\tau \in (K_i \nu(y))_\tau \cap (K_j \nu(y))_\tau\},$$

where $(\cdot)_\tau$ denotes the orthogonal projection to the hyperplane tangent to $\varGamma_{ij}$ at y. Notice that $K(y)$ is a cylinder in M^N_D. We take the view that this is a constraint on the vector $(\Sigma_D \nu(y))_\tau$, rather than on the matrix Σ_D. Set

$$K_\varGamma(y) := (K_i \nu(y))_\tau \cap (K_j \nu(y))_\tau \subseteq \mathbb{R}^N. \tag{2.15}$$

In that way, $\mathbb{I}_{K_\varGamma(y)}$ is the Legendre transform of the map $a \mapsto H(y, a \odot \nu(y))$ with $a \perp \nu(y)$, and conversely.

Coming to the periodic multiphase material, we consider the dissipation potential

$$H_\varepsilon : (\varOmega \cup \varGamma_d) \times \mathrm{M}^N_D \to [0, +\infty]$$

defined as (see (1.3))

$$H_\varepsilon(x, \xi) := H(x/\varepsilon, \xi).$$

For every $p \in \mathcal{M}_b(\varOmega \cup \varGamma_d; \mathrm{M}^N_D)$ we define the dissipation functional to be

$$\mathcal{H}_\varepsilon(p) := \int_{\varOmega \cup \varGamma_d} H_\varepsilon(x, p/|p|)\, d|p|, \tag{2.16}$$

where, from now onward, for any bounded Radon measure q on $\mathbb{R}^N$, $q/|q|$ denotes the Radon–Nikodym derivative of q with respect to its total variation $|q|$.

If $t \mapsto p(t)$ is a map from $[0, T]$ to $\mathcal{M}_b(\varOmega \cup \varGamma_d; \mathrm{M}^N_D)$, we finally define the total dissipation over an interval $[a, b] \subseteq [0, T]$ to be

$$\mathcal{D}_\varepsilon(a, b; p) := \sup\Big\{\sum_{j=1}^{k} \mathcal{H}_\varepsilon(p(t_j) - p(t_{j-1})) : a = t_0 < t_1 < \cdots < t_k = b\Big\}.$$

Quasistatic evolutions. We prescribe the boundary displacement w on $\varGamma_d$ as the trace on $\varGamma_d$ of

$$w \in AC(0, T; H^1(\mathbb{R}^N; \mathbb{R}^N)). \tag{2.17}$$

We now have all the ingredients for defining a quasi-static evolution as follows.

Definition 2.5 (Quasistatic evolution). We say that $t \mapsto (u_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t)) \in \mathcal{A}(w(t))$ is an *ε-quasi-static evolution* relative to w provided that the following conditions hold for every $t \in [0, T]$.

(a) Global stability: for every $(v, \eta, q) \in \mathcal{A}(w(t))$,

$$\mathcal{Q}_\varepsilon(e_\varepsilon(t)) \leq \mathcal{Q}_\varepsilon(\eta) + \mathcal{H}_\varepsilon(q - p_\varepsilon(t)). \tag{2.18}$$

(b) Energy equality: $t \mapsto p_\varepsilon(t)$ has bounded variation from $[0, T]$ to $\mathcal{M}_b(\Omega \cup \Gamma_d; \mathrm{M}^N_D)$ and

$$\mathcal{Q}_\varepsilon(e(t)) + \mathcal{D}_\varepsilon(0, t; p_\varepsilon) = \mathcal{Q}_\varepsilon(e(0)) + \int_0^t \int_\Omega \sigma_\varepsilon(\tau) \cdot E\dot{w}(\tau)\, dx\, d\tau \quad \text{with } \sigma_\varepsilon(t) := \mathbb{C}_\varepsilon e_\varepsilon(t).$$

The following existence result has been established in [11, Theorem 2.7].

Theorem 2.6 (Existence of a heterogeneous evolution). *Assume that* (2.5), (2.6), (2.11), (2.12), (2.13), (2.17) *are satisfied. Let* $(u^0_\varepsilon, e^0_\varepsilon, p^0_\varepsilon) \in \mathcal{A}(w(0))$ *satisfy the global stability condition* (2.18). *Then there exists a quasi-static evolution* $t \mapsto (u_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t))$ *relative to the boundary displacement* w *such that* $(u_\varepsilon(0), e_\varepsilon(0), p_\varepsilon(0)) = (u^0_\varepsilon, e^0_\varepsilon, p^0_\varepsilon)$.

Remark 2.7 (Balance equations). According to [11, Theorem 3.6], $\sigma_\varepsilon(t)$ satisfies the balance equation and the admissibility conditions, i.e.,

$$\begin{gathered} \operatorname{div} \sigma_\varepsilon(t) = 0 \quad \text{in } \Omega, \qquad \sigma_\varepsilon(t)\nu = 0 \quad \text{on } \partial\Omega \setminus \bar{\Gamma}_d, \\ (\sigma_\varepsilon)_D(t, x) \in K_\varepsilon(x) \quad \text{for a.e. } x \in \Omega. \end{gathered}$$

We set

$$\mathcal{K}_\varepsilon := \{\sigma \in L^2(\Omega; \mathrm{M}^N_{\mathrm{sym}}) : \operatorname{div} \sigma = 0 \text{ in } \Omega,\ \sigma\nu = 0 \text{ on } \partial\Omega \setminus \bar{\Gamma}_d, \\ \sigma_D(x) \in K_\varepsilon(x) \text{ for a.e. } x \in \Omega\}, \tag{2.19}$$

and we refer to $\mathcal{K}_\varepsilon$ as the family of ε-*statically admissible* stress fields.

3. Elasto-plasticity on the periodic torus

In this section, we collect a few results which are consequences of [11] in a periodic setting: they will be useful when dealing with the homogenization of quasi-static evolutions in periodic heterogeneous materials.

Let $\mathcal{Y}$ be a geometrically admissible multiphase torus according to Section 2.

Definition 3.1 (Periodic admissible configurations). The family $\mathcal{A}_\mathcal{Y}$ of *admissible configurations* on $\mathcal{Y}$ is given by the set of triplets

$$u \in BD(\mathcal{Y}), \qquad E \in L^2(\mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}}), \qquad P \in \mathcal{M}_b(\mathcal{Y}; \mathrm{M}^N_D)$$

such that

$$E_y u = E + P \quad \text{on } \mathcal{Y}.$$

We set

$$\Pi_\mathcal{Y} := \{P \in \mathcal{M}_b(\mathcal{Y}; \mathrm{M}^N_D) : \exists (u, E) \text{ such that } (u, E, P) \in \mathcal{A}_\mathcal{Y}\}.$$

Recalling (2.9), we adopt the following

Definition 3.2 (Periodic statically admissible stresses). $\Sigma \in L^2(\mathcal{Y}; \mathbb{M}^N_{\mathrm{sym}})$ is said to be a *statically admissible stress* on the torus if

$$\operatorname{div}_y \Sigma = 0 \quad \text{on } \mathcal{Y}$$

and

$$\Sigma_D(y) \in K(y) \quad \text{for a.e. } y \in \mathcal{Y}.$$

We denote the set of all such stresses by $\mathcal{K}_\mathcal{Y}$.

If $\Sigma \in \mathcal{K}_\mathcal{Y}$, then in particular $\Sigma_D \in L^\infty(\mathcal{Y}; \mathbb{M}^N_{\mathrm{sym}})$, from which it is deduced (see [11, Proposition 6.1]) that $\Sigma \in L^r(\mathcal{Y}; \mathbb{M}^N_{\mathrm{sym}})$ for every $1 \leq r < \infty$ with

$$\|\Sigma\|_r \leq C_r(\|\Sigma\|_2 + \|\Sigma_D\|_\infty) \tag{3.1}$$

for some $C_r > 0$.

Moreover, considering the interfaces Γ, it is possible to define a tangential trace for $\Sigma\nu$ on $\Gamma \setminus \mathcal{S}$,

$$(\Sigma\nu)_\tau \in L^\infty(\Gamma; \mathbb{R}^N),$$

in the following way. Consider a smooth approximation $\Sigma_n \in C^\infty(\mathcal{Y}; \mathbb{M}^N_{\mathrm{sym}})$ such that

$$\begin{cases} \Sigma_n \to \Sigma & \text{strongly in } L^2(\mathcal{Y}; \mathbb{M}^N_{\mathrm{sym}}), \\ \operatorname{div}_y \Sigma_n \to 0 & \text{strongly in } L^2(\mathcal{Y}; \mathbb{R}^N), \\ \|(\Sigma_n)_D\|_\infty \leq \|\Sigma_D\|_\infty, & \end{cases}$$

and consider $(\Sigma_n\nu)_\tau := (\Sigma_n)\nu - ((\Sigma_n)\nu \cdot \nu)\nu$ (the tangential component of $(\Sigma_n)_D$ is defined analogously). It is then immediate that $(\Sigma_n\nu)_\tau = ((\Sigma_n)_D\nu)_\tau$. Since $y \mapsto \nu(y)$ is an $L^\infty(\Gamma; \mathbb{R}^N)$-mapping, there exists an $L^\infty(\Gamma; \mathbb{R}^N)$-function $(\Sigma\nu)_\tau$ such that, up to a subsequence,

$$(\Sigma_n\nu)_\tau \overset{*}{\rightharpoonup} (\Sigma\nu)_\tau \quad \text{weakly* in } L^\infty(\Gamma; \mathbb{R}^N).$$

$(\Sigma\nu)_\tau$ is only a function of $\{(\Sigma_n)_D\nu\}_{n\in\mathbb{N}}$ which we will denote henceforth by $(\Sigma_D\nu)_\tau$. Notice that $(\Sigma_D\nu)_\tau$ may depend upon the approximation sequence $\{\Sigma_n\}_{n\in\mathbb{N}}$ (or at least upon $\{(\Sigma_n)_D\}_{n\in\mathbb{N}}$). If $\Gamma \setminus \mathcal{S}$ is a C^2-hypersurface, then $(\Sigma_D\nu)_\tau$ is uniquely determined as an element of $L^\infty(\Gamma; \mathbb{R}^N)$. Indeed, considering Γ_{ij}, for every $\varphi \in H^{1/2}_{00}(\Gamma_{ij}; \mathbb{R}^N)$, it is readily seen that

$$\int_{\Gamma_{ij}} (\Sigma\nu)_\tau \cdot \varphi \, d\mathcal{H}^{N-1} = \langle \Sigma\nu, \varphi\rangle - \langle (\Sigma\nu)_\nu, \varphi\rangle, \quad \text{where} \quad \langle (\Sigma\nu)_\nu, \varphi\rangle := \langle \Sigma\nu, (\varphi \cdot \nu)\nu\rangle.$$

Since the normal component $(\varphi \cdot \nu)\nu$ of φ with respect to Γ_{ij} belongs to $H^{1/2}_{00}(\Gamma_{ij}; \mathbb{R}^N)$ in view of the regularity of ν, the definition of $(\Sigma\nu)_\nu$ is meaningful.

The following result is a consequence of [11, Section 6 and Lemma 3.8].

Theorem 3.3 (Duality). *Let $P \in \Pi_{\mathcal{Y}}$ and $\Sigma \in \mathcal{K}_{\mathcal{Y}}$. Then the distribution*

$$\langle \Sigma_D, P\rangle(\psi) := -\int_{\mathcal{Y}} \psi(y)\Sigma \cdot E\, dy - \int_{\mathcal{Y}} \Sigma \cdot [u \odot \nabla\psi]\, dy, \qquad \psi \in C^1(\mathcal{Y}), \tag{3.2}$$

is a bounded Radon measure on $\mathcal{Y}$ such that

$$|\langle \Sigma_D, P\rangle| \leq \|\Sigma_D\|_\infty |P|.$$

Moreover, for every $i \neq j$, and for every tangential trace $(\Sigma_D \nu)_\tau$,

$$\langle \Sigma_D, P\rangle \lfloor_{\Gamma_{ij}} = (\Sigma_D \nu)_\tau \cdot (u^i - u^j)\mathcal{H}^{N-1}\lfloor_{\Gamma_{ij}},$$

where ν points from $\mathcal{Y}_j$ to $\mathcal{Y}_i$, and u^i, u^j are the traces on Γ_{ij} of the restrictions of u on $\mathcal{Y}_i$ and $\mathcal{Y}_j$.

Remark 3.4. Note that the proof of Lemma 3.8 in [11] only requires that $\Sigma_D \in L^\infty(\mathcal{Y}; \mathrm{M}^N_D)$ and thus the requirement that $\Sigma \in \mathcal{K}_{\mathcal{Y}}$ in the previous theorem can be weakened to $\Sigma \in L^2(\mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}})$ with $\mathrm{div}_y\, \Sigma = 0$ on $\mathcal{Y}$ and $\Sigma_D \in L^\infty(\mathcal{Y}; \mathrm{M}^N_D)$.

The following result holds true (see [11, Proposition 3.9 and Theorem 3.13]).

Proposition 3.5. *Let $(u, E, P) \in \mathcal{A}_{\mathcal{Y}}$, $\Sigma \in \mathcal{K}_{\mathcal{Y}}$, and let $\mathcal{Y}$ be a C^2-admissible multi-phase torus. Then*

$$H(y, P/|P|)|P| \geq \langle \Sigma_D, P\rangle \quad \text{as measures on } \mathcal{Y}.$$

If moreover equality holds, then

$$\frac{P}{|P|}(y) \in N_{K(y)}(\Sigma_D(y)) \quad \text{for } \mathcal{L}^N\text{-a.e. } y \in \{|P| > 0\},$$

where $N_{K(y)}(\Sigma_D(y))$ denotes the normal cone to $K(y)$ at $\Sigma_D(y)$, and, for every $i \neq j$,

$$\frac{u^i - u^j}{|u^i - u^j|} \in \vec{N}_{K_\Gamma(y)}((\Sigma_D \nu)_\tau(y)) \quad \text{for } \mathcal{H}^{N-1}\text{-a.e. } y \in \{u^i \neq u^j\},$$

where ν points from $\mathcal{Y}_j$ to $\mathcal{Y}_i$, u^i, u^j are the traces on Γ_{ij} of the restrictions of u on $\mathcal{Y}_i$ and $\mathcal{Y}_j$, and $\vec{N}_{K_\Gamma(y)}(\tau)$ denotes the normal cone—a cone of vectors—to $K_\Gamma(y)$ at a vector $\tau \perp \nu(y)$.

4. Two-scale convergence of measures

In this section we recall the definition and the main properties of two-scale convergence for Radon measures proved in [2]. We also prove a structure result for the two-scale limit of symmetrized gradients of weakly* converging sequences of *BD* functions.

4.1. *Definitions and basic properties*

We adopt the following

Definition 4.1 (Two-scale measure convergence). Let $\Omega \subseteq \mathbb{R}^N$ be an open set, $\{\mu_\varepsilon\}_{\varepsilon>0}$ be a family in $\mathcal{M}_b(\Omega)$ and consider $\mu \in \mathcal{M}_b(\Omega \times \mathcal{Y})$. Then

$$\mu_\varepsilon \xrightarrow{w^*\text{-}2} \mu_0 \quad \text{two-scale weakly* in } \mathcal{M}_b(\Omega \times \mathcal{Y})$$

iff, for every $\chi \in C_0^0(\Omega \times \mathcal{Y})$,

$$\lim_{\varepsilon \to 0} \int_\Omega \chi(x, x/\varepsilon)\, d\mu_\varepsilon(x) = \int_{\Omega \times \mathcal{Y}} \chi(x, y)\, d\mu(x, y).$$

The convergence is called *two-scale weak* convergence*.

Remark 4.2. Notice that the family $\{\mu_\varepsilon\}_{\varepsilon>0}$ determines the family of measures $\{\lambda_\varepsilon\}_{\varepsilon>0} \subset \mathcal{M}_b(\Omega \times \mathcal{Y})$ obtained by setting

$$\int_{\Omega \times \mathcal{Y}} \chi(x, y)\, d\lambda_\varepsilon(x, y) := \int_\Omega \chi(x, x/\varepsilon)\, d\mu_\varepsilon(x)$$

for every $\chi \in C_0^0(\Omega \times \mathcal{Y})$. Thus μ_0 is simply the weak* limit in $\mathcal{M}_b(\Omega \times \mathcal{Y})$ of a suitable subsequence of $\{\lambda_\varepsilon\}_{\varepsilon>0}$.

Remark 4.3. Let $\mathcal{D} \subseteq \mathcal{Y}$, and assume that μ_ε has its support on $\Omega \cap \mathcal{D}_\varepsilon$, and $\mu_\varepsilon \xrightarrow{w^*\text{-}2} \mu_0$ two-scale weakly* in $\mathcal{M}_b(\Omega \times \mathcal{Y})$. Then $\operatorname{supp} \mu_0 \subset \Omega \times \bar{\mathcal{D}}$.

In view of Remark 4.2, two-scale weak* convergence has the following compactness property.

Proposition 4.4 (Two-scale compactness). *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and $\{\mu_\varepsilon\}_{\varepsilon>0}$ be a bounded family in $\mathcal{M}_b(\Omega)$. Then there exist $\mu_0 \in \mathcal{M}_b(\Omega \times \mathcal{Y})$ and $\varepsilon_n \to 0$ such that*

$$\mu_{\varepsilon_n} \xrightarrow{w^*\text{-}2} \mu_0 \quad \textit{two-scale weakly* in } \mathcal{M}_b(\Omega \times \mathcal{Y}).$$

Remark 4.5. The notion of two-scale weak* convergence can be easily adapted to measures in $\mathcal{M}_b(\Omega; X)$, where X is a finite-dimensional space. For our applications in plasticity, X will be either $\mathbb{R}^N$, or the spaces of matrices $\mathbb{M}^N_{\mathrm{sym}}$ and $\mathbb{M}^N_D$.

The following lower semicontinuity lemma is a two-scale analogue of Reshetnyak's lower semicontinuity theorem ([4, Theorem 2.38] or [19, Theorem 1.7]).

Lemma 4.6. *Let Ω be an open subset of $\mathbb{R}^N$, X a finite-dimensional linear space, and let $H : X \to [0, +\infty)$ be a convex and positively one-homogeneous function. If $\{\mu_\varepsilon\}_{\varepsilon>0}$ is a bounded family of measures in $\mathcal{M}_b(\Omega; X)$ such that*

$$\mu_\varepsilon \xrightarrow{w^*\text{-}2} \mu_0 \quad \textit{two-scale weakly* in } \mathcal{M}_b(\Omega \times \mathcal{Y}; X),$$

then

$$\liminf_{\varepsilon} \int_\Omega H(\mu_\varepsilon/|\mu_\varepsilon|)\, d|\mu_\varepsilon| \geq \int_{\Omega \times \mathcal{Y}} H(\mu_0/|\mu_0|)\, d|\mu_0|.$$

Proof. We can endow X with an inner product. Since H is convex and positively one-homogeneous,

$$H(\xi) = \sup_{m\in\mathbb{N}}\{a_m \cdot \xi : a_m \in X\}.$$

Let us extract a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ such that, setting $\mu_n := \mu_{\varepsilon_n}$,

$$\liminf_{\varepsilon}\int_{\Omega} H(\mu_\varepsilon/|\mu_\varepsilon|)\, d|\mu_\varepsilon| = \lim_{n}\int_{\Omega} H(\mu_n/|\mu_n|)\, d|\mu_n|.$$

Denote by $\mathcal{H} \in \mathcal{M}_b(\Omega \times \mathcal{Y})$ the two-scale weak* limit of (a subsequence of)

$$H(\mu_n/|\mu_n|)|\mu_n|$$

(still indexed by n). We will show that

$$\frac{\mathcal{H}}{|\mu_0|}(x_0, y_0) \geq H\left(\frac{\mu_0}{|\mu_0|}(x_0, y_0)\right) \quad \text{for } |\mu_0|\text{-a.e. } (x_0, y_0) \text{ in } \Omega \times \mathcal{Y}. \tag{4.1}$$

Then, by the very definition of two-scale convergence, for any $0 \leq \varphi \leq 1 \in C_c^0(\Omega)$,

$$\begin{aligned}\lim_{n}\int_{\Omega} H(\mu_n/|\mu_n|)\, d|\mu_n| &\geq \int_{\Omega\times\mathcal{Y}} \varphi(x)\, d\mathcal{H}(x, y)\\ &\geq \int_{\Omega\times\mathcal{Y}} \varphi(x) H\left(\frac{\mu_0}{|\mu_0|}(x, y)\right) d|\mu_0|(x, y).\end{aligned}$$

Letting $\varphi \nearrow 1$ on Ω, we get the result by Lebesgue's dominated convergence theorem.

Take (x_0, y_0) to be a Lebesgue point for $\mu_0/|\mu_0|$ with respect to $|\mu_0|$. Since we can argue locally, Besicovitch's derivation theorem allows us to choose (x_0, y_0) such that, if $B_r(x_0, y_0)$ denotes the open ball of center (x_0, y_0) and radius r in $\mathbb{R}^N \times \mathcal{Y}$,

$$\frac{\mathcal{H}}{|\mu_0|}(x_0, y_0) = \lim_{r\to 0^+} \frac{\mathcal{H}(B_r(x_0, y_0))}{|\mu_0|(B_r(x_0, y_0))}.$$

Choose a sequence $\{r_k \searrow 0\}$ and $\varphi_{k,l} \in C_c^0(B_{r_k}(x_0, y_0))$ with $0 \leq \varphi_{k,l} \overset{l}{\nearrow} 1_{B_{r_k}(x_0,y_0)}$. Then, by monotone convergence,

$$\begin{aligned}\frac{\mathcal{H}}{|\mu_0|}(x_0, y_0) &= \lim_{k} \frac{1}{|\mu_0|(B_{r_k}(x_0, y_0))} \lim_{l} \int_{\Omega\times\mathcal{Y}} \varphi_{k,l}(x, y)\, d\mathcal{H}(x, y)\\ &= \lim_{k} \frac{1}{|\mu_0|(B_{r_k}(x_0, y_0))} \lim_{l}\lim_{n} \int_{\Omega} \varphi_{k,l}(x, x/\varepsilon_n) H\left(\frac{\mu_n}{|\mu_n|}(x)\right) d|\mu_n|(x)\\ &\geq \liminf_{k} \frac{1}{|\mu_0|(B_{r_k}(x_0, y_0))} \liminf_{l}\lim_{n} \int_{\Omega} \varphi_{k,l}(x, x/\varepsilon_n) a_m \cdot d\mu_n(x)\\ &= \liminf_{k} \frac{1}{|\mu_0|(B_{r_k}(x_0, y_0))} \liminf_{l} \int_{\Omega\times\mathcal{Y}} \varphi_{k,l}(x, y) a_m \cdot d\mu_0(x, y).\end{aligned}$$

Lebesgue's dominated convergence theorem finally yields

$$\frac{\mathcal{H}}{|\mu_0|}(x_0, y_0) \geq \liminf_k \frac{1}{|\mu_0|(B_{r_k}(x_0, y_0))} \int_{B_{r_k}(x_0,y_0)} a_m \cdot d\mu_0 = a_m \cdot \frac{\mu_0}{|\mu_0|}(x_0, y_0).$$

Taking the supremum of the right hand-side of the above inequality over $m \in \mathbb{N}$ yields (4.1). ☐

4.2. Two-scale limits of symmetrized gradients of BD functions

For our homogenization problem in plasticity, we will need to consider two-scale weak* limits of measures which are also symmetrized gradients of *BD* functions. For $\Omega \subseteq \mathbb{R}^N$ open, set

$$\begin{aligned}\boldsymbol{\mathcal{X}}(\Omega) := \{\mu \in \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{R}^N) : E_y\mu \in \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}}),\\ \mu(F \times \mathcal{Y}) = 0 \text{ for every Borel set } F \subseteq \Omega\}, \qquad (4.2)\end{aligned}$$

where $E_y\mu$ denotes the distributional symmetrized gradient of μ with respect to y. The following proposition enumerates the main properties of $\boldsymbol{\mathcal{X}}(\Omega)$ that will be used in what follows.

Proposition 4.7. *Let $\mu \in \boldsymbol{\mathcal{X}}(\Omega)$. Then:*

(a) *There exist $\eta \in \mathcal{M}_b^+(\Omega)$ and a Borel map $(x, y) \in \Omega \times \mathcal{Y} \mapsto \mu_x(y) \in \mathbb{R}^N$ such that, for η-a.e. $x \in \Omega$,*

$$\mu_x \in BD(\mathcal{Y}), \qquad \int_{\mathcal{Y}} \mu_x(y)\, dy = 0, \qquad |E_y\mu_x|(\mathcal{Y}) \neq 0, \qquad (4.3)$$

and

$$\mu = \mu_x(y)(\eta \otimes \mathcal{L}^N_y).$$

Moreover, the map $x \mapsto E_y\mu_x \in \mathcal{M}_b(\mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}})$ is η-measurable and

$$E_y\mu = \eta \overset{\text{gen.}}{\otimes} E_y\mu_x.$$

(b) *For any C^1-hypersurface $\mathcal{D} \subseteq \mathcal{Y}$, if ν denotes a continuous unit normal vector field to $\mathcal{D}$, then*

$$E_y\mu \lfloor_{\Omega\times\mathcal{D}} = a(x, y) \odot \nu(y)(\eta \otimes (\mathcal{H}^{N-1} \lfloor_{\mathcal{D}})), \qquad (4.4)$$

where $a : \Omega \times \mathcal{D} \to \mathbb{R}^N$ is a Borel function.

Proof. Let us prove item (a). By [4, Theorem 2.28 and Corollary 2.29] we know that μ and $\lambda := E_y\mu$ can be disintegrated with respect to $\mathrm{proj}_\# |\mu|$ and $\mathrm{proj}_\# |\lambda|$ respectively, proj denoting the projection of $\Omega \times \mathcal{Y}$ on the first factor, and $\mathrm{proj}_\#$ the associated push forward of measures. Setting

$$\eta := \mathrm{proj}_\# |\mu| + \mathrm{proj}_\# |\lambda|$$

we obtain the disintegrations

$$\mu = \eta \overset{\text{gen.}}{\otimes} \mu_x \quad \text{and} \quad \lambda = \eta \overset{\text{gen.}}{\otimes} \lambda_x \tag{4.5}$$

with $\mu_x \in \mathcal{M}_b(\mathcal{Y}; \mathbb{R}^N)$ and $\lambda_x \in \mathcal{M}_b(\mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}})$. Further, if $F := \{x \in \Omega : |\lambda_x|(\mathcal{Y}) \neq 0\}$, then, obviously, $\lambda = \eta\lfloor_F \overset{\text{gen.}}{\otimes} \lambda_x$.

For every $g \in C^1(\mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}})$ and $f \in C^1_c(\Omega)$,

$$\begin{aligned}\int_\Omega f(x)\langle \mu_x, \mathrm{div}_y\, g\rangle\, d\eta(x) &= \langle \eta \overset{\text{gen.}}{\otimes} \mu_x, f(x)\,\mathrm{div}_y\, g\rangle = \langle \mu, \mathrm{div}_y(f(x)g(y))\rangle \\ &= -\langle E_y\mu, f(x)g(y)\rangle = -\langle \eta\lfloor_F \overset{\text{gen.}}{\otimes} \lambda_x, f(x)g(y)\rangle \\ &= -\int_\Omega f(x)1_F(x)\langle \lambda_x, g(y)\rangle\, d\eta(x).\end{aligned}$$

Letting g vary in a countable and dense set (by Fourier series for example), we deduce that, for η-a.e. $x \in \Omega$ and for all $h \in C^1(\mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}})$,

$$\langle \mu_x, \mathrm{div}_y\, h\rangle = -\langle 1_F(x)\lambda_x, h(y)\rangle,$$

i.e., using a regularization argument through convolution,

$$\mu_x \in BD(\mathcal{Y}) \quad \text{and} \quad E_y\mu_x = 1_F(x)\lambda_x. \tag{4.6}$$

Finally, since $\mu(G \times \mathcal{Y}) = 0$ for every Borel set $G \subseteq \Omega$ we get, for every $f \in C^0_c(\Omega)$,

$$0 = \langle \mu, f(x)\rangle = \int_\Omega f(x)\mu_x(\mathcal{Y})\, d\eta(x),$$

so that, for η-a.e. $x \in \Omega$,

$$\mu_x(\mathcal{Y}) = 0. \tag{4.7}$$

In particular, for η-a.e. x in $\Omega \setminus F$, μ_x is a rigid body motion on $\mathcal{Y}$ that satisfies (4.7), hence $\mu_x \equiv 0$ and we can thus replace η by $\eta\lfloor_F$ in both equalities in (4.5). We still denote the new measure by η from now onward.

In order to complete the proof of item (a), it suffices to show that it is not restrictive to assume that $(x, y) \mapsto \mu_x(y)$ is a Borel map. From (4.5) and (4.6), we infer that μ is absolutely continuous with respect to $\eta \otimes \mathcal{L}^N_y$. Consequently, there exists a Borel map $h : \Omega \times \mathcal{Y} \to \mathbb{R}^N$ such that $\mu = h(x, y)(\eta \otimes \mathcal{L}^N_y)$. Moreover for η-a.e. $x \in \Omega$ there exists $\mathcal{S}_x \subseteq \mathcal{Y}$ with $\mathcal{L}^N_y(\mathcal{S}_x) = 0$ and such that

$$h(x, y) = \mu_x(y) \qquad \text{for every } y \notin \mathcal{S}_x.$$

This is sufficient for replacing μ_x with $h(x, \cdot)\,\mathcal{L}^N_y$ in (4.5), so that (a) follows.

Let us come to item (b). By (a), the map $x \mapsto E_y\mu_x\lfloor_{\mathcal{D}}$ is η-measurable with

$$E_y\mu\lfloor_{\Omega\times\mathcal{D}} = \eta \overset{\text{gen.}}{\otimes} (E_y\mu_x\lfloor_{\mathcal{D}}).$$

Thanks to the structure of symmetrized gradients of *BD* functions, for η-a.e. $x \in \Omega$,

$$E_y\mu_x \lfloor_{\mathcal{D}} = b(x, y) \odot \nu(y)\mathcal{H}^{N-1}\lfloor_{\mathcal{D}}$$

for a suitable $b(x, y) \in \mathbb{R}^N$. We thus infer that $E_y\mu\lfloor_{\Omega\times\mathcal{D}}$ is absolutely continuous with respect to the measure $\zeta := \eta \otimes (\mathcal{H}^{N-1}\lfloor_{\mathcal{D}})$. By Radon–Nikodym's theorem, we deduce that

$$E_y\mu\lfloor_{\Omega\times\mathcal{D}} = \eta \overset{\text{gen.}}{\otimes} [b(x, y) \odot \nu(y)\mathcal{H}^{N-1}\lfloor_{\mathcal{D}}] = f(x, y)\zeta \tag{4.8}$$

for a suitable Borel function $f : \Omega\times\mathcal{D} \to \mathbb{M}^N_{\text{sym}}$. As previously noted in the introduction, this equality is not sufficient to infer that $f(x, y) = b(x, y) \odot \nu(y)$, ζ-a.e. on $\Omega \times \mathcal{D}$, from which the conclusion would easily follow. From (4.8) we can only infer, as above, that, for η-a.e. $x \in \Omega$, there exists $\mathcal{N}_x \subseteq \mathcal{D}$ with $\mathcal{H}^{N-1}(\mathcal{N}_x) = 0$, and such that

$$f(x, y) = b(x, y) \odot \nu(y) \qquad \text{for every } y \notin \mathcal{N}_x. \tag{4.9}$$

Let us show that there exists a map $a : \Omega \times \mathcal{D} \to \mathbb{R}^N$ such that

$$f(x, y) = a(x, y) \odot \nu(y) \qquad \text{for } \zeta\text{-a.e. } (x, y) \in \Omega \times \mathcal{D}. \tag{4.10}$$

For every $y \in \mathcal{D}$, we consider $\Pi(y) := \{\xi \odot \nu(y) : \xi \in \mathbb{R}^N\} \subseteq \mathbb{M}^N_{\text{sym}}$ and the Borel set $B := \{(x, y) \in \Omega \times \mathcal{D} : \text{dist}(f(x, y), \Pi(y)) \neq 0\}$. That set is readily seen to be ζ-negligible in view of (4.9) and of Fubini's theorem. Then (4.10) follows. Finally, we can assume that a is Borel regular since ν is continuous and does not vanish on $\mathcal{D}$, so that the proof of item (b) is concluded. □

The following result will be useful.

Lemma 4.8. *The space*

$$\mathcal{E} := \{E_y\mu : \mu \in \boldsymbol{\mathcal{X}}(\Omega)\}$$

is weakly closed in* $\mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{M}^N_{\text{sym}})$.

Proof. In view of the Krein–Šmulian theorem and since $C^0_0(\Omega\times\mathcal{Y}; \mathbb{M}^N_{\text{sym}})$ is separable, it is enough to show sequential weak*-closedness. Assume that $\{\lambda_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{E}$ such that

$$\lambda_n \overset{*}{\rightharpoonup} \lambda \qquad \text{weakly* in } \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{M}^N_{\text{sym}}).$$

By assumption there exists a measure $\mu_n \in \boldsymbol{\mathcal{X}}(\Omega)$ such that $E_y\mu_n = \lambda_n$. Note that $\{\mu_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{R}^N)$: indeed Proposition 4.7(a) implies that

$$\mu_n = \mu^n_x\,(\eta_n \otimes \mathcal{L}^N_y), \qquad E_y\mu_n = \eta_n \overset{\text{gen.}}{\otimes} E_y\mu^n_x,$$

with $\eta_n \in \mathcal{M}^+_b(\Omega)$ and $\mu^n_x \in BD(\mathcal{Y})$ satisfying (4.3) for η_n-a.e. $x \in \Omega$. Taking into account Poincaré–Korn's inequality in $BD(\mathcal{Y})$ and applying [4, Corollary 2.29], we obtain

$$\begin{aligned} |\mu_n|(\Omega \times \mathcal{Y}) &= \int_\Omega \left[\int_{\mathcal{Y}} |\mu^n_x(y)|\,dy\right] d\eta_n(x) \leq C \int_\Omega |E_y\mu^n_x|(\mathcal{Y})\,d\eta_n(x) \\ &= C|\lambda_n|(\Omega \times \mathcal{Y}) \leq C' \end{aligned}$$

for some constant C'. Up to a subsequence, there exists $\mu \in \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{R}^N)$ with

$$\mu_n \overset{*}{\rightharpoonup} \mu \quad \text{weakly* in } \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{R}^N).$$

Clearly $E_y\mu = \lambda$. Moreover, passing to the limit in the equality

$$\int_{\Omega \times \mathcal{Y}} f(x)\, d\mu_n(x, y) = 0, \qquad f \in C_c^0(\Omega),$$

we get, by standard approximation arguments, $\mu(F \times \mathcal{Y}) = 0$ for every Borel set $F \subseteq \Omega$, so that $\lambda \in \mathcal{E}$. □

The following lemma is essential in the study of two-scale weak* limits of symmetrized gradients of *BD* functions.

Lemma 4.9. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and $\lambda \in \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}})$. The following items are equivalent:*

(a) *For every $\chi \in C_0^0(\Omega \times \mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}})$ with $\operatorname{div}_y \chi(x, y) = 0$ (in the sense of distributions),*

$$\int_{\Omega \times \mathcal{Y}} \chi(x, y)\, d\lambda(x, y) = 0.$$

(b) *There exists $\mu \in \boldsymbol{\mathcal{X}}(\Omega)$ such that $\lambda = E_y\mu$.*

Proof. The fact that (b) implies (a) follows by integration by parts and a density argument. Let us assume that (a) holds. By Lemma 4.8, $\mathcal{E} := \{E_y\mu : \mu \in \boldsymbol{\mathcal{X}}(\Omega)\}$ is weakly* closed in $\mathcal{M}_b(\Omega \times \mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}})$. Then, if by contradiction (b) is not true, i.e., $\lambda \notin \mathcal{E}$, the Hahn–Banach theorem—which is applied here to $\mathcal{M}_b(\Omega \times \mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}})$ equipped with its weak* topology—yields the existence of $\chi \in C_0^0(\Omega \times \mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}})$ such that

$$\int_{\Omega \times \mathcal{Y}} \chi(x, y)\, d\lambda(x, y) = 1, \tag{4.11}$$

and, for every $\mu \in \boldsymbol{\mathcal{X}}(\Omega)$,

$$\int_{\Omega \times \mathcal{Y}} \chi(x, y)\, dE_y\mu(x, y) = 0. \tag{4.12}$$

In particular, if we choose μ to be a smooth function, (4.12) implies that $\operatorname{div}_y \chi(x, y) = 0$ (in the sense of distributions). As a consequence, (4.11) contradicts (a), and the result follows. □

The previous results combine into a structure result for two-scale weak* limits of symmetrized gradients of *BD* functions.

Proposition 4.10 (Symmetrized gradients). *Let* $\Omega \subseteq \mathbb{R}^N$ *be open, and let* $\{u_\varepsilon\}_{\varepsilon>0}$ *be a bounded family in* $BD(\Omega)$ *such that*

$$u_\varepsilon \overset{*}{\rightharpoonup} u \quad \text{weakly}^* \text{ in } BD(\Omega)$$

for some $u \in BD(\Omega)$ *as* $\varepsilon \to 0$. *Let*

$$Eu_\varepsilon \xrightarrow{w^*\text{-}2} \lambda \quad \text{two-scale weakly}^* \text{ in } \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}}).$$

Then there exists $\mu \in \boldsymbol{\mathcal{X}}(\Omega)$ *such that*

$$\lambda = Eu \otimes \mathcal{L}^N_y + E_y\mu.$$

Proof. Since $u_\varepsilon \to u$ strongly in $L^1(\Omega; \mathbb{R}^N)$,

$$u_\varepsilon \mathcal{L}^N_x \xrightarrow{w^*\text{-}2} u(x)\,(\mathcal{L}^N_x \otimes \mathcal{L}^N_y) \quad \text{two-scale weakly}^* \text{ in } \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{R}^N).$$

By compactness, there exist $\varepsilon_n \to 0$ and $\lambda \in \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}})$ such that

$$Eu_{\varepsilon_n} \xrightarrow{w^*\text{-}2} \lambda \quad \text{two-scale weakly}^* \text{ in } \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}}).$$

Considering $\chi \in C^1_c(\Omega \times \mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}})$ with $\mathrm{div}_y\, \chi = 0$, from the equality

$$\int_\Omega \chi(x, x/\varepsilon)\, dEu_\varepsilon(x) = -\int_\Omega \mathrm{div}_x\, \chi(x, x/\varepsilon) u_\varepsilon(x)\, dx$$

we get, as $\varepsilon \to 0$,

$$\int_{\Omega\times\mathcal{Y}} \chi(x, y)\, d\lambda(x, y) = -\int_{\Omega\times\mathcal{Y}} \mathrm{div}_x\, \chi(x, y) u(x)\, dx\, dy = \int_{\Omega\times\mathcal{Y}} \chi(x, y)\, d(Eu \otimes \mathcal{L}^N_y).$$

By a density argument, we infer that

$$\int_{\Omega\times Y} \chi\, d[\lambda - Eu \otimes \mathcal{L}^N_y] = 0$$

for every $\chi \in C^0_0(\Omega \times \mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}})$ with $\mathrm{div}_y\, \chi = 0$ in the sense of distributions. The result now follows by Lemma 4.9. □

4.3. *Unfolding of sequences of symmetrized gradients of BD functions*

In the following we adapt the unfolding method originally developed for sequences of L^p functions in [8, 9] to the setting at hand.

For every $\varepsilon > 0$ let

$$Q^i_\varepsilon := \left\{ x \in \mathbb{R}^N : \frac{x - \varepsilon i}{\varepsilon} \in [0, 1)^N \right\} \quad \text{and} \quad x^i_\varepsilon := \varepsilon i.$$

Clearly $\mathbb{R}^N = \bigcup_{i\in\mathbb{Z}^N} Q^i_\varepsilon$. Given $\Omega \subseteq \mathbb{R}^N$ open, we set

$$I_\varepsilon(\Omega) := \{i \in \mathbb{Z}^N : Q^i_\varepsilon \subset \Omega\}. \tag{4.13}$$

For $\mu_\varepsilon \in \mathcal{M}_b(\Omega)$ and $Q^i_\varepsilon \subset \Omega$ we let $\mu^i_\varepsilon \in \mathcal{M}_b(\mathcal{Y})$ be the measure defined as

$$\int_{\mathcal{Y}} \psi(y)\, d\mu^i_\varepsilon(y) := \frac{1}{\varepsilon^N}\int_{Q^i_\varepsilon} \psi(x/\varepsilon)\, d\mu_\varepsilon(x), \qquad \psi \in C^0(\mathcal{Y}). \tag{4.14}$$

Then set $\tilde\lambda_\varepsilon \in \mathcal{M}_b(\Omega \times \mathcal{Y})$, *the unfolded measures associated with* μ_ε, to be

$$\tilde\lambda_\varepsilon := \sum_{i\in I_\varepsilon(\Omega)} (\mathcal{L}^N_x \lfloor_{Q^i_\varepsilon}) \otimes \mu^i_\varepsilon. \tag{4.15}$$

Proposition 4.11 (Unfolding). *Let $\Omega \subseteq \mathbb{R}^N$ be open and $\{\mu_\varepsilon\}_{\varepsilon>0}$ be a bounded family in $\mathcal{M}_b(\Omega)$ such that*

$$\mu_\varepsilon \xrightarrow{w^*\text{-}2} \mu_0 \quad \textit{two-scale weakly* in } \mathcal{M}_b(\Omega\times\mathcal{Y}).$$

Let $\{\tilde\lambda_\varepsilon\}_{\varepsilon>0} \subset \mathcal{M}_b(\Omega\times\mathcal{Y})$ be the associated family of unfolded measures according to (4.15). *Then*

$$\tilde\lambda_\varepsilon \overset{*}{\rightharpoonup} \mu_0 \quad \textit{weakly* in } \mathcal{M}_b(\Omega\times\mathcal{Y}).$$

Proof. It suffices to show that, for every $\chi \in C^0_c(\Omega\times\mathcal{Y})$,

$$\lim_{\varepsilon\to 0}\int_{\Omega\times\mathcal{Y}} \chi\, d\tilde\lambda_\varepsilon = \lim_{\varepsilon\to 0}\int_{\Omega\times\mathcal{Y}} \chi\, d\lambda_\varepsilon$$

where λ_ε has been introduced in Remark 4.2. Let $\tilde\Omega \subset \mathbb{R}^N$ be open, bounded and such that $\operatorname{supp}(\chi) \subset\subset \tilde\Omega \times \mathcal{Y}$.

Note that

$$\lim_\varepsilon \varepsilon^N \#(I_\varepsilon(\tilde\Omega)) = \mathcal{L}^N(\tilde\Omega). \tag{4.16}$$

Then, for ε small enough,

$$\int_{\tilde\Omega\times\mathcal{Y}} \chi(x,y)\, d\tilde\lambda_\varepsilon = \frac{1}{\varepsilon^N}\sum_{i\in I_\varepsilon(\tilde\Omega)} \int_{Q^i_\varepsilon\times Q^i_\varepsilon} \chi(z, x/\varepsilon)\, d\mu_\varepsilon(x) dz,$$

so that, with (4.16),

$$\begin{aligned}
&\left|\int_{\Omega\times\mathcal{Y}} \chi(x,y)\, d\lambda_\varepsilon - \int_{\Omega\times\mathcal{Y}} \chi(x,y)\, d\tilde\lambda_\varepsilon\right| = \left|\int_{\tilde\Omega\times\mathcal{Y}} \chi(x,y)\, d\lambda_\varepsilon - \int_{\tilde\Omega\times\mathcal{Y}} \chi(x,y)\, d\tilde\lambda_\varepsilon\right| \\
&\le \|\chi\|_\infty\big(\mathcal{L}^N(\tilde\Omega) - \varepsilon^N\#(I_\varepsilon(\tilde\Omega))\big) + \sum_{i\in I_\varepsilon(\tilde\Omega)} \int_{Q^i_\varepsilon} \left|\chi(x, x/\varepsilon) - \frac{1}{\varepsilon^N}\int_{Q^i_\varepsilon} \chi(z, x/\varepsilon)\, dz\right| d|\mu_\varepsilon| \\
&\le O(\varepsilon) + \delta_\varepsilon |\mu_\varepsilon|(\tilde\Omega),
\end{aligned}$$

with

$$\delta_\varepsilon := \sup_{|x_1-x_2|<\varepsilon\sqrt{N},\, y\in\mathcal{Y}} |\chi(x_1,y) - \chi(x_2,y)| \to 0.$$

Hence the result upon letting ε go to 0. □

Remark 4.12 (Two-scale convergence in Lebesgue spaces). Unfolding provides an easy link between two-scale weak* convergence of measures and two-scale convergence of L^p functions. Let $\Omega \subset \mathbb{R}^N$ be open and bounded and $\{u_\varepsilon\}_{\varepsilon>0}$ be a bounded family in $L^p(\Omega)$ for some $p \in (1, \infty)$ such that

$$u_\varepsilon \mathcal{L}^N \xrightarrow{w^*\text{-}2} \mu_0 \quad \text{two-scale weakly* in } \mathcal{M}_b(\Omega \times \mathcal{Y}).$$

Then there exists $u_0 \in L^p(\Omega \times \mathcal{Y})$ such that

$$\mu_0 = u_0(x, y)(\mathcal{L}_x^N \otimes \mathcal{L}_y^N). \tag{4.17}$$

Indeed, according to (4.14), for every $i \in I_\varepsilon(\Omega)$,

$$\mu_\varepsilon^i = v_\varepsilon^i(y)\, \mathcal{L}_y^N$$

where $v_\varepsilon^i(y) := u_\varepsilon(x_\varepsilon^i + \varepsilon\mathcal{I}(y))$. Consequently,

$$\tilde{\lambda}_\varepsilon = v_\varepsilon(x, y)(\mathcal{L}_x^N \otimes \mathcal{L}_y^N) \quad \text{with} \quad v_\varepsilon(x, y) := \sum_{i \in I_\varepsilon(\Omega)} 1_{Q_\varepsilon^i}(x) v_\varepsilon^i(y).$$

A direct computation shows that

$$\int_{\Omega\times\mathcal{Y}} |v_\varepsilon(x, y)|^p\, dx\, dy = \int_{\bigcup_{i\in I_\varepsilon(\Omega)} Q_\varepsilon^i} |u_\varepsilon(x)|^p\, dx \le \int_\Omega |u_\varepsilon|^p\, dx.$$

By weak compactness of $L^p(\Omega \times \mathcal{Y})$ we infer immediately that (4.17) holds true.

We will say that

$$u_\varepsilon \xrightarrow{w\text{-}2} u_0 \quad \text{two-scale weakly in } L^p(\Omega \times \mathcal{Y}).$$

If further

$$\lim_{\varepsilon\to 0} \int_\Omega |u_\varepsilon|^p\, dx = \int_{\Omega\times\mathcal{Y}} |u_0|^p\, dx\, dy,$$

we will say that

$$u_\varepsilon \overset{s\text{-}2}{\to} u_0 \quad \text{two-scale strongly in } L^p(\Omega \times \mathcal{Y}).$$

In the context of unfolding, sequences of symmetrized gradients of *BD* functions will satisfy the following proposition which will be used in the proof of Theorem 5.7.

Proposition 4.13. *Let $\Omega \subseteq \mathbb{R}^N$ be open and let $\mathcal{B} \subseteq \mathcal{Y}$ be an open set with Lipschitz boundary. If $u_\varepsilon \in BD(\Omega)$, then the unfolded measure associated with $Eu_\varepsilon \llcorner_{\mathcal{B}_\varepsilon\setminus\mathcal{C}_\varepsilon}$ according to (4.15) is given by*

$$\sum_{i\in I_\varepsilon(\Omega)} (\mathcal{L}_x^N \llcorner_{Q_\varepsilon^i}) \otimes E_y\hat{u}_\varepsilon^i \llcorner_{\mathcal{B}\setminus\mathcal{C}}, \tag{4.18}$$

where $\mathcal{C}$ is defined in (1.1) and $\hat{u}_\varepsilon^i$ is a suitable function in $BD(\mathcal{Y})$ such that

$$\int_{\partial\mathcal{B}} |\hat{u}_\varepsilon^i|\, d\mathcal{H}^{N-1} + |E_y\hat{u}_\varepsilon^i|(\mathcal{B}\cap\mathcal{C}) \le \frac{C}{\varepsilon^N}|Eu_\varepsilon|(\mathrm{int}(Q_\varepsilon^i)) \tag{4.19}$$

for some constant C independent of i and ε.

Proof. Note that $\mathcal{C}_\varepsilon = (\bigcup_i \partial Q^i_\varepsilon) \cap \Omega$. Accordingly, for $i \in I_\varepsilon(\Omega)$ and $\psi \in C^1(\mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}})$,

$$\int_{Q^i_\varepsilon} \psi(x/\varepsilon) \cdot dEu_\varepsilon \lfloor_{\mathcal{B}_\varepsilon \setminus \mathcal{C}_\varepsilon} = \int_{\mathrm{int}(Q^i_\varepsilon)} \psi(x/\varepsilon) \cdot dEu_\varepsilon \lfloor_{\mathcal{B}_\varepsilon}.$$

Since $\mathcal{B}_\varepsilon$ has a Lipschitz boundary, $u_\varepsilon 1_{\mathcal{B}_\varepsilon} \in BD_{\mathrm{loc}}(\Omega)$ with

$$Eu_\varepsilon \lfloor_{\mathcal{B}_\varepsilon} = E(u_\varepsilon 1_{\mathcal{B}_\varepsilon}) + (u_\varepsilon)\lfloor_{\partial \mathcal{B}_\varepsilon} \odot \nu \mathcal{H}^{N-1} \lfloor_{\partial \mathcal{B}_\varepsilon},$$

where, from now onward in this proof, for any open Lipschitz domain $A \subset\subset \Omega$ and any $u \in BD(\Omega)$, $u\lfloor_{\partial A}$ denotes the trace of $u 1_A$ on ∂A, while ν is the exterior normal to ∂A. Then

$$\int_{\mathrm{int}(Q^i_\varepsilon)} \psi(x/\varepsilon) \cdot dEu_\varepsilon \lfloor_{\mathcal{B}_\varepsilon}$$
$$= \int_{\mathrm{int}(Q^i_\varepsilon)} \psi(x/\varepsilon) \cdot dE(u_\varepsilon 1_{\mathcal{B}_\varepsilon}) + \int_{\mathrm{int}(Q^i_\varepsilon)} \psi(x/\varepsilon) \cdot [(u_\varepsilon)\lfloor_{\partial \mathcal{B}_\varepsilon} \odot \nu] \, d\mathcal{H}^{N-1} \lfloor_{\partial \mathcal{B}_\varepsilon}.$$

If we set $v^i_\varepsilon(z) := u_\varepsilon(x^i_\varepsilon + \varepsilon z)$ for $z \in (0,1)^N$, then $v^i_\varepsilon \in BD((0,1)^N)$ and, thanks to the periodicity of ψ, the definition of $\mathcal{B}_\varepsilon$, and Remark 1.1,

$$\int_{Q^i_\varepsilon} \psi(x/\varepsilon) \cdot dEu_\varepsilon \lfloor_{\mathcal{B}_\varepsilon \setminus \mathcal{C}_\varepsilon} = \varepsilon^{N-1} \int_{(0,1)^N} \psi(z) \cdot dE(v^i_\varepsilon 1_{\mathcal{I}(\mathcal{B})})(z)$$
$$+ \varepsilon^{N-1} \int_{(0,1)^N} \psi(z) \cdot [(v^i_\varepsilon)\lfloor_{\partial \mathcal{I}(\mathcal{B})}(z) \odot \nu(z)] \, d\mathcal{H}^{N-1}(z). \tag{4.20}$$

Adding a rigid body motion to u_ε on Q^i_ε does not change Eu_ε on $\mathcal{B}_\varepsilon \setminus \mathcal{C}_\varepsilon$, hence it does not modify the computation in (4.20). But then, by Poincaré–Korn's inequality, we may as well assume that

$$\int_{\partial(0,1)^N} |(v^i_\varepsilon)\lfloor_{\partial(0,1)^N}| \, d\mathcal{H}^{N-1} \le C|Ev^i_\varepsilon|((0,1)^N) = \frac{C}{\varepsilon^{N-1}} |Eu^i_\varepsilon|(\mathrm{int}(Q^i_\varepsilon)) \tag{4.21}$$

for some constant $C > 0$ independent of i and ε.

Let $\hat{u}^i_\varepsilon \in BD(\mathcal{Y})$ be such that

$$\hat{u}^i_\varepsilon(y) := \frac{1}{\varepsilon} v^i_\varepsilon(\mathcal{I}(y)).$$

From (4.21) and through the identification of the opposite sides of $\partial(0,1)^N$ when passing to $\mathcal{Y}$, we obtain

$$|E_y \hat{u}^i_\varepsilon|(\mathcal{Y}) \le \frac{C+1}{\varepsilon^N} |Eu^i_\varepsilon|(\mathrm{int}(Q^i_\varepsilon)). \tag{4.22}$$

Moreover,

$$\int_{(0,1)^N} \psi \cdot dE(v^i_\varepsilon 1_{\mathcal{I}(\mathcal{B})}) = \varepsilon \int_{\mathcal{Y} \setminus \mathcal{C}} \psi \cdot dE(\hat{u}^i_\varepsilon 1_{\mathcal{B}})$$

while

$$\int_{(0,1)^N} \psi \cdot [(v^i_\varepsilon)\lfloor_{\partial\mathcal{I}(\mathcal{B})} \odot \nu]\, d\mathcal{H}^{N-1} = \varepsilon \int_{\partial\mathcal{B}\setminus\mathcal{C}} \psi \cdot [(\hat{u}^i_\varepsilon)\lfloor_{\partial\mathcal{B}} \odot \nu]\, d\mathcal{H}^{N-1},$$

where $(\hat{u}^i_\varepsilon)\lfloor_{\partial\mathcal{B}}$ denotes the trace on $\partial\mathcal{B}$ of the restriction of $\hat{u}^i_\varepsilon$ to $\mathcal{B}$. Therefore (4.20) reads as

$$\frac{1}{\varepsilon^N}\int_{Q^i_\varepsilon} \psi(x/\varepsilon)\cdot dEu_\varepsilon\lfloor_{\mathcal{B}_\varepsilon\setminus\mathcal{C}_\varepsilon} = \int_{\mathcal{Y}\setminus\mathcal{C}} \psi\cdot dE(\hat{u}^i_\varepsilon 1_{\mathcal{B}}) + \int_{\partial\mathcal{B}\setminus\mathcal{C}} \psi\cdot[(\hat{u}^i_\varepsilon)\lfloor_{\partial\mathcal{B}}\odot\nu]\, d\mathcal{H}^{N-1}.$$

Now,

$$E(\hat{u}^i_\varepsilon 1_{\mathcal{B}}) = E\hat{u}^i_\varepsilon\lfloor_{\mathcal{B}} - (\hat{u}^i_\varepsilon)\lfloor_{\partial\mathcal{B}}\odot\nu\mathcal{H}^{N-1}\lfloor_{\partial\mathcal{B}},$$

thus (4.20) finally reads

$$\frac{1}{\varepsilon^N}\int_{Q^i_\varepsilon} \psi(x/\varepsilon)\, dEu_\varepsilon\lfloor_{\mathcal{B}_\varepsilon\setminus\mathcal{C}_\varepsilon} = \int_{\mathcal{Y}} \psi\, dE_y\hat{u}^i_\varepsilon\lfloor_{\mathcal{B}\setminus\mathcal{C}}. \tag{4.23}$$

Note that we can add to $\hat{u}^i_\varepsilon$ rigid body motions on the finitely many connected components of $\mathcal{B}$ with no effect on the preceding equality, nor on $E_y\hat{u}^i_\varepsilon\lfloor_{\mathcal{B}\cap\mathcal{C}}$ (since rigid body motions on $\mathcal{B}$ are continuous on $\mathcal{B}$). As a consequence, thanks to Poincaré–Korn's inequality on $BD(\mathcal{Y})$, and in view of (4.22), we can assume that

$$\begin{aligned}&\int_{\partial\mathcal{B}} |\hat{u}^i_\varepsilon|\, d\mathcal{H}^{N-1} + |E_y\hat{u}^i_\varepsilon|(\mathcal{B}\cap\mathcal{C})\\ &\quad\le C'|E_y\hat{u}^i_\varepsilon|(\mathcal{B}) + |E_y\hat{u}^i_\varepsilon|(\mathcal{B}\cap\mathcal{C}) \le (C'+1)|E_y\hat{u}^i_\varepsilon|(\mathcal{Y}) \le \frac{C''}{\varepsilon^N}|Eu^i_\varepsilon|(\mathrm{int}(Q^i_\varepsilon))\end{aligned}$$

for some C', C'' independent of i and ε, so that (4.19) follows. □

5. Two-scale kinematics and two-scale statics

This section, the most technical of the paper, is devoted to an investigation of the disintegration and duality properties of the two-scale limits of the kinematically admissible fields $u_\varepsilon, e_\varepsilon, p_\varepsilon$ and of the statically admissible fields σ_ε associated with the heterogeneous evolution. We will also discuss the lower semicontinuity properties of the various energies involved in that evolution.

5.1. Two-scale kinematics and lower semicontinuity

In this subsection, we define the set of admissible two-scale (kinematically admissible) configurations and proceed, for future use, to disintegrate them in a manner such that almost every x-fiber (with respect to a suitable measure) is actually an element of $\mathcal{A}_\mathcal{Y}$ (see Definition 3.1). We then show that two-scale kinematically admissible configurations arise from a natural compactness argument. We finally establish a lower semicontinuity result for the ε-dissipation potentials $\mathcal{H}_\varepsilon$ resulting in a homogenized dissipation potential $\mathcal{H}^{\mathrm{hom}}$.

In order to handle the Dirichlet boundary condition, it proves convenient to consider $\Omega' \subseteq \mathbb{R}^N$ open bounded and such that $\Omega \subset \Omega'$ and $\partial\Omega \cap \Omega' = \Gamma_d$. Given a boundary displacement $w \in H^1(\mathbb{R}^N;\mathbb{R}^N)$, and a configuration $(u,e,p) \in \mathcal{A}(w)$, we may extend u, e, p to Ω' by setting

$$u = w, \qquad e = Ew, \qquad p = 0 \quad \text{on } \Omega' \setminus \overline{\Omega}. \tag{5.1}$$

It is readily checked that the admissibility conditions (2.4) become

$$Eu = e + p \quad \text{on } \Omega'. \tag{5.2}$$

Then the family of admissible configurations for w can be described as

$$\mathcal{A}(w) = \{(u,e,p) \in BD(\Omega') \times L^2(\Omega'; \mathbb{M}^N_{\rm sym}) \times \mathcal{M}_b(\Omega'; \mathbb{M}^N_D) : \text{(5.1) and (5.2) are satisfied}\}. \tag{5.3}$$

Coming to a two-scale setting, we adopt the following

Definition 5.1 (Kinematically admissible two-scale configurations). $\mathcal{A}^{\rm hom}(w)$, the family of *admissible two-scale configurations* relative to w, is the set of triplets (u, E, P) with

$$u \in BD(\Omega'), \qquad E \in L^2(\Omega' \times \mathcal{Y}; \mathbb{M}^N_{\rm sym}), \qquad P \in \mathcal{M}_b(\Omega' \times \mathcal{Y}; \mathbb{M}^N_D),$$

such that

$$u = w, \qquad E = Ew, \qquad P = 0 \quad \text{on } (\Omega' \setminus \overline{\Omega}) \times \mathcal{Y}, \tag{5.4}$$

and also such that there exists $\mu \in \boldsymbol{\mathcal{X}}(\Omega')$ (see (4.2)) with

$$E(x,y)\,(\mathcal{L}^N_x \otimes \mathcal{L}^N_y) + P - Eu \otimes \mathcal{L}^N_y = E_y\mu \quad \text{in } \Omega' \times \mathcal{Y}. \tag{5.5}$$

Further, set

$$\Pi(w) := \{P \in \mathcal{M}_b(\Omega' \times \mathcal{Y}; \mathbb{M}^N_D) : \exists (u,E) \text{ such that } (u,E,P) \in \mathcal{A}^{\rm hom}(w)\}.$$

Remark 5.2. The element $\mu \in \boldsymbol{\mathcal{X}}(\Omega')$ associated with (u, E, P) according to the previous definition is uniquely determined. Indeed, (5.5) implies that $E_y\mu$ is uniquely determined. The disintegrations $\mu = \mu_x(y)(\eta \otimes \mathcal{L}^N_y)$ and $E_y\mu = \eta \overset{\rm gen.}{\otimes} E_y\mu_x$ for a suitable $\eta \in \mathcal{M}^+_b(\Omega')$ given by Proposition 4.7 are such that $\mu_x \in BD(\mathcal{Y})$ and $\int_\mathcal{Y} \mu_x\,dy = 0$ for η-a.e. $x \in \Omega'$. Thus Poincaré–Korn's inequality on $BD(\mathcal{Y})$ yields

$$\begin{aligned}|\mu|(\Omega' \times \mathcal{Y}) &= \int_{\Omega'} \left[\int_\mathcal{Y} |\mu_x(y)|\,dy\right] d\eta(x) \le C \int_{\Omega'} |E_y\mu_x(y)|(\mathcal{Y})\,d\eta(x)\\ &= |E_y\mu|(\Omega' \times \mathcal{Y}),\end{aligned}$$

from which the uniqueness of μ follows.

Remark 5.3. If $\mathcal{T} \subseteq \mathcal{Y}$ is such that $\mathcal{H}^{N-1}(\mathcal{T}) = 0$, then

$$P\lfloor_{\Omega' \times \mathcal{T}} = 0.$$

Indeed, $P\lfloor_{\Omega' \times \mathcal{T}} = E_y\mu\lfloor_{\Omega' \times \mathcal{T}}$, and the conclusion results from Proposition 4.7(a).

The following disintegration result then holds:

Lemma 5.4 (Admissible configurations and disintegration). *Let* $(u, E, P) \in \mathcal{A}^{\mathrm{hom}}(w)$ *with associated* $\mu \in \boldsymbol{\mathcal{X}}(\Omega')$, *and set*

$$\eta := \mathcal{L}^N_x + (\mathrm{proj}_\# |P|)^s \in \mathcal{M}^+_b(\Omega').$$

The following disintegrations hold true:

$$Eu \otimes \mathcal{L}^N_y = A(x)(\eta \otimes \mathcal{L}^N_y), \tag{5.6}$$

$$E(x, y)(\mathcal{L}^N_x \otimes \mathcal{L}^N_y) = C(x)E(x, y)(\eta \otimes \mathcal{L}^N_y), \tag{5.7}$$

$$P = \eta \overset{\text{gen.}}{\otimes} P_x, \tag{5.8}$$

and we can choose a Borel map $(x, y) \mapsto \mu_x(y) \in \mathbb{R}^N$ *such that*

$$\mu = \mu_x(y)(\eta \otimes \mathcal{L}^N_y), \qquad E_y\mu = \eta \overset{\text{gen.}}{\otimes} E_y\mu_x. \tag{5.9}$$

Above, $A : \Omega' \to \mathbb{M}^N_{\mathrm{sym}}$ *and* $C : \Omega' \to [0, +\infty)$ *are the respective Radon–Nikodym derivatives of* Eu *and* $\mathcal{L}^N_x$ *with respect to* η, $E(x, y)$ *is a Borel representative of* E, *while* $\mu_x \in BD(\mathcal{Y})$, $\int_{\mathcal{Y}} \mu_x \, dy = 0$, *and* $P_x \in \mathcal{M}_b(\mathcal{Y}; \mathbb{M}^N_D)$ *for* η*-a.e.* $x \in \Omega'$.

In particular, for η*-a.e.* $x \in \Omega'$, *the measure* $P_x \in \mathcal{M}_b(\mathcal{Y}; \mathbb{M}^N_D)$ *is the plastic strain of the element of* $\mathcal{A}_{\mathcal{Y}}$ *given by*

$$(\mu_x, C(x)E(x, \cdot) - A(x), P_x).$$

Proof. Since $\mathrm{proj}_\#(E_y\mu) = 0$, from (5.5) we get

$$Eu = \left(\int_{\mathcal{Y}} E(x, y)\, dy\right)\mathcal{L}^N_x + \mathrm{proj}_\#(P) = e(x)\mathcal{L}^N_x + \mathrm{proj}_\#(P) \quad \text{ on } \Omega',$$

where $e(x) := \int_{\mathcal{Y}} E(x, y)\, dy \in L^2(\Omega'; \mathbb{M}^N_{\mathrm{sym}})$. Consequently, the measure Eu is absolutely continuous with respect to η. We can thus write

$$Eu \otimes \mathcal{L}^N_y = A(x)(\eta \otimes \mathcal{L}^N_y),$$

where $A : \Omega' \to \mathbb{M}^N_{\mathrm{sym}}$ is the Radon–Nikodym derivative of Eu with respect to η, so that (5.6) follows. If $C : \Omega' \to [0, +\infty)$ is the Radon–Nikodym derivative of $\mathcal{L}^N_x$ with respect to η, and $E(x, y)$ is a Borel representative of E, it is immediate that

$$E(x, y)\, (\mathcal{L}^N_x \otimes \mathcal{L}^N_y) = C(x)E(x, y)(\eta \otimes \mathcal{L}^N_y),$$

so that (5.7) holds true. Finally, by [4, Theorem 2.28], the measure P can be disintegrated with respect to $\mathrm{proj}_\# |P|$ which is absolutely continuous with respect to η, so that the disintegration (5.8) follows.

Let us come to (5.9). By Proposition 4.7(a),

$$\mu = \tilde{\mu}_x(y)(\zeta \otimes \mathcal{L}^N_y), \qquad E_y\mu = \zeta \overset{\text{gen.}}{\otimes} E_y\tilde{\mu}_x$$

for a suitable measure $\zeta \in \mathcal{M}^+_b(\Omega')$, and a suitable Borel function $(x, y) \mapsto \tilde{\mu}_x(y) \in \mathbb{R}^N$ with $\tilde{\mu}_x \in BD(\mathcal{Y})$, $\int_{\mathcal{Y}} \tilde{\mu}_x \, dy = 0$ and

$$|E_y\tilde{\mu}_x|(\mathcal{Y}) \neq 0$$

for ζ-a.e. $x \in \Omega'$. At the expense of replacing ζ with $|E_y\tilde{\mu}_x|(\mathcal{Y})\zeta$, it is not restrictive to assume that $|E_y\tilde{\mu}_x|(\mathcal{Y}) = 1$ for ζ-a.e. $x \in \Omega'$.

Since, by [4, Corollary 2.29], $\mathrm{proj}_\# |E_y\mu| = \zeta$, while, in view of the above,

$$\mathrm{proj}_\# |E_y\mu| = \left\{ \int_{\mathcal{Y}} |C(x)E(x, y) - A(x)| \, dy + |P_x|(\mathcal{Y}) \right\} \eta,$$

ζ is absolutely continuous with respect to η. Thus, $\zeta = D(x)\eta$, where $D : \Omega' \to [0, +\infty[$ can be chosen to be a Borel map. The disintegration (5.9) follows upon setting

$$\mu_x(y) := D(x)\tilde{\mu}_x(y).$$

Finally, note that, for η-a.e. $x \in \Omega'$,

$$E_y\mu_x = (C(x)E(x, \cdot) - A(x))\, \mathcal{L}^N_y + P_x.$$

Moreover, in view of the very definition of η, we have $C(x) \in [0, 1]$, so that

$$\int_{\Omega'} \left[\int_{\mathcal{Y}} |C(x)E(x, y)|^2 \, dy \right] d\eta \leq \int_{\Omega'} \left[\int_{\mathcal{Y}} |E(x, y)|^2 \, dy \right] dx < \infty.$$

Thus, $C(x)E(x, \cdot) - A(x) \in L^2(\mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}})$ for η-a.e. $x \in \Omega'$, and this proves the last assertion of the lemma. □

Remark 5.5. Since $|P| = \eta \overset{\text{gen.}}{\otimes} |P_x|$, we have

$$\eta \overset{\text{gen.}}{\otimes} \frac{P_x}{|P_x|} |P_x| = \eta \overset{\text{gen.}}{\otimes} P_x = P = \frac{P}{|P|} |P| = \eta \overset{\text{gen.}}{\otimes} \frac{P}{|P|} |P_x|,$$

so that, for η-a.e. $x \in \Omega'$,

$$\frac{P}{|P|}(x, \cdot) = \frac{P_x}{|P_x|} \qquad |P_x|\text{-a.e. on } \mathcal{Y}. \tag{5.10}$$

The definition of the class of admissible two-scale configurations is motivated by the following compactness result.

Lemma 5.6 (Compactness). *Let* $\{(u_\varepsilon, e_\varepsilon, p_\varepsilon)\}_{\varepsilon>0} \subset \mathcal{A}(w)$ *be such that*

$$\|u_\varepsilon\|_{BD(\Omega')} + \|e_\varepsilon\|_{L^2(\Omega';\mathbb{M}^N_{\mathrm{sym}})} + \|p_\varepsilon\|_{\mathcal{M}_b(\Omega';\mathbb{M}^N_D)} \leq C$$

and

$$\begin{aligned} &u_\varepsilon \rightharpoonup u && \textit{weakly* in } BD(\Omega'),\\ &e_\varepsilon \xrightarrow{w\text{-}2} E && \textit{two-scale weakly in } L^2(\Omega' \times \mathcal{Y}; \mathbb{M}^N_{\mathrm{sym}}),\\ &p_\varepsilon \xrightarrow{w^*\text{-}2} P && \textit{two-scale weakly* in } \mathcal{M}_b(\Omega' \times \mathcal{Y}; \mathbb{M}^N_D). \end{aligned}$$

Then $(u, E, P) \in \mathcal{A}^{\mathrm{hom}}(w)$.

Proof. Since $(u_\varepsilon, e_\varepsilon, p_\varepsilon) = (w, Ew, 0)$ on $\Omega' \setminus \overline{\Omega}$, it is immediate that (5.4) holds.

By compactness of the canonical injection of BD into L^1,

$$u_\varepsilon \to u \qquad \text{strongly in } L^1(\Omega'; \mathbb{R}^N),$$

so that

$$u_\varepsilon \mathcal{L}^N_x \xrightarrow{w^*\text{-}2} u(\mathcal{L}^N_x \otimes \mathcal{L}^N_y) \qquad \text{two-scale weakly* in } \mathcal{M}_b(\Omega' \times \mathcal{Y}; \mathbb{R}^N).$$

From the compatibility condition $Eu_\varepsilon = e_\varepsilon + p_\varepsilon$ on Ω' we deduce, in view of Proposition 4.10, the existence of $\mu \in \mathcal{X}(\Omega')$ such that

$$Eu(x) \otimes \mathcal{L}^N_y + E_y\mu = E(x, y)\,(\mathcal{L}^N_x \otimes \mathcal{L}^N_y) + P,$$

and the result follows. □

For $(u, E, P) \in \mathcal{A}^{\mathrm{hom}}(w)$ we set

$$\mathcal{Q}^{\mathrm{hom}}(E) := \frac{1}{2}\int_{\Omega\times\mathcal{Y}} \mathbb{C}(y)E \cdot E\,dx\,dy \tag{5.11}$$

$$\mathcal{H}^{\mathrm{hom}}(P) := \int_{(\Omega\cup\Gamma_d)\times\mathcal{Y}} H(y, P/|P|)\,d|P|. \tag{5.12}$$

We call $\mathcal{Q}^{\mathrm{hom}}$ the *homogenized elastic energy*, and $\mathcal{H}^{\mathrm{hom}}$ the *homogenized dissipation*. The domain of integration in the definition of $\mathcal{H}^{\mathrm{hom}}$ can be extended to Ω' since $P = 0$ on $(\Omega' \setminus \overline{\Omega}) \times \mathcal{Y}$.

The following lower semicontinuity result holds.

Theorem 5.7 (Lower semicontinuity). *Let* $(u_\varepsilon, e_\varepsilon, p_\varepsilon) \in \mathcal{A}(w)$ *be such that*

$$\begin{aligned} &u_\varepsilon \rightharpoonup u && \textit{weakly* in } BD(\Omega'),\\ &e_\varepsilon \xrightarrow{w\text{-}2} E && \textit{two-scale weakly in } L^2(\Omega' \times \mathcal{Y}; \mathbb{M}^N_{\mathrm{sym}}),\\ &p_\varepsilon \xrightarrow{w^*\text{-}2} P && \textit{two-scale weakly* in } \mathcal{M}_b(\Omega' \times \mathcal{Y}; \mathbb{M}^N_D), \end{aligned} \tag{5.13}$$

with $(u, E, P) \in \mathcal{A}^{\text{hom}}(w)$. *Then, for* $\mathcal{Q}_\varepsilon$ *and* $\mathcal{H}_\varepsilon$ *as in* (2.7) *and* (2.16) *respectively, we get*

$$\mathcal{Q}^{\text{hom}}(E) \le \liminf_{\varepsilon} \mathcal{Q}_\varepsilon(e_\varepsilon), \tag{5.14}$$

$$\mathcal{H}^{\text{hom}}(P) \le \liminf_{\varepsilon} \mathcal{H}_\varepsilon(p_\varepsilon). \tag{5.15}$$

Proof. We first prove (5.14). In view of Remark 4.12, it is readily seen that

$$\mathbb{C}_\varepsilon e_\varepsilon \xrightarrow{w\text{-}2} \mathbb{C}(y)E \quad \text{two-scale weakly in } L^2(\Omega' \times \mathcal{Y}; \mathrm{M}^N_{\text{sym}}).$$

Given $\Phi \in C^\infty_c(\Omega \times \mathcal{Y}; \mathrm{M}^N_{\text{sym}})$, and passing to the limit in the inequality

$$0 \le \frac{1}{2} \int_\Omega \mathbb{C}_\varepsilon(x)(e_\varepsilon - \Phi(x, x/\varepsilon)) \cdot (e_\varepsilon - \Phi(x, x/\varepsilon))\, dx$$

we obtain

$$\int_{\Omega\times\mathcal{Y}} \mathbb{C}(y)E \cdot \Phi(x, y)\, dx\, dy - \frac{1}{2}\int_{\Omega\times\mathcal{Y}} \mathbb{C}(y)\Phi(x, y) \cdot \Phi(x, y)\, dx\, dy \le \liminf_{\varepsilon} \mathcal{Q}_\varepsilon(e_\varepsilon).$$

Letting Φ converge to E strongly in $L^2(\Omega \times \mathcal{Y}; \mathrm{M}^N_{\text{sym}})$ yields (5.14).

The proof of (5.15) is more delicate, and we proceed in two steps.

Step 1. As a first step, consider $\mathcal{B} \subseteq \mathcal{Y}$, an open set with Lipschitz boundary, and also such that $\partial\mathcal{B} \setminus \mathcal{T}$ is C^1 for some compact set $\mathcal{T}$ with $\mathcal{H}^{N-1}(\mathcal{T}) = 0$. Assume also that $\partial\mathcal{B} \cap \mathcal{C} \subseteq \mathcal{T}$, where $\mathcal{C}$ has been introduced in (1.1).

Let $v_\varepsilon \in BD(\Omega')$ be such that

$$v_\varepsilon \overset{*}{\rightharpoonup} v \quad \text{weakly}^* \text{ in } BD(\Omega'),$$

and (see (1.2))

$$Ev_\varepsilon \lfloor_{\Omega' \cap \mathcal{B}_\varepsilon} \xrightarrow{w^*\text{-}2} \pi \quad \text{two-scale weakly}^* \text{ in } \mathcal{M}_b(\Omega' \times \mathcal{Y}; \mathrm{M}^N_{\text{sym}}).$$

We claim that π is supported in $\Omega' \times \bar{\mathcal{B}}$ and that

$$\pi\lfloor_{\Omega' \times (\partial\mathcal{B}\setminus\mathcal{T})} = a(x, y) \odot \nu(y)\, \zeta, \tag{5.16}$$

where $\zeta \in \mathcal{M}^+_b(\Omega' \times (\partial\mathcal{B} \setminus \mathcal{T}))$, $a : \Omega' \times (\partial\mathcal{B} \setminus \mathcal{T}) \to \mathbb{R}^N$ is a Borel map, and ν is the exterior normal to $\partial\mathcal{B}$.

Indeed, in view of Remark 4.3, the two-scale weak* limits (up to subsequences) of

$$Ev_\varepsilon \lfloor_{\Omega' \cap \mathcal{B}_\varepsilon \cap \mathcal{C}_\varepsilon} \in \mathcal{M}_b(\Omega'; \mathrm{M}^N_{\text{sym}})$$

have support concentrated on $\Omega' \times \overline{\mathcal{B} \cap \mathcal{C}}$. Since by assumption $\partial\mathcal{B} \cap \mathcal{C} \subseteq \mathcal{T}$, they do not contribute to the behaviour of π on $\Omega' \times (\partial\mathcal{B} \setminus \mathcal{T})$. We can therefore focus on the two-scale weak* limit $\tilde{\pi}$ (up to subsequences) of

$$Ev_\varepsilon \lfloor_{\Omega' \cap (\mathcal{B}_\varepsilon \setminus \mathcal{C}_\varepsilon)} \in \mathcal{M}_b(\Omega'; \mathrm{M}^N_{\text{sym}})$$

as

$$\pi\lfloor_{\Omega'\times(\partial\mathcal{B}\setminus\mathcal{T})}=\tilde{\pi}\lfloor_{\Omega'\times(\partial\mathcal{B}\setminus\mathcal{T})}.$$

Let

$$\sum_{i\in I_\varepsilon(\Omega')}(\mathcal{L}^N_x\lfloor_{Q^i_\varepsilon})\otimes\mu^i_\varepsilon\in\mathcal{M}_b(\Omega'\times\mathcal{Y};\mathrm{M}^N_{\mathrm{sym}})$$

be the unfolded measure associated with $Ev_\varepsilon\lfloor_{\Omega'\cap(\mathcal{B}_\varepsilon\setminus\mathcal{C}_\varepsilon)}$ according to (4.15). Then, appealing to Proposition 4.13, we get, for every $\chi\in C^1_c(\Omega'\times\mathcal{Y};\mathrm{M}^N_{\mathrm{sym}})$ with $\operatorname{div}_y\chi(x,y)=0$,

$$\begin{aligned}\int_{\Omega'\times\mathcal{Y}}\chi(x,y)\,d\tilde{\pi}(x,y)&=\lim_{\varepsilon\to0}\sum_{i\in I_\varepsilon(\Omega')}\int_{Q^i_\varepsilon}\left[\int_{\mathcal{B}\setminus\mathcal{C}}\chi(x,y)\cdot dE_y\hat{v}^i_\varepsilon\right]dx\\&=\lim_{\varepsilon\to0}\sum_{i\in I_\varepsilon(\Omega')}\int_{Q^i_\varepsilon}\left[\int_{\partial\mathcal{B}}\chi(x,y)\cdot(\hat{v}^i_\varepsilon(y)\odot\nu(y))\,d\mathcal{H}^{N-1}(y)-\int_{\mathcal{C}\cap\mathcal{B}}\chi(x,y)\cdot dE\hat{v}^i_\varepsilon\right]dx\end{aligned}\tag{5.17}$$

for a suitable $\hat{v}^i_\varepsilon\in BD(\mathcal{Y})$ such that

$$\int_{\partial\mathcal{B}}|\hat{v}^i_\varepsilon|\,d\mathcal{H}^{N-1}+|E_y\hat{v}^i_\varepsilon|(\mathcal{C}\cap\mathcal{B})\leq\frac{C}{\varepsilon^N}|Ev_\varepsilon|(\operatorname{int}(Q^i_\varepsilon)),\tag{5.18}$$

where $C>0$ independent of i and ε.

In view of (5.18) a density argument allows us to rewrite (5.17) as

$$\int_{\Omega'\times\mathcal{Y}}\chi\,d\tilde{\pi}=\lim_{\varepsilon\to0}\int_{\Omega'\times\mathcal{Y}}\chi\,d\lambda^1_\varepsilon+\int_{\Omega'\times\mathcal{Y}}\chi\,d\lambda^2_\varepsilon,\quad\chi\in C^0_0(\Omega'\times\mathcal{Y};\mathrm{M}^N_{\mathrm{sym}}),\ \operatorname{div}_y\chi=0,\tag{5.19}$$

with $\lambda^1_\varepsilon,\lambda^2_\varepsilon\in\mathcal{M}_b(\Omega'\times\mathcal{Y};\mathrm{M}^N_{\mathrm{sym}})$ such that (up to a subsequence)

$$\lambda^1_\varepsilon\overset{*}{\rightharpoonup}\lambda^1,\qquad\lambda^2_\varepsilon\overset{*}{\rightharpoonup}\lambda^2\qquad\text{weakly* in }\mathcal{M}_b(\Omega'\times\mathcal{Y};\mathrm{M}^N_{\mathrm{sym}}).$$

Moreover $\operatorname{supp}(\lambda^1)\subseteq\Omega'\times\partial\mathcal{B}$ and $\operatorname{supp}(\lambda^2)\subseteq\Omega'\times\overline{\mathcal{C}\cap\mathcal{B}}$. In view of (5.19), Lemma 4.9 implies the existence of $\mu\in\boldsymbol{\mathcal{X}}(\Omega')$ such that

$$\tilde{\pi}=\lambda^1+\lambda^2+E_y\mu.$$

Recalling that $\partial\mathcal{B}\cap\mathcal{C}\subseteq\mathcal{T}$,

$$\tilde{\pi}\lfloor_{\partial\mathcal{B}\setminus\mathcal{T}}=\lambda^1\lfloor_{\partial\mathcal{B}\setminus\mathcal{T}}+E_y\mu\lfloor_{\partial\mathcal{B}\setminus\mathcal{T}}.$$

Thanks to Proposition 4.7(b), the proof is complete if we show the analogue of (5.16) for $\lambda^1\lfloor_{\partial\mathcal{B}\setminus\mathcal{T}}$.

Consider

$$\eta_\varepsilon:=\hat{v}_\varepsilon(x,y)(\mathcal{L}^N_x\otimes(\mathcal{H}^{N-1}_y\lfloor_{\partial\mathcal{B}}))\in\mathcal{M}_b(\Omega'\times\mathcal{Y};\mathbb{R}^N)$$

with

$$\hat{v}_\varepsilon(x,y) := \sum_{i\in I_\varepsilon(\Omega')} 1_{Q^i_\varepsilon}(x)\hat{v}^i_\varepsilon(y),$$

so that $\lambda^1_\varepsilon = \eta_\varepsilon(x,y)\odot\nu(y)$ for any Borel extension of ν to $\mathcal{Y}$. In view of (5.18), up to a subsequence,

$$\eta_\varepsilon \overset{*}{\rightharpoonup} \eta \quad \text{weakly* in } \mathcal{M}_b(\Omega'\times\mathcal{Y};\mathbb{R}^N)$$

for some $\eta\in\mathcal{M}_b(\Omega'\times\mathcal{Y};\mathbb{R}^N)$. Since ν is continuous along $\partial\mathcal{B}\setminus\mathcal{T}$, we immediately get

$$\lambda^1\lfloor_{\partial\mathcal{B}\setminus\mathcal{T}} = \frac{\eta}{|\eta|}\odot\nu|\eta|\lfloor_{\partial\mathcal{B}\setminus\mathcal{T}},$$

so that claim (5.16) follows because $\eta/|\eta|$ is a Borel function.

Step 2. We now prove (5.15), assuming, with no loss of generality, that

$$\liminf_\varepsilon \mathcal{H}_\varepsilon(p_\varepsilon) < \infty. \tag{5.20}$$

We decompose p_ε as

$$p_\varepsilon = \sum_i p^i_\varepsilon + \sum_{i\neq j} p^{ij}_\varepsilon$$

where, since p_ε does not charge $\mathcal{H}^{N-1}$-negligible sets,

$$p^i_\varepsilon := p_\varepsilon\lfloor_{\Omega'\cap(\mathcal{Y}_i)_\varepsilon} \quad\text{and}\quad p^{ij}_\varepsilon := p_\varepsilon\lfloor_{\Omega'\cap(\Gamma_{ij}\setminus\mathcal{S})_\varepsilon}.$$

Up to a subsequence,

$$p^i_\varepsilon \xrightarrow{w^*\text{-}2} P^i \qquad \text{two-scale weakly* in } \mathcal{M}_b(\Omega'\times\mathcal{Y};\mathrm{M}^N_D),$$

$$p^{ij}_\varepsilon \xrightarrow{w^*\text{-}2} P^{ij} \qquad \text{two-scale weakly* in } \mathcal{M}_b(\Omega'\times\mathcal{Y};\mathrm{M}^N_D).$$

Clearly

$$P = \sum_i P^i + \sum_{i\neq j} P^{ij} \tag{5.21}$$

with $\mathrm{supp}(P^i)\subseteq\bar{\Omega}\times\bar{\mathcal{Y}}_i$ and, thanks to Remark 4.3, $\mathrm{supp}(P^{ij})\subseteq\bar{\Omega}\times\Gamma_{ij}$.

Invoking Lemma 4.6 we get

$$\begin{aligned}
\liminf_\varepsilon \int_{\Omega\cup\Gamma_d} H_\varepsilon(x, p^i_\varepsilon/|p^i_\varepsilon|)\,d|p^i_\varepsilon| &= \liminf_\varepsilon \int_{\Omega'} H(x/\varepsilon, p^i_\varepsilon/|p^i_\varepsilon|)\,d|p^i_\varepsilon| \\
&= \liminf_\varepsilon \int_{\Omega'} H_i(p^i_\varepsilon/|p^i_\varepsilon|)\,d|p^i_\varepsilon| \ge \int_{\Omega'\times\mathcal{Y}} H_i(P^i/|P^i|)\,d|P^i| \\
&= \int_{\Omega'\times\mathcal{Y}_i} H_i(P^i/|P^i|)\,d|P^i| + \int_{\Omega'\times\Gamma} H_i(P^i/|P^i|)\,d|P^i| \\
&\ge \int_{\Omega'\times\mathcal{Y}_i} H(y, P^i/|P^i|)\,d|P^i| + \sum_{j\neq i}\int_{\Omega'\times(\Gamma_{ij}\setminus\mathcal{S})} H_i(P^i/|P^i|)\,d|P^i|.
\end{aligned}$$

By (5.13) $e_\varepsilon \xrightarrow{w\text{-}2} E$ two-scale weakly in $L^2(\Omega' \times \mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}})$, so that

$$Eu_\varepsilon \lfloor_{\Omega' \cap (\mathcal{Y}_i)_\varepsilon} \xrightarrow{w^*\text{-}2} E 1_{\Omega' \times \mathcal{Y}_i} (\mathcal{L}^N_x \otimes \mathcal{L}^N_y) + P^i \quad \text{two-scale weakly* in } \mathcal{M}_b(\Omega' \times \mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}}).$$

We denote by ν the normal to Γ_{ij} pointing from $\mathcal{Y}_j$ to $\mathcal{Y}_i$. Since, according to (2.3), $\mathcal{H}^{N-1}(\Gamma \cap \mathcal{C}) = 0$, so that we may as well identify $\mathcal{S}$ with $\mathcal{S} \cup (\Gamma \cap \mathcal{C})$, ensuring that $\Gamma \cap \mathcal{C} \subset \mathcal{S}$, the first step of the proof implies that, for every $j \neq i$,

$$P^i \lfloor_{\Omega \times (\Gamma_{ij} \setminus \mathcal{S})} = -(a^{ij} \odot \nu)\eta^{ij} \tag{5.22}$$

for a suitable $\eta^{ij} \in \mathcal{M}^+_b(\Omega' \times (\Gamma_{ij} \setminus \mathcal{S}))$, and suitable Borel functions $a^{ij} : \Omega' \times (\Gamma_{ij} \setminus \mathcal{S}) \to \mathbb{R}^N$ such that $a^{ij}(x) \perp \nu(x)$ for η^{ij}-a.e. $(x, y) \in \Omega \times (\Gamma_{ij} \setminus \mathcal{S})$ (recall that P^i has values in M^N_D). Thus,

$$\begin{aligned}\liminf_\varepsilon \int_{\Omega \cup \Gamma_d} & H_\varepsilon(x, p^i_\varepsilon/|p^i_\varepsilon|)\, d|p^i_\varepsilon| \\ & \geq \int_{\Omega' \times \mathcal{Y}_i} H(y, P^i/|P^i|)\, d|P^i| + \sum_{j \neq i} \int_{\Omega' \times (\Gamma_{ij} \setminus \mathcal{S})} H_i(-a^{ij} \odot \nu)\, d\eta^{ij}. \end{aligned} \tag{5.23}$$

As to p^{ij}_ε,

$$p^{ij}_\varepsilon = (u^i_\varepsilon - u^j_\varepsilon) \odot \nu(x/\varepsilon)\, \mathcal{H}^{N-1} \lfloor_{(\Gamma_{ij} \setminus \mathcal{S})_\varepsilon},$$

where u^i_ε and u^j_ε are the traces of u_ε on $\Omega' \cap (\Gamma_{ij} \setminus \mathcal{S})_\varepsilon$ coming from $(\mathcal{Y}_i)_\varepsilon$ and $(\mathcal{Y}_j)_\varepsilon$ respectively. In view of the definition of H on $\Gamma_{ij} \setminus \mathcal{S}$ (see (2.13)), and since the inf-convolution is indeed attained as a minimum, we get

$$\begin{aligned}\int_{\Omega \cup \Gamma_d} H_\varepsilon(x, p^{ij}_\varepsilon/|p^{ij}_\varepsilon|)\, d|p^{ij}_\varepsilon| &= \int_{\Omega' \cap (\Gamma_{ij} \setminus \mathcal{S})_\varepsilon} H_\varepsilon(x, p^{ij}_\varepsilon/|p^{ij}_\varepsilon|)\, d|p^{ij}_\varepsilon| \\ &= \int_{\Omega' \cap (\Gamma_{ij} \setminus \mathcal{S})_\varepsilon} H_\varepsilon(x, (u^i_\varepsilon - u^j_\varepsilon)(x) \odot \nu(x/\varepsilon))\, d\mathcal{H}^{N-1} \\ &= \int_{\Omega' \cap (\Gamma_{ij} \setminus \mathcal{S})_\varepsilon} [H_i(b^{ij}_{i,\varepsilon}(x) \odot \nu(x/\varepsilon)) + H_j(-b^{ij}_{j,\varepsilon}(x) \odot \nu(x/\varepsilon))]\, d\mathcal{H}^{N-1} \end{aligned} \tag{5.24}$$

for suitable Borel functions $b^{ij}_{i,\varepsilon}, b^{ij}_{j,\varepsilon} : \Omega' \cap (\Gamma_{ij} \setminus \mathcal{S})_\varepsilon \to \mathbb{R}^N$ such that

$$b^{ij}_{i,\varepsilon}(x) - b^{ij}_{j,\varepsilon}(x) = u^i_\varepsilon(x) - u^j_\varepsilon(x) \quad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in (\Gamma_{ij} \setminus \mathcal{S})_\varepsilon$$

with

$$b^{ij}_{i,\varepsilon}(x) \perp \nu(x/\varepsilon), b^{ij}_{j,\varepsilon}(x) \perp \nu(x/\varepsilon) \quad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in (\Gamma_{ij} \setminus \mathcal{S})_\varepsilon.$$

That the functions $b^{ij}_{i,\varepsilon}, b^{ij}_{j,\varepsilon}$ are Borel can be proved by approximating $u^i_\varepsilon - u^j_\varepsilon$ along $(\Gamma_{ij} \setminus \mathcal{S})_\varepsilon$ by simple functions, and recalling that ν is continuous.

In view of the coercivity estimate (2.12) and of the bound (5.20) we obtain

$$\int_{\Omega' \cap (\Gamma_{ij} \setminus \mathcal{S})_\varepsilon} [|b^{ij}_{i,\varepsilon}(x) \odot \nu(x/\varepsilon)| + |b^{ij}_{j,\varepsilon}(x) \odot \nu(x/\varepsilon)|]\, d\mathcal{H}^{N-1}(x) \leq C$$

for a suitable constant $C > 0$. The bound above actually implies that the measures

$$\eta^{ij}_{i,\varepsilon} := b^{ij}_{i,\varepsilon}\mathcal{H}^{N-1}\lfloor_{\Omega'\cap(\Gamma_{ij}\setminus\mathcal{S})_\varepsilon} \quad \text{and} \quad \eta^{ij}_{j,\varepsilon} := b^{ij}_{j,\varepsilon}\mathcal{H}^{N-1}\lfloor_{\Omega'\cap(\Gamma_{ij}\setminus\mathcal{S})_\varepsilon}$$

are bounded in ε. Thus, recalling Remark 4.3, we can assume that, up to a subsequence that will not be relabeled,

$$\begin{cases} b^{ij}_{i,\varepsilon} \odot \nu(x/\varepsilon)\,\mathcal{H}^{N-1}\lfloor_{\Omega'\cap(\Gamma_{ij}\setminus\mathcal{S})_\varepsilon} \xrightarrow{w^*\text{-}2} \lambda^{ij} & \text{two-scale weakly* in } \mathcal{M}_b(\Omega' \times \mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}}), \\ b^{ij}_{j,\varepsilon} \odot \nu(x/\varepsilon)\,\mathcal{H}^{N-1}\lfloor_{\Omega'\cap(\Gamma_{ij}\setminus\mathcal{S})_\varepsilon} \xrightarrow{w^*\text{-}2} \lambda^{ji} & \text{two-scale weakly* in } \mathcal{M}_b(\Omega' \times \mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}}), \end{cases}$$

and

$$\begin{cases} \eta^{ij}_{i,\varepsilon} \xrightarrow{w^*\text{-}2} \eta^{ij}_i = b^{ij}_i|\eta^{ij}_i| & \text{two-scale weakly* in } \mathcal{M}_b(\Omega' \times \mathcal{Y}; \mathbb{R}^N), \\ \eta^{ij}_{j,\varepsilon} \xrightarrow{w^*\text{-}2} \eta^{ij}_j = b^{ij}_j|\eta^{ij}_j| & \text{two-scale weakly* in } \mathcal{M}_b(\Omega' \times \mathcal{Y}; \mathbb{R}^N), \end{cases}$$

with $\lambda^{ij}, \lambda^{ji} \in \mathcal{M}_b(\Omega' \times \mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}})$ and $\eta^{ij}_i, \eta^{ij}_j \in \mathcal{M}_b(\Omega' \times \mathcal{Y}; \mathbb{R}^N)$ such that

$$\mathrm{supp}(\lambda^{ij}), \mathrm{supp}(\lambda^{ji}), \mathrm{supp}(\eta^{ij}_i), \mathrm{supp}(\eta^{ij}_j) \subseteq \bar{\Omega} \times \Gamma_{ij}.$$

Since the normal vector field ν is continuous on $\Gamma_{ij} \setminus \mathcal{S}$, we get

$$\lambda^{ij} = (b^{ij}_i \odot \nu)|\eta^{ij}_i| \quad \text{and} \quad \lambda^{ji} = (b^{ij}_j \odot \nu)|\eta^{ij}_j| \quad \text{on } \Omega' \times (\Gamma_{ij} \setminus \mathcal{S}).$$

In view of Lemma 4.6 we obtain

$$\begin{aligned} &\liminf_\varepsilon \int_{(\Omega\cup\Gamma_d)\cap(\Gamma_{ij}\setminus\mathcal{S})_\varepsilon} H_\varepsilon(x, p^{ij}_\varepsilon/|p^{ij}_\varepsilon|)\,d|p^{ij}_\varepsilon| \\ &\quad = \liminf_\varepsilon \int_{\Omega'\cap(\Gamma_{ij}\setminus\mathcal{S})_\varepsilon} H_\varepsilon(x, p^{ij}_\varepsilon/|p^{ij}_\varepsilon|)\,d|p^{ij}_\varepsilon| \\ &\quad = \liminf_\varepsilon \int_{\Omega'\cap(\Gamma_{ij}\setminus\mathcal{S})_\varepsilon} [H_i(b^{ij}_{i,\varepsilon}(x) \odot \nu(x/\varepsilon)) + H_j(-b^{ij}_{j,\varepsilon}(x) \odot \nu(x/\varepsilon))]\,d\mathcal{H}^{N-1}(x) \\ &\quad \geq \int_{\Omega'\times(\Gamma_{ij}\setminus\mathcal{S})} H_i(b^{ij}_i \odot \nu(y))\,d|\eta^{ij}_i| + \int_{\Omega'\times(\Gamma_{ij}\setminus\mathcal{S})} H_j(-b^{ij}_j \odot \nu(y))\,d|\eta^{ij}_j|. \end{aligned} \tag{5.25}$$

Recalling (5.21) and (5.22), the previous analysis shows that

$$\begin{aligned} P\lfloor_{\Omega'\times(\Gamma_{ij}\setminus\mathcal{S})} &= -(a^{ij} \odot \nu)\eta^{ij} + (a^{ji} \odot \nu)\eta^{ji} + (b^{ij}_i \odot \nu)|\eta^{ij}_i| - (b^{ij}_j \odot \nu)|\eta^{ij}_j| \\ &= [(c^i - c^j) \odot \nu]\,\zeta^{ij}, \end{aligned} \tag{5.26}$$

where $\zeta^{ij} := \eta^{ij} + \eta^{ji} + |\eta^{ij}_i| + |\eta^{ij}_j|$, and c^i, c^j are suitable Borel functions on $\Omega' \times (\Gamma_{ij}\setminus\mathcal{S})$ with values in $\mathbb{R}^N$ such that

$$(c^i \odot \nu)\zeta^{ij} = -(a^{ij} \odot \nu)\lambda^{ij} + (b^{ij}_i \odot \nu)|\eta^{ij}_i|,$$

idem for c^j. Further,

$$c^i(x,y)\perp \nu(y),\quad c^j(x,y)\perp \nu(y)\quad \text{for } \zeta^{ij}\text{-a.e. } (x,y)\in \Omega'\times(\Gamma_{ij}\backslash\mathcal{S}).$$

Since

$$\begin{aligned}\mathcal{H}_\varepsilon(p_\varepsilon)=&\sum_i\int_{\Omega\cup\Gamma_d} H_\varepsilon(x,p^i_\varepsilon/|p^i_\varepsilon|)\,d|p^i_\varepsilon|\\&+\sum_{i\neq j}\int_{(\Omega\cup\Gamma_d)\cap(\Gamma_{ij}\backslash\mathcal{S})_\varepsilon} H_\varepsilon(x,p^{ij}_\varepsilon/|p^{ij}_\varepsilon|)\,d|p^{ij}_\varepsilon|,\end{aligned}$$

we get, thanks to (5.23) and (5.25),

$$\begin{aligned}\liminf_\varepsilon \mathcal{H}_\varepsilon(p_\varepsilon)\geq&\sum_i\liminf_\varepsilon\int_{\Omega\cup\Gamma_d} H_\varepsilon(x,p^i_\varepsilon/|p^i_\varepsilon|)\,d|p^i_\varepsilon|\\&+\sum_{i\neq j}\liminf_\varepsilon\int_{(\Omega\cup\Gamma_d)\cap(\Gamma_{ij}\backslash\mathcal{S})_\varepsilon} H_\varepsilon(x,p^{ij}_\varepsilon/|p^{ij}_\varepsilon|)\,d|p^{ij}_\varepsilon|\\ \geq&\sum_i\Bigg(\int_{\Omega'\times\mathcal{Y}_i} H(y,P^i/|P^i|)\,d|P^i|+\sum_{j\neq i}\int_{\Omega'\times(\Gamma_{ij}\backslash\mathcal{S})} H_i(-a^{ij}\odot\nu)\,d\eta^{ij}\Bigg)\\&+\sum_{i\neq j}\Bigg(\int_{\Omega'\times(\Gamma_{ij}\backslash\mathcal{S})} H_i(b^{ij}_i\odot\nu)\,d|\eta^{ij}_i|+\int_{\Omega'\times(\Gamma_{ij}\backslash\mathcal{S})} H_j(-b^{ij}_j\odot\nu)\,d|\eta^{ij}_j|\Bigg)\\ =&\int_{\Omega'\times\bigcup_i\mathcal{Y}_i} H(y,p/|p|)\,d|p|\\&+\sum_{i\neq j}\Bigg(\int_{\Omega'\times(\Gamma_{ij}\backslash\mathcal{S})} H_i(-a^{ij}\odot\nu)\,d\eta^{ij}+\int_{\Omega'\times(\Gamma_{ij}\backslash\mathcal{S})} H_j(a^{ji}\odot\nu)\,d\eta^{ji}\\&+\int_{\Omega'\times(\Gamma_{ij}\backslash\mathcal{S})} H_i(b^{ij}_i\odot\nu)\,d|\eta^{ij}_i|+\int_{\Omega'\times(\Gamma_{ij}\backslash\mathcal{S})} H_j(-b^{ij}_j\odot\nu)\,d|\eta^{ij}_j|\Bigg).\end{aligned}$$

In view of (5.26), by the definition of H on $\Omega'\times(\Gamma_{ij}\backslash\mathcal{S})$ and the subadditive character of H_i and H_j, and since, in view of Remark 5.3, P does not charge $\Omega'\times\mathcal{S}$, we deduce that

$$\begin{aligned}\liminf_\varepsilon \mathcal{H}_\varepsilon(p_\varepsilon)\geq&\int_{\Omega'\times\bigcup_i\mathcal{Y}_i} H(y,P/|P|)\,d|P|\\&+\sum_{i\neq j}\int_{\Omega'\times(\Gamma_{ij}\backslash\mathcal{S})}[H_i(c^i(x,y)\odot\nu(y))+H_j(-c^j(x,y)\odot\nu(y))]\,d\zeta^{ij}(x,y)\\ \geq&\int_{\Omega'\times\bigcup_i\mathcal{Y}_i} H(y,P/|P|)\,d|P|+\sum_{i\neq j}\int_{\Omega'\times(\Gamma_{ij}\backslash\mathcal{S})} H(y,(c^i-c^j)\odot\nu)\,d\zeta^{ij}\\ =&\int_{\Omega'\times\bigcup_i\mathcal{Y}_i} H(y,P/|P|)\,d|P|+\sum_{i\neq j}\int_{\Omega'\times(\Gamma_{ij}\backslash\mathcal{S})} H(y,P/|P|)\,d|P|=\mathcal{H}^{\text{hom}}(P),\end{aligned}$$

which concludes the proof. □

5.2. *Two-scale statics and duality*

In this subsection we define two-scale (statically admissible) stress configurations, investigate the duality between those and elements of $\mathcal{A}^{\mathrm{hom}}(w)$ in the spirit of Theorem 3.3 and Proposition 3.5, and show that they naturally arise as two-scale weak limits of statically admissible stress fields.

We adopt the following

Definition 5.8 (Two-scale static admissibility). An element $\Sigma \in L^2(\Omega \times \mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}})$ such that

$$\operatorname{div}_y \Sigma = 0 \quad \text{on } \Omega \times \mathcal{Y}, \qquad \Sigma_D(x, y) \in K(y) \quad \text{for } \mathcal{L}^N_x \otimes \mathcal{L}^N_y\text{-a.e. } (x, y) \in \Omega \times \mathcal{Y}$$

and

$$\operatorname{div}_x \sigma = 0 \quad \text{in } \Omega, \qquad \sigma \cdot \nu = 0 \quad \text{on } \partial\Omega \setminus \overline{\Gamma}_d, \tag{5.27}$$

where $\sigma(x) := \int_{\mathcal{Y}} \Sigma(x, y)\, dy$, is said to be *two-scale statically admissible*; we denote by $\mathcal{K}^{\mathrm{hom}}$ the set of all such stresses.

Remark 5.9. Recalling Definition 3.2, if $\Sigma \in \mathcal{K}^{\mathrm{hom}}$, then, for a.e. $x \in \Omega$,

$$\Sigma(x, \cdot) \in \mathcal{K}_{\mathcal{Y}}.$$

According to (3.1), there exists, for every $1 \le r < \infty$, a constant $C_r > 0$ (independent of x) such that

$$\|\Sigma(x, \cdot)\|_{L^r(\mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}})} \le C_r[\|\Sigma(x, \cdot)\|_{L^2(\mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}})} + \|\Sigma_D(x, \cdot)\|_{L^\infty(\mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}})}].$$

Let $P \in \Pi^{\mathrm{hom}}(w)$ and $\Sigma \in \mathcal{K}^{\mathrm{hom}}$. In view of Lemma 5.4, $P = \eta \overset{\text{gen.}}{\otimes} P_x$, P_x being a plastic strain for an admissible configuration on $\mathcal{Y}$ for η-a.e. $x \in \Omega'$. On the other hand, according to Remark 5.9, for $\mathcal{L}^N_x$-a.e. $x \in \Omega$, $\Sigma_x := \Sigma(x, \cdot) \in L^2(\mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}})$ is a statically admissible stress field on $\mathcal{Y}$. Thus it would be tempting to conclude that, recalling Theorem 3.3, a coupling between P_x and Σ_x is available on *almost every fiber* with base in Ω. But there is a snag: the measure η can have concentrated parts, while Σ_x is only well defined almost everywhere with respect to the Lebesgue measure. To overcome this difficulty, we will have to construct in a first step an adequate approximation of Σ (see Lemma 5.10), then use that approximation to define in turn a (disintegrated) two-scale analogue of the duality measure $\langle \Sigma_D, P \rangle$ defined in (3.2) (see Proposition 5.11) and to obtain the analogue of Proposition 3.5 (see Theorem 5.12).

Lemma 5.10 (Approximation of stresses). *Let* $\Sigma \in \mathcal{K}^{\mathrm{hom}}$. *There exists a mapping* $\Sigma_n : \mathbb{R}^N \times \mathcal{Y} \to \mathrm{M}^N_{\mathrm{sym}}$ *with*

$$\Sigma_n \in L^2(\mathbb{R}^N \times \mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}}), \tag{5.28}$$

and such that the following holds:

(a) $\Sigma_n(x, y) \in C^\infty(\mathbb{R}^N; L^2(\mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}}))$;

(b) $\operatorname{div}_y \Sigma_n(x,\cdot) = 0$ *on* $\mathcal{Y}$ *for every* $x \in \mathbb{R}^N$, *and*

$$\|\Sigma_n(x,y)\|_{L^2(\mathcal{Y};\mathbb{M}^N_{\mathrm{sym}})} \le \tilde{C}_n \|\Sigma\|_{L^2(\Omega\times\mathcal{Y};\mathbb{M}^N_{\mathrm{sym}})},$$

where $\tilde{C}_n$ *does not depend on* x. *Moreover*

$$\sup_n \|(\Sigma_n)_D(x,\cdot)\|_\infty < \infty$$

and for every $1 \le r < \infty$ *there exists* $C_n > 0$ *such that*

$$\|\Sigma_n(x,\cdot)\|_{L^r(\mathcal{Y};\mathbb{M}^N_{\mathrm{sym}})} \le C_n;$$

(c) *for every* $\varepsilon > 0$, *there exists* N_ε *such that, for* $n \ge N_\varepsilon$ *and for every* $x \in \mathbb{R}^N$,

$$(\Sigma_n(x,y))_D \in (1+\varepsilon)K(y) \quad \text{for a.e. } y \in \mathcal{Y};$$

(d) $\Sigma_n \to \Sigma$ *strongly in* $L^2(\Omega\times\mathcal{Y};\mathbb{M}^N_{\mathrm{sym}})$; *and*

(e) *setting* $\sigma_n(x) := \int_{\mathcal{Y}} \Sigma_n(x,y)\,dy$ *and* $\sigma(x) := \int_{\mathcal{Y}} \Sigma(x,y)\,dy$, $\sigma_n \in C^\infty(\mathbb{R}^N;\mathbb{M}^N_{\mathrm{sym}})$,

$$\begin{aligned} &\sup \|(\sigma_n)_D\|_\infty < \infty, \\ &\sigma_n \to \sigma \qquad \text{strongly in } L^2(\Omega;\mathbb{M}^N_{\mathrm{sym}}), \\ &\operatorname{div}\sigma_n \to 0 \qquad \text{strongly in } L^N(\Omega;\mathbb{R}^N), \\ &\sigma_n \to \sigma \qquad \text{strongly in } L^r(\Omega;\mathbb{M}^N_{\mathrm{sym}}) \text{ for every } 1 \le r < \infty. \end{aligned}$$

Proof. Let us extend Σ to $\mathbb{R}^N \times \mathcal{Y}$ by setting $\Sigma = 0$ outside Ω. For every $x \in \partial\Omega$, consider an open neighborhood U such that $\partial\Omega \cap U$ is a Lipschitz subgraph with respect to a suitable coordinate system. We cover $\partial\Omega$ with finitely many open sets $U_1,\ldots,U_m$ associated with $x_1,\ldots,x_m \in \partial\Omega$, and assume that there exist $\tau_i \in \mathbb{R}^N$ such that

$$(U_i \cap \overline{\Omega}) + a\tau_i \subset\subset \Omega, \qquad 0 < a < 1. \tag{5.29}$$

Let $\{\psi_i\}_{i=1}^m$ be a partition of unity of $\partial\Omega$ subordinated to $\{U_i\}_{i=1}^m$. Write

$$\Sigma = \sum_{i=1}^m \psi_i \Sigma + \Big(1 - \sum_{i=1}^m \psi_i\Big)\Sigma := \sum_{i=1}^m \Sigma_i + \Sigma_0, \tag{5.30}$$

the last term having compact support in $\Omega \times \mathcal{Y}$.

The approximation Σ_n is obtained by infinitesimally translating each Σ_i in the direction $-\tau_i$ and taking a convolution with respect to x, while Σ_0 is simply regularized by convolution with respect to x. We then use a diagonal argument.

Indeed, (5.28) and items (a) and (d) immediately follow, while (b) follows by the definition of $\mathcal{K}^{\mathrm{hom}}$ and the continuity of the ψ_i's if one further takes Remark 5.9 into account. As far as (c) is concerned, the definition of $\mathcal{K}^{\mathrm{hom}}$ implies that, for a.e. $x \in \mathbb{R}^N$ and a.e. $y \in \mathcal{Y}$,

$$(\Sigma_i)_D(x,y) \in \psi_i(x)K(y) \quad (i = 1,\ldots,m), \qquad (\Sigma_0)_D(x,y) \in \Big(1 - \sum_{i=1}^m \psi_i(x)\Big)K(y).$$

Given $\varepsilon > 0$, in view of the continuity of ψ_i and of the convexity of $K(y)$, the construction above shows that, for n large enough, and for every $x \in \mathbb{R}^N$ and a.e. $y \in \mathcal{Y}$,

$$(\Sigma_i^n)_D(x, y) \in (\psi_i(x) + \varepsilon)K(y), \quad (\Sigma_0^n)_D(x, y) \in \Big(1 + \varepsilon - \sum_{i=1}^{m} \psi_i(x)\Big)K(y),$$

so that, using the convexity of $K(y)$ once more, $(\Sigma_n(x, y))_D \in (1 + (m + 1)\varepsilon)K(y)$ for a.e. $y \in \mathcal{Y}$. Item (c) thus follows in view of the arbitrariness of ε.

Finally, to prove (e) we need only justify the convergence of $\operatorname{div} \sigma_n$, the first two properties being a consequence of the previous items modulo an integration in y, while the last statement is a consequence of the inequality in Remark 2.2. From (5.30) we deduce, by integrating in y,

$$\sigma = \sum_{i=1}^{m} \psi_i \sigma + \Big(1 - \sum_{i=1}^{m} \psi_i\Big)\sigma.$$

The associated approximation obtained by translations and convolutions can be written explicitly as

$$\sigma_n(x) = \rho_n(x) \star \Big[\sum_{i=1}^{m} \psi_i(x + a_n\tau_i)\sigma(x + a_n\tau_i) + \Big(1 - \sum_{i=1}^{m} \psi_i(x)\Big)\sigma(x)\Big]$$

with $a_n \searrow 0$ and $\{\rho_n\}_{n\in\mathbb{N}}$ suitable convolution kernels. Since $\operatorname{div}(\sigma(x+a_n\tau_i)) = 0$ thanks to (5.29), the convergence follows from (5.27) and Remark 2.2, which imply that σ is in $L^r(\Omega; \mathrm{M}^N_{\mathrm{sym}})$ for $1 \le r < \infty$. □

Proposition 5.11 (Two-scale duality). *Let $\Sigma \in \mathcal{K}^{\mathrm{hom}}$, and $(u, E, P) \in \mathcal{A}^{\mathrm{hom}}(w)$. Let $\eta \in \mathcal{M}_b^+(\Omega')$ be the measure such that $P = \eta \overset{\text{gen.}}{\otimes} P_x$, with $P_x \in \mathcal{M}_b(\mathcal{Y}; \mathrm{M}^N_D)$, according to Lemma 5.4.*

(a) *If $\{\Sigma_n\}_{n\in\mathbb{N}}$ is the sequence given by Lemma 5.10, the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ defined as*

$$\lambda_n := \eta \overset{\text{gen.}}{\otimes} \langle (\Sigma_n)_D(x, \cdot), P_x\rangle$$

(where $\langle (\Sigma_n)_D(x, \cdot), P_x\rangle$ is the measure on $\mathcal{Y}$ associated with the duality between the stress $\Sigma_n(x, \cdot)$ and the plastic strain P_x according to Remark 3.4) is a bounded sequence of elements of $\mathcal{M}_b(\Omega' \times \mathcal{Y})$.

(b) *There exists a subsequence of $\{\lambda_n\}_{n\in\mathbb{N}}$ (still indexed by n) and an element $\lambda \in \mathcal{M}_b(\Omega' \times \mathcal{Y})$ such that*

$$\lambda_n \overset{*}{\rightharpoonup} \lambda \quad \textit{weakly}^* \textit{ in } \Omega' \times \mathcal{Y},$$

with

$$\lambda = (\mathcal{L}^N_x \llcorner_{\Omega}) \overset{\text{gen.}}{\otimes} \langle \Sigma_D(x, \cdot), P_x\rangle + \lambda^s, \tag{5.31}$$

where $\langle \Sigma_D(x, \cdot), P_x\rangle \in \mathcal{M}_b(\mathcal{Y})$ denotes the duality between the stress $\Sigma_D(x, \cdot) \in \mathcal{K}_{\mathcal{Y}}$ and the plastic strain $P_x \in \Pi_{\mathcal{Y}}$, and where $\lambda^s \in \mathcal{M}_b(\Omega' \times \mathcal{Y})$ is such that

$$|\lambda^s| \ll \eta^s \overset{\text{gen.}}{\otimes} |P_x|.$$

Finally, if $\partial\lfloor_{\partial\Omega}\Gamma_d$ *is admissible in the sense of Definition* 2.1, *the mass of* λ *is given by*

$$\lambda(\Omega'\times\mathcal{Y}) = -\int_{\Omega\times\mathcal{Y}} \Sigma\cdot E\,dx\,dy + \int_{\Omega}\sigma\cdot Ew\,dx. \tag{5.32}$$

Proof. (a) By Lemma 5.4, for η-a.e. $x\in\Omega'$ the measure $P_x\in\mathcal{M}_b(\mathcal{Y};\mathbb{M}^N_D)$ is the plastic strain of the admissible configuration on $\mathcal{Y}$ given by $(\mu_x, C(x)E(x,\cdot)-A(x), P_x)$, where $\mu_x\in BD(\mathcal{Y})$, while $C:\Omega'\to[0,1]$ and $A:\Omega'\to\mathbb{M}^N_{\mathrm{sym}}$ are the Radon–Nikodym derivatives of $\mathcal{L}^N_x$ and Eu with respect to η, respectively. Thanks to Lemma 5.10,

$$\Sigma_n(x,\cdot)\in L^2(\mathcal{Y};\mathbb{M}^N_{\mathrm{sym}}),\qquad (\Sigma_n)_D(x,\cdot)\in L^\infty(\mathcal{Y};\mathbb{M}^N_D),\qquad \mathrm{div}_y\,\Sigma_n(x,\cdot)=0$$

for *every* $x\in\Omega'$. We conclude that the duality $\langle(\Sigma_n)_D(x,\cdot),P_x\rangle$ is well defined as an element in $\mathcal{M}_b(\mathcal{Y})$ for η-a.e. $x\in\Omega'$.

By definition of $\langle(\Sigma_n)_D(x,\cdot),P_x\rangle$,

$$\begin{aligned}\langle(\Sigma_n)_D(x,\cdot),P_x\rangle(\psi) = &-\int_{\mathcal{Y}}\psi(y)\Sigma_n(x,y)\cdot[C(x)E(x,y)-A(x)]\,dy\\ &-\int_{\mathcal{Y}}\Sigma_n(x,y)\cdot[\mu_x(y)\odot\nabla\psi(y)]\,dy\end{aligned} \tag{5.33}$$

for every $\psi\in C^1(\mathcal{Y})$. The η-a.e. defined map

$$x\mapsto\langle(\Sigma_n)_D(x,\cdot),P_x\rangle(\psi)\text{ is }\eta\text{-measurable on }\Omega'. \tag{5.34}$$

Indeed, a direct computation shows that the maps $f(x,y):=\psi(y)\Sigma_n(x,y)\cdot[C(x)E(x,y)-A(x)]$ and $g(x,y):=\Sigma_n(x,y)\cdot[\mu_x(y)\odot\nabla\psi(y)]$ are summable with respect to the measure $\eta\otimes\mathcal{L}^N_y$. Then (5.34) follows by Fubini's theorem.

Through a standard approximation argument, we infer that $x\mapsto\langle\Sigma_n(x,\cdot),P_x\rangle(F)$ is η-measurable for every Borel set $F\subseteq\mathcal{Y}$. Since, in view of Lemma 5.10(b),

$$|\langle(\Sigma_n)_D(x,\cdot),P_x\rangle|\le\|(\Sigma_n)_D(x,\cdot)\|_\infty|P_x|\le C|P_x|,$$

we deduce from the actual definition of generalized products (see Subsection 1.2 or [4, Definition 2.27]) that $\lambda_n=\eta\overset{\text{gen.}}{\otimes}\langle\Sigma_n(x,\cdot),P_x\rangle$ is well defined as an element of $\mathcal{M}_b(\Omega'\times\mathcal{Y})$.

Since

$$|\lambda_n|=\eta\overset{\text{gen.}}{\otimes}|[(\Sigma_n)_D(x,\cdot),P_x]|\le\eta\overset{\text{gen.}}{\otimes}\|(\Sigma_n)_D(x,\cdot)\|_\infty|P_x|\le C|P|$$

with C independent of n, we infer that $\{\lambda_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{M}_b(\Omega'\times\mathcal{Y})$.

(b) Up to a subsequence,

$$\lambda_n\overset{*}{\rightharpoonup}\lambda\quad\text{weakly* in }\mathcal{M}_b(\Omega'\times\mathcal{Y})$$

for a suitable $\lambda \in \mathcal{M}_b(\Omega' \times \mathcal{Y})$. For every $\varphi \in C^0_c(\Omega')$, the very definition of λ_n yields

$$\begin{aligned}\langle \lambda_n, \varphi\rangle &= -\int_{\Omega'\times\mathcal{Y}} \varphi(x)\Sigma_n(x,y)\cdot C(x)E(x,y)\,d(\eta\otimes\mathcal{L}^N_y)\\ &\quad + \int_{\Omega'} \varphi(x)\sigma_n(x)\cdot A(x)\,d\eta(x)\\ &= -\int_{\Omega'} \varphi(x)\Sigma_n(x,y)\cdot E(x,y)\,dx\,dy + \int_{\Omega'} \varphi(x)\sigma_n(x)\,dEu(x).\end{aligned}$$

But σ_n is continuous, so

$$\begin{aligned}\int_{\Omega'} \varphi(x)\sigma_n(x)\cdot dEu(x) &= \int_{\Omega'} \varphi(x)\sigma_n(x)\cdot e(x)\,dx + \int_{\Omega'} \varphi(x)\sigma_n(x)\cdot dp(x)\\ &= \int_{\Omega'} \varphi(x)\sigma_n(x)\cdot e(x)\,dx + \int_{\Omega'} \varphi(x)(\sigma_n)_D(x)\cdot dp(x),\end{aligned}$$

hence

$$\langle \lambda_n, \varphi\rangle = -\int_{\Omega'\times\mathcal{Y}} \varphi(x)\Sigma_n\cdot E\,dx\,dy + \int_{\Omega'} \varphi\sigma_n\cdot e\,dx + \int_{\Omega'} \varphi(\sigma_n)_D\cdot dp,$$

where $e(x) := \int_{\mathcal{Y}} E(x,y)\,dy \in L^2(\Omega'; \mathrm{M}^N_{\mathrm{sym}})$ and $p := \mathrm{proj}_\# P \in \mathcal{M}_b(\Omega'; \mathrm{M}^N_D)$.

Since $\sigma_n \in C^\infty(\mathbb{R}^N; \mathrm{M}^N_{\mathrm{sym}})$ we have, recalling Remark 2.2,

$$(\sigma_n)_D p = \langle(\sigma_n)_D, p\rangle \qquad \text{as measures on } \Omega'.$$

Appealing to the convergences of σ_n to σ in Lemma 5.10(e) we deduce from the definition of the duality product in (2.2) and the facts that $\varphi \equiv 0$ on $\bar{\Gamma}_t$ while $p \equiv 0$ on $\Omega' \setminus \bar{\Omega}$ that

$$\langle(\sigma_n)_D, p\rangle \xrightharpoonup{*} \langle(\sigma)_D, p\rangle \qquad \text{weakly* in } \mathcal{M}_b(\Omega')$$

(and thus $\langle(\sigma)_D, p\rangle \in \mathcal{M}_b(\Omega')$), with, for every $\varphi \in C^1_c(\Omega')$,

$$\langle(\sigma)_D, p\rangle(\varphi) = -\int_\Omega \varphi\sigma\cdot(e - Ew)\,dx - \int_\Omega \sigma\cdot[(u-w)\odot\nabla\varphi]\,dx.$$

Using Lemma 5.10(d), and since $e \equiv E \equiv Ew$ outside Ω, we deduce that

$$\begin{aligned}\langle\lambda, \varphi\rangle &= \lim_n \langle\lambda_n, \varphi\rangle\\ &= \lim_n \left[-\int_{\Omega'\times\mathcal{Y}} \varphi(x)\Sigma_n\cdot E\,dx\,dy + \int_{\Omega'} \varphi(x)\sigma_n\cdot e\,dx + \langle(\sigma_n)_D, p\rangle(\varphi)\right]\\ &= \lim_n \left[-\int_{\Omega\times\mathcal{Y}} \varphi(x)\Sigma_n\cdot E\,dx\,dy + \int_{\Omega} \varphi(x)\sigma_n\cdot e\,dx + \langle(\sigma_n)_D, p\rangle(\varphi)\right]\\ &= -\int_{\Omega\times\mathcal{Y}} \varphi(x)\Sigma\cdot E\,dx\,dy + \int_\Omega \varphi(x)\sigma\cdot e\,dx + \langle\sigma_D, p\rangle(\varphi).\end{aligned}$$

If $\partial \lfloor_{\partial\Omega} \Gamma_d$ is admissible, letting $\varphi \nearrow 1_{\Omega'}$ we get

$$\begin{aligned}\lambda(\Omega' \times \mathcal{Y}) &= -\int_{\Omega\times\mathcal{Y}} \Sigma \cdot E\,dx\,dy + \int_{\Omega} \sigma \cdot e\,dx + \langle \sigma_D, p\rangle(\Omega)\\ &= -\int_{\Omega\times\mathcal{Y}} \Sigma \cdot E\,dx\,dy + \int_{\Omega} \sigma \cdot e\,dx - \int_{\Omega} \sigma \cdot (e - Ew)\,dx\\ &= -\int_{\Omega\times\mathcal{Y}} \Sigma \cdot E\,dx\,dy + \int_{\Omega} \sigma \cdot Ew\,dx,\end{aligned}$$

which is (5.32).

It now remains to establish the precise form (5.31) of λ. Note that, since $P_x = 0$ for $\mathcal{L}^N$-a.e. $x \in \Omega' \setminus \Omega$ and $\eta = \mathcal{L}^N_x + \eta^s$,

$$\begin{aligned}\lambda_n &= \mathcal{L}^N_x \overset{\text{gen.}}{\otimes} \langle(\Sigma_n)_D(x,\cdot), P_x\rangle + \eta^s \overset{\text{gen.}}{\otimes} \langle(\Sigma_n)_D(x,\cdot), P_x\rangle\\ &= (\mathcal{L}^N_x \lfloor_{\Omega}) \overset{\text{gen.}}{\otimes} \langle(\Sigma_n)_D(x,\cdot), P_x\rangle + \eta^s \overset{\text{gen.}}{\otimes} \langle(\Sigma_n)_D(x,\cdot), P_x\rangle =: \lambda^1_n + \lambda^2_n.\end{aligned}$$

In view of Lemma 5.10(b),

$$|\lambda^1_n| \le C(\mathcal{L}^N_x \lfloor_{\Omega}) \overset{\text{gen.}}{\otimes} |P_x| \le C|P| \quad \text{and} \quad |\lambda^2_n| \le C\eta^s \overset{\text{gen.}}{\otimes} |P_x| \le C|P|,$$

with C independent of n. As a consequence, we may assume that, up to extracting a further subsequence,

$$\begin{aligned}\lambda^1_n &\overset{*}{\rightharpoonup} \lambda^1 \quad \text{weakly* in } \mathcal{M}_b(\Omega' \times \mathcal{Y}),\\ \lambda^2_n &\overset{*}{\rightharpoonup} \lambda^2 \quad \text{weakly* in } \mathcal{M}_b(\Omega' \times \mathcal{Y}),\end{aligned}$$

with

$$|\lambda^1| \le C(\mathcal{L}^N_x \lfloor_{\Omega}) \overset{\text{gen.}}{\otimes} |P_x| \quad \text{and} \quad |\lambda^2| \le C\eta^s \overset{\text{gen.}}{\otimes} |P_x|$$

as measures on $\Omega' \times \mathcal{Y}$.

In view of Lemma 5.10(b) & (d), and taking into account Remark 5.9,

$$\Sigma_n(x,\cdot) \to \Sigma(x,\cdot) \quad \text{strongly in } L^r(\mathcal{Y}; \mathbb{M}^N_{\text{sym}}) \text{ for a.e. } x \in \Omega,\ 1 \le r < \infty.$$

Since, according to Lemma 5.4, $(\mu_x, C(x)E(x,\cdot) - A(x), P_x) \in \mathcal{A}_{\mathcal{Y}}$, hence $(C(x)E(x,\cdot) - A(x)) \in L^2(\mathcal{Y}, \mathbb{M}^N_{\text{sym}})$ for η-a.e. $x \in \Omega$, we immediately pass to the limit in (5.33) and conclude that

$$\langle(\Sigma_n)_D(x,\cdot), P_x\rangle \overset{*}{\rightharpoonup} \langle(\Sigma)_D(x,\cdot), P_x\rangle \quad \text{weakly* in } \mathcal{M}_b(\mathcal{Y}).$$

By the very definition of a generalized product, we finally obtain

$$\lambda^1 = (\mathcal{L}^N_x \lfloor_{\Omega}) \overset{\text{gen.}}{\otimes} \langle(\Sigma)_D(x,\cdot), P_x\rangle.$$

Since $\lambda = \lambda^1 + \lambda^2$, (5.31) follows and the proof is complete. □

We now establish the two-scale analogue of Proposition 3.5.

Theorem 5.12. *Assume that $\mathcal{Y}$ is a C^2-admissible multiphase torus. Then, for every $\Sigma \in \mathcal{K}^{\mathrm{hom}}$ and $(u, E, P) \in \mathcal{A}^{\mathrm{hom}}(w)$,*

$$H(y, P/|P|)|P| \geq \lambda,$$

with λ defined in (5.31). Further, if equality holds, then for $\mathcal{L}^N_x$-a.e. $x \in \Omega$,

$$\frac{P_x}{|P_x|}(y) \in N_{K(y)}(\Sigma_D(x, y)) \quad \text{for } \mathcal{L}^N_y\text{-a.e. } y \in \{|P_x| > 0\};$$

and, letting $\mu \in \boldsymbol{\mathcal{X}}(\Omega')$ be the measure associated with (u, E, P) and using the disintegration (5.9), we get, for $\mathcal{L}^N_x$-a.e. $x \in \Omega$ and for every $i \neq j$,

$$\frac{\mu^i_x(y) - \mu^j_x(y)}{|\mu^i_x(y) - \mu^j_x(y)|} \in \vec{N}_{K_\Gamma(y)}((\Sigma_D(x, \cdot)\nu)_\tau(y)) \quad \text{for } \mathcal{H}^{N-1}\text{-a.e. } y \in \{\mu^i_x \neq \mu^j_x\},$$

where μ^i_x and μ^j_x are the traces on Γ_{ij} of the restrictions of μ_x to $\mathcal{Y}_i$ and $\mathcal{Y}_j$ respectively, assuming that ν points from $\mathcal{Y}_j$ to $\mathcal{Y}_i$, and where $\vec{N}_{K_\Gamma(y)}(\tau)$ denotes the normal cone (a cone of vectors) to $K_\Gamma(y)$ at a vector $\tau \perp \nu(y)$.

Proof. Let $\{\Sigma_n\}_{n\in\mathbb{N}}$ be the sequence given by Lemma 5.10, and let $\{\lambda_n\}_{n\in\mathbb{N}}$ be the associated measures defined in Proposition 5.11. Given $\varepsilon > 0$, Lemma 5.10(c) implies that, for n large enough,

$$(\Sigma_n)_D(x, \cdot) \in (1 + \varepsilon)K(y) \quad \text{for a.e. } y \in \mathcal{Y} \text{ and for every } x \in \Omega'.$$

By Proposition 3.5, we deduce that, for η-a.e. $x \in \Omega'$,

$$H(y, P_x/|P_x|)|P_x| \geq \frac{1}{1 + \varepsilon}\langle(\Sigma_n)_D(x, \cdot), P_x\rangle \quad \text{as measures on } \mathcal{Y}.$$

Consequently, in view of (5.10) and item (a) in Proposition 5.11,

$$H(y, P/|P|)|P| = \eta \overset{\text{gen.}}{\otimes} H(y, P/|P|)|P_x| = \eta \overset{\text{gen.}}{\otimes} H(y, P_x/|P_x|)|P_x| \geq \frac{1}{1 + \varepsilon}\lambda_n.$$

Proposition 5.11(b) implies the desired inequality upon passing to the limit in n, then in ε.

If, further, equality holds, then the decomposition $P = \eta \overset{\text{gen.}}{\otimes} P_x$, with $\eta := \mathcal{L}^N_x + (\mathrm{proj}_\# |P|)^s$ given by Lemma 5.4 implies, in view of (5.31), that

$$(\mathcal{L}^N_x \lfloor_\Omega) \overset{\text{gen.}}{\otimes} H(y, P/|P|)|P_x| = (\mathcal{L}^N_x \lfloor_\Omega) \overset{\text{gen.}}{\otimes} \langle\Sigma_D(x, \cdot), P_x\rangle$$

so that, recalling (5.10),

$$H(y, P_x/|P_x|)|P_x| = \langle\Sigma_D(x, \cdot), P_x\rangle \quad \text{as measures on } \mathcal{Y},$$

and this for $\mathcal{L}^N_x$-a.e. $x \in \Omega$. The result now follows from Proposition 3.5 once it is recalled that, thanks to Lemma 5.4, P_x is the plastic strain of the *BD* deformation μ_x on $\mathcal{Y}$. □

Remark 5.13. Assuming that $\partial\lfloor_{\partial\Omega}\Gamma_d$ is admissible in the sense of Definition 2.1, the previous theorem together with (5.32) immediately implies the two-scale version of the principle of maximum plastic work, that is, for any $\Sigma \in \mathcal{K}^{\text{hom}}$ and any triplet $(u, E, P) \in \mathcal{A}^{\text{hom}}(w)$,

$$\mathcal{H}^{\text{hom}}(P) \geq [\Sigma \mid P] := -\int_{\Omega\times\mathcal{Y}} \Sigma \cdot E\, dx\, dy + \int_{\Omega} \sigma \cdot Ew\, dx.$$

As a final remark in this subsection, two-scale statically admissible fields naturally arise as two-scale weak limits of ε-statically admissible stress fields (see (2.19)). Indeed,

Proposition 5.14. *Let $(\sigma_\varepsilon)_{\varepsilon>0}$ be a bounded family in $L^2(\Omega; \mathbb{M}^N_{\text{sym}})$ such that $\sigma_\varepsilon \in \mathcal{K}_\varepsilon$ and*

$$\sigma_\varepsilon \xrightarrow{w\text{-}2} \Sigma \quad \textit{two-scale weakly in } L^2(\Omega \times \mathcal{Y}; \mathbb{M}^N_{\text{sym}}).$$

Then $\Sigma \in \mathcal{K}^{\text{hom}}$.

Proof. Since $\sigma(x) := \int_{\mathcal{Y}} \Sigma(x, y)\, dy$ is the weak L^2-limit of σ_ε, it is immediate that

$$\operatorname{div}_x \sigma = 0 \quad \text{in } \Omega, \qquad \sigma \cdot \nu = 0 \quad \text{on } \partial\Omega \setminus \overline{\Gamma}_d.$$

Applying the definition of two-scale weak convergence it is readily seen that

$$\operatorname{div}_y \Sigma = 0 \quad \text{on } \mathcal{Y}.$$

In order to prove our assertion, we appeal to Remark 4.12. The function Σ is the weak limit in $L^2(\Omega \times \mathcal{Y}; \mathbb{M}^N_{\text{sym}})$ of the functions

$$\Sigma_\varepsilon(x, y) := \sum_{i\in I_\varepsilon(\Omega)} 1_{Q^i_\varepsilon}(x)\sigma^i_\varepsilon(y),$$

where $I_\varepsilon(\Omega)$ is defined in (4.13), and $\sigma^i_\varepsilon(y) := \sigma_\varepsilon(x^i_\varepsilon + \varepsilon\mathcal{I}(y))$. Since $\sigma_\varepsilon \in \mathcal{K}_\varepsilon$, we deduce that

$$\Sigma_\varepsilon \in \{\Xi \in L^2(\Omega \times \mathcal{Y}; \mathbb{M}^N_{\text{sym}}) : \Xi_D(x, y) \in K(y) \text{ for a.e. } (x, y) \in \Omega \times \mathcal{Y}\}.$$

But this set is convex and closed in the strong topology of $L^2(\Omega \times \mathcal{Y}; \mathbb{M}^N_{\text{sym}})$, hence weakly closed, and this concludes the proof. □

6. Two-scale homogenization of the quasi-static evolution

In this last section, we address in the first subsection the two-scale limit of the heterogeneous quasi-static evolution, while we derive the corresponding generalized flow rule in the second subsection.

6.1. Two-scale quasi-static evolutions and the homogenization result

For any $t \mapsto P(t) \in \mathcal{M}_b(\Omega' \times \mathcal{Y}; \mathrm{M}^N_D)$, $t \in [0, T]$, we define the *homogenized total dissipation* on $[a, b] \subseteq [0, T]$ to be

$$\mathcal{D}^{\mathrm{hom}}(a, b; P) := \sup\Big\{\sum_{i=1}^{I} \mathcal{H}^{\mathrm{hom}}(P(t_i) - P(t_{i-1})) : a = t_0 \le t_1 \le \cdots \le t_I = b\Big\},$$

where $\mathcal{H}^{\mathrm{hom}}$ was defined in (5.12).

Recalling the definitions of $\mathcal{A}^{\mathrm{hom}}(w)$ and of $\mathcal{Q}^{\mathrm{hom}}$ (see Definition 5.1 and (5.11)), we are now in a position to formulate a notion of quasi-static elasto-plastic evolution in a two-scale setting.

Definition 6.1 (Two-scale quasi-static evolution). We say that

$$t \mapsto (u(t), E(t), P(t)) \in \mathcal{A}^{\mathrm{hom}}(w(t))$$

is a *two-scale quasi-static evolution* relative to w iff the following conditions hold for every $t \in [0, T]$:

(a) Global stability: for every $(v, \Xi, Q) \in \mathcal{A}^{\mathrm{hom}}(w(t))$,

$$\mathcal{Q}^{\mathrm{hom}}(E(t)) \le \mathcal{Q}^{\mathrm{hom}}(\Xi) + \mathcal{H}^{\mathrm{hom}}(Q - P(t)).$$

(b) Energy equality: $t \mapsto P(t)$ has bounded variation from $[0, T]$ to $\mathcal{M}_b(\Omega' \times \mathcal{Y}; \mathrm{M}^N_D)$ and

$$\mathcal{Q}^{\mathrm{hom}}(E(t)) + \mathcal{D}^{\mathrm{hom}}(0, t; P) = \mathcal{Q}^{\mathrm{hom}}(E(0)) + \int_0^t \int_\Omega \sigma(\tau) \cdot E\dot{w}(\tau)\, dx\, d\tau,$$

where $\sigma(t, x) := \int_{\mathcal{Y}} \mathbb{C}(y) E(t, x, y)\, dy$ for a.e. $x \in \Omega$.

As will be seen shortly, two-scale quasi-static evolutions naturally arise in the description of the behavior of quasi-static evolutions in periodic heterogeneous materials as the size of the microstructure goes to zero.

For every $\varepsilon > 0$, let $(u^0_\varepsilon, e^0_\varepsilon, p^0_\varepsilon) \in \mathcal{A}(w(0))$ be globally stable initial configurations such that

$$\begin{cases} u^0_\varepsilon \overset{*}{\rightharpoonup} u_0 & \text{weakly* in } BD(\Omega'), \\ e^0_\varepsilon \overset{s\text{-}2}{\to} E_0 & \text{two-scale strongly in } L^2(\Omega' \times \mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}}), \\ p^0_\varepsilon \xrightharpoonup{w^*\text{-}2} P_0 & \text{two-scale weakly* in } \mathcal{M}_b(\Omega' \times \mathcal{Y}; \mathrm{M}^N_D), \end{cases} \tag{6.1}$$

for some $(u_0, E_0, P_0) \in \mathcal{A}^{\mathrm{hom}}(w(0))$. In particular,

$$\lim_{\varepsilon \to 0} Q_\varepsilon(e^0_\varepsilon) = \mathcal{Q}^{\mathrm{hom}}(E_0). \tag{6.2}$$

In view of the above assumptions on $(u^0_\varepsilon, e^0_\varepsilon, p^0_\varepsilon)$, Theorem 2.6 applies to the evolution at fixed ε and yields a quasi-static evolution in the sense of Definition 2.5. The following homogenization result holds.

Theorem 6.2 (Two-scale homogenization of a quasi-static evolution). *Assume that*

- $\partial\lfloor_{\partial\Omega}\Gamma_d$ *is admissible in the sense of Definition* 2.1*;*
- *relations* (2.5)*,* (2.6)*,* (2.11)*,* (2.12)*,* (2.13)*,* (2.17) *hold; and*
- *for every* $\varepsilon > 0$*,* $(u^0_\varepsilon, e^0_\varepsilon, p^0_\varepsilon) \in \mathcal{A}_\varepsilon(w(0))$ *are globally stable configurations satisfying* (6.1).

Let $t \mapsto (u_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t))$ *be a quasi-static evolution relative to the boundary displacement* w *such that*

$$(u_\varepsilon(0), e_\varepsilon(0), p_\varepsilon(0)) = (u^0_\varepsilon, e^0_\varepsilon, p^0_\varepsilon).$$

Then there exists $\varepsilon_n \to 0$ *and a two-scale quasi-static evolution* $t \mapsto (u(t), E(t), P(t))$ *relative to the boundary displacement* w *such that*

$$(u(0), E(0), P(0)) = (u_0, E_0, P_0)$$

and such that, upon setting $(u_n, e_n, p_n) := (u_{\varepsilon_n}, e_{\varepsilon_n}, p_{\varepsilon_n})$*,*

$$\begin{cases} u_n(t) \overset{*}{\rightharpoonup} u(t) & \text{weakly* in } BD(\Omega'), \\ e_n(t) \xrightharpoonup{w\text{-}2} E(t) & \text{two-scale weakly in } L^2(\Omega' \times \mathcal{Y}; \mathbb{M}^N_{\mathrm{sym}}), \\ p_n(t) \xrightharpoonup{w^*\text{-}2} P(t) & \text{two-scale weakly* in } \mathcal{M}_b(\Omega' \times \mathcal{Y}; \mathbb{M}^N_D), \end{cases} \tag{6.3}$$

for every $t \in [0, T]$.

Proof. We divide the proof into several steps.

Step 1: Compactness. From the energy balance at fixed ε and upon application of [21, Chapter II, Proposition 2.4]—taking $\int_{\Omega'\setminus\overline{\Omega}} |u|\,dx$ as a continuous seminorm on $BD(\Omega')$—we deduce the existence of a constant $C > 0$ such that, for every $\varepsilon > 0$ and $t \in [0, T]$,

$$\|u_\varepsilon(t)\|_{BD(\Omega')} + \|e_\varepsilon(t)\|_{L^2(\Omega';\mathbb{M}^N_{\mathrm{sym}})} + \mathcal{V}_{\mathcal{M}_b(\Omega';\mathbb{M}^N_D)}(0, t; p_\varepsilon) \le C. \tag{6.4}$$

In view of Proposition 4.4 and of Remark 4.2, application of [15, Theorem 3.2] yields a sequence $\{\varepsilon_n \searrow 0\}$ and $P \in BV(0, T; \mathcal{M}_b(\Omega' \times \mathcal{Y}; \mathbb{M}^N_D))$ such that, for every $t \in [0, T]$,

$$p_n(t) \xrightharpoonup{w^*\text{-}2} P(t) \quad \text{two-scale weakly* in } \mathcal{M}_b(\Omega' \times \mathcal{Y}; \mathbb{M}^N_D).$$

Further, for a possibly t-dependent subsequence $\{\varepsilon_{n_t}\}_{n_t\in\mathbb{N}}$ of $\{\varepsilon_n\}_{n\in\mathbb{N}}$,

$$\begin{cases} u_{n_t}(t) \overset{*}{\rightharpoonup} u(t) & \text{weakly* in } BD(\Omega'), \\ e_{n_t}(t) \xrightharpoonup{w\text{-}2} E(t) & \text{two-scale weakly in } L^2(\Omega' \times \mathcal{Y}; \mathbb{M}^N_{\mathrm{sym}}), \end{cases} \tag{6.5}$$

and, according to Lemma 5.6, $(u(t), E(t), P(t)) \in \mathcal{A}^{\mathrm{hom}}(w(t))$.

Finally, in view of Remark 4.12, we can choose $\{\varepsilon_{n_t}\}_{n_t\in\mathbb{N}}$ such that

$$\sigma_{n_t}(t) := \mathbb{C}_{\varepsilon_{n_t}} e_{n_t}(t) \xrightarrow{w\text{-}2} \Sigma(t) := \mathbb{C}(y)E(t) \quad \text{two-scale weakly in } L^2(\Omega\times\mathcal{Y};\mathrm{M}^N_{\mathrm{sym}});$$

consequently,

$$\sigma_{n_t}(t) \rightharpoonup \sigma(t) \quad \text{weakly in } L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}}) \tag{6.6}$$

where $\sigma(t,x) := \int_{\mathcal{Y}} \Sigma(t,x,y)\,dy$ for a.e. $x\in\Omega$. By Proposition 5.14, $\Sigma(t)\in\mathcal{K}^{\mathrm{hom}}$ because, in view of Remark 2.7, $\sigma_{n_t}(t)\in\mathcal{K}_{\varepsilon_{n_t}}$.

Step 2: Global stability. Since $(u(t),E(t),P(t))\in\mathcal{A}^{\mathrm{hom}}(w(t))$ (with associated $\mu(t)\in\boldsymbol{\mathcal{X}}(\Omega')$), it follows that, for every $(v,\Xi,Q)\in\mathcal{A}^{\mathrm{hom}}(w(t))$ (with associated $\nu\in\boldsymbol{\mathcal{X}}(\Omega')$), $(v-u(t),\Xi-E(t),Q-P(t))$ belongs to $\mathcal{A}^{\mathrm{hom}}(0)$. Since $\Sigma(t)\in\mathcal{K}^{\mathrm{hom}}$, Remark 5.13 implies that

$$\mathcal{H}^{\mathrm{hom}}(Q-P(t)) \ge -\int_{\Omega\times\mathcal{Y}} \Sigma\cdot(\Xi-E(t))\,dx\,dy = -\int_{\Omega\times\mathcal{Y}} \mathbb{C}(y)E(t)\cdot(\Xi-E(t))\,dx\,dy,$$

from which it is immediately deduced that

$$\mathcal{H}^{\mathrm{hom}}(Q-P(t)) + \mathcal{Q}^{\mathrm{hom}}(\Xi) \ge \mathcal{Q}^{\mathrm{hom}}(E(t)) + \mathcal{Q}^{\mathrm{hom}}(\Xi-E(t)) \ge \mathcal{Q}^{\mathrm{hom}}(E(t)),$$

hence the global stability.

Assume that $(u'(t),E'(t),P(t))\in\mathcal{A}^{\mathrm{hom}}(w(t))$, with associated $\mu'(t)\in\boldsymbol{\mathcal{X}}(\Omega')$, also satisfies global stability. Then, by the convexity of the set $\mathcal{A}^{\mathrm{hom}}(w(t))$ and the strict convexity of $\mathcal{Q}^{\mathrm{hom}}$, it is immediate that

$$E'(t) = E(t).$$

From the admissibility condition (5.5) we infer

$$Eu(t)\otimes\mathcal{L}^N_y + E_y\mu(t) = Eu'(t)\otimes\mathcal{L}^N_y + E_y\mu'(t) \quad \text{on } \Omega'\times\mathcal{Y},$$

so that taking the average with respect to y we obtain

$$Eu(t) = Eu'(t) \quad \text{in } \Omega'.$$

Since $u(t)=u'(t)=w(t)$ on $\Omega'\setminus\overline{\Omega}$, using again [21, Chapter II, Proposition 2.4] with $\int_{\Omega'\setminus\overline{\Omega}}|u|\,dx$ as a continuous seminorm on $BD(\Omega')$, we infer $u(t)=u'(t)$ on Ω'. Therefore, there is no need to extract a subsequence $\{\varepsilon_{n_t}\}_{n_t\in\mathbb{N}}$ from $\{\varepsilon_n\}_{n\in\mathbb{N}}$ in (6.5), so that the whole sequences $\{u_n(t)\}_{n\in\mathbb{N}}$, $\{E_n(t)\}_{n\in\mathbb{N}}$ converge, which establishes (6.3).

Step 3: Energy balance. We start with the energy balance at fixed ε. It states in particular (see Theorem 2.6) that for any partition $0\le t_1\le\cdots\le t_m=t$ of $[0,t]$,

$$\mathcal{Q}_{\varepsilon_n}(e_n(t)) + \sum_{i=0}^{m-1}\mathcal{H}_{\varepsilon_n}(p_n(t_{i+1})-p_n(t_i)) \le \mathcal{Q}_{\varepsilon_n}(e_n(0)) + \int_0^t\int_\Omega \sigma_n(s)\cdot E\dot{w}(s)\,dx\,ds.$$

Pass to the limit as $n \nearrow \infty$. For the left-hand side, Theorem 5.7 yields

$$\mathcal{Q}^{\mathrm{hom}}(E(t)) + \sum_{i=0}^{m-1} \mathcal{H}^{\mathrm{hom}}(P(t_{i+1}) - P(t_i)) \\ \leq \liminf_n \Big[\mathcal{Q}_{\varepsilon_n}(e_n(t)) + \sum_{i=0}^{m-1} \mathcal{H}_{\varepsilon_n}(p_n(t_{i+1}) - p_n(t_i)) \Big].$$

In view of (6.4) and of (6.6), Lebesgue's dominated convergence theorem entails that the limit of the second term on the right hand-side is by $\int_0^t \int_\Omega \sigma(s) \cdot E\dot{w}(s)\, dx\, ds$. In view of (6.2),

$$\lim_n \mathcal{Q}_{\varepsilon_n}(e_n(0)) = \mathcal{Q}^{\mathrm{hom}}(E_0).$$

Recalling all limits, we finally obtain

$$\mathcal{Q}^{\mathrm{hom}}(E(t)) + \sum_{i=0}^{m-1} \mathcal{H}^{\mathrm{hom}}(P(t_{i+1}) - P(t_i)) \leq \mathcal{Q}^{\mathrm{hom}}(E_0) + \int_0^t \int_\Omega \sigma(s) \cdot E\dot{w}(s)\, dx\, ds.$$

Taking the supremum over all partitions $0 \leq t_1 \leq \cdots \leq t_m = t$ of $[0, t]$ then yields

$$\mathcal{Q}^{\mathrm{hom}}(E(t)) + \mathcal{D}^{\mathrm{hom}}(0, t; P) \leq \mathcal{Q}^{\mathrm{hom}}(E_0) + \int_0^t \int_\Omega \sigma(s) \cdot E\dot{w}(s)\, dx\, ds. \tag{6.7}$$

Deriving the reverse inequality in (6.7) is straightforward. Indeed, the argument is identical to that at the end of the proof of [11, Theorem 2.7] upon replacing $\mathcal{Q}, \mathcal{D}, \mathcal{H}$ by $\mathcal{Q}^{\mathrm{hom}}, \mathcal{D}^{\mathrm{hom}}, \mathcal{H}^{\mathrm{hom}}$, respectively, and replacing the global minimality statement used there by item (a) in Definition 6.1. It simply consists in testing, at time t_i, the global minimality of the triplet $(u(t_i), E(t_i), P(t_i))$ by $(u(t_{i+1}) + w(t_i) - w(t_{i+1}), E(t_{i+1}) + (Ew(t_i) - Ew(t_{i+1})), P(t_{i+1})) \in \mathcal{A}^{\mathrm{hom}}(w(t_i))$ and passing to the limit in the time step in the resulting inequality upon remarking that the BV regularity in time for P implies that $t \mapsto \Sigma(t) \in L^2(\Omega \times \mathcal{Y}; \mathbb{M}^N_{\mathrm{sym}})$ can only have a countable number of discontinuity points; see [11, Remark 2.6 and Theorem 2.7] for details. □

6.2. *Flow rule for two-scale quasi-static evolutions*

This subsection is devoted to the analysis of the flow rule for a two-scale quasi-static evolution. To this end, we need to interpret the energy equality for a two-scale quasi-static evolution in terms of a more classical flow rule with respect to the variable y.

Lemma 6.3 (Static admissibility). *Let $t \mapsto (u(t), E(t), P(t)) \in \mathcal{A}^{\mathrm{hom}}(w(t))$ be a two-scale quasi-static evolution according to Definition 6.1. Then, for every $t \in [0, T]$,*

$$\Sigma(t) := \mathbb{C}E(t) \in \mathcal{K}^{\mathrm{hom}},$$

where $\mathcal{K}^{\mathrm{hom}}$ is the set of two-scale statically admissible stresses (see Definition 5.8).

Proof. Take $(v, \Xi, Q) \in \mathcal{A}^{\hom}(0)$. From global stability with $(u(t) + v, E(t) + \Xi, P(t) + Q)$ as test field, it is immediate that

$$\int_{\Omega\times\mathcal{Y}} \Sigma(t)\cdot\Xi\,dx\,dy + \mathcal{H}^{\hom}(Q) \geq 0$$

so that

$$-\mathcal{H}^{\hom}(Q) \leq \int_{\Omega\times\mathcal{Y}} \Sigma(t)\cdot\Xi\,dx\,dy \leq \mathcal{H}^{\hom}(-Q).$$

Considering $(0, E_y\Phi(x,y), 0) \in \mathcal{A}^{\hom}(0)$ where $\Phi(x,y) \in C_c^\infty(\Omega\times\mathcal{Y};\mathbb{R}^N)$ (with associated $\mu := (\Phi(x,y) - \int_{\mathcal{Y}}\Phi(x,y)\,dy)(\mathcal{L}^N_x \otimes \mathcal{L}^N_y) \in \boldsymbol{\mathcal{X}}(\Omega')$), the previous inequality entails that

$$\operatorname{div}_y \Sigma = 0 \quad \text{on } \Omega\times\mathcal{Y}.$$

Given $\mathcal{B}_1 \subseteq \Omega$ and $\mathcal{B}_2 \subseteq \mathcal{Y}$ Borel sets, and an arbitrary $\xi \in \mathrm{M}^N_D$, then

$$(0, \xi 1_{\mathcal{B}_1\times\mathcal{B}_2}(x,y), -\xi 1_{\mathcal{B}_1\times\mathcal{B}_2}(x,y)) \in \mathcal{A}^{\hom}(0)$$

(with associated $\mu := 0 \in \boldsymbol{\mathcal{X}}(\Omega')$). Thus, for $\mathcal{L}^N_x \otimes \mathcal{L}^N_y$-a.e. $(x,y) \in \Omega\times\mathcal{Y}$, $H(y,\xi) \geq \Sigma_D(t,x,y)\cdot\xi$, so that, by the definition (2.10) of H and the arbitrariness of ξ, we conclude that

$$\Sigma_D(x,y) \in K(y).$$

Finally, by considering $(v, E_x v, 0) \in \mathcal{A}^{\hom}(0)$ with $v \in C^1(\overline{\Omega})$ and $v = 0$ on $\Omega' \setminus \overline{\Omega}$, we get

$$\operatorname{div}_x \sigma = 0 \quad \text{in } \Omega, \qquad \sigma\cdot\nu = 0 \quad \text{on } \partial\Omega \setminus \overline{\Gamma}_d,$$

so that $\Sigma(t) \in \mathcal{K}^{\hom}$. □

A proof completely analogous to that of [10, Theorem 5.2], in the two-scale setting and modulo the absence of external loads, would entail the following

Proposition 6.4 (Regularity in time). *If $t \mapsto (u(t), E(t), P(t))$ is a two-scale quasi-static evolution, then*

$$(u, E, P) \in AC(0, T; BD(\Omega') \times L^2(\Omega'\times\mathcal{Y};\mathrm{M}^N_{\mathrm{sym}}) \times \mathcal{M}_b(\Omega'\times\mathcal{Y};\mathrm{M}^N_D)).$$

Moreover, the following limits exist for a.e. $t \in [0,T]$:

$$\dot u(t) := \lim_{s\to t}\frac{u(s)-u(t)}{s-t} \qquad \textit{weakly* in } BD(\Omega'),$$

$$\dot E(t) := \lim_{s\to t}\frac{E(s)-E(t)}{s-t} \qquad \textit{strongly in } L^2(\Omega'\times\mathcal{Y};\mathrm{M}^N_{\mathrm{sym}}),$$

$$\dot P(t) := \lim_{s\to t}\frac{P(s)-P(t)}{s-t} \qquad \textit{weakly* in } \mathcal{M}_b(\Omega'\times\mathcal{Y};\mathrm{M}^N_D),$$

with $(\dot u(t), \dot E(t), \dot P(t)) \in \mathcal{A}(\dot w(t))$. Finally $\mathcal{D}^{\hom}(0,t;P) \in AC(0,T)$ and, for a.e. $t \in [0,T]$,

$$\dot{\mathcal{D}}^{\hom}(0,t;P) = -\int_{\Omega\times\mathcal{Y}} \Sigma(t)\cdot\dot E(t)\,dx\,dy + \int_\Omega \sigma(t)\cdot E\dot w(t)\,dx.$$

We need the following lower semicontinuity result for the two-scale dissipation potential $\mathcal{H}^{\mathrm{hom}}$.

Proposition 6.5 (Lower semicontinuity of $\mathcal{H}^{\mathrm{hom}}$). *Let* $(u_n, E_n, P_n) \in \mathcal{A}^{\mathrm{hom}}(w_n)$ *be such that*

$$\begin{aligned} u_n &\overset{*}{\rightharpoonup} u && \textit{weakly* in } BD(\Omega'), \\ E_n &\rightharpoonup E && \textit{weakly in } L^2(\Omega' \times \mathcal{Y}; \mathbb{M}^N_{\mathrm{sym}}), \\ P_n &\overset{*}{\rightharpoonup} P && \textit{weakly* in } \mathcal{M}_b(\Omega' \times \mathcal{Y}; \mathbb{M}^N_D), \\ w_n &\to w && \textit{strongly in } H^1(\mathbb{R}^N; \mathbb{R}^N). \end{aligned} \tag{6.8}$$

Then $(u, E, P) \in \mathcal{A}^{\mathrm{hom}}(w)$ *and*

$$\mathcal{H}^{\mathrm{hom}}(P) \le \liminf_n \mathcal{H}^{\mathrm{hom}}(P_n). \tag{6.9}$$

Proof. Since

$$Eu_n \otimes \mathcal{L}^N_y + E_y \mu^n = E_n \,(\mathcal{L}^N_x \otimes \mathcal{L}^N_y) + P_n \quad \text{on } \Omega' \times \mathcal{Y} \tag{6.10}$$

and in view of Lemma 4.8, we immediately infer that $(u, E, P) \in \mathcal{A}^{\mathrm{hom}}(w)$.

The lower semicontinuity (6.9) follows by an argument identical to Step 2 in the proof of Theorem 5.7 provided that we establish the following result. Let $\mathcal{B} \subseteq \mathcal{Y}$ be an open set with Lipschitz boundary and exterior normal denoted by ν, such that $\partial\mathcal{B} \setminus \mathcal{T}$ is of class C^1 for some closed set $\mathcal{T} \subseteq \partial\mathcal{B}$ with $\mathcal{H}^{N-1}(\mathcal{T}) = 0$. If

$$P_n \llcorner_{\Omega' \times \mathcal{B}} \overset{*}{\rightharpoonup} \lambda \quad \text{weakly* in } \mathcal{M}_b(\Omega' \times \mathcal{Y}; \mathbb{M}^N_D),$$

then

$$\lambda \llcorner_{\Omega' \times (\partial\mathcal{B} \setminus \mathcal{T})} = (a(x, y) \odot \nu(y))\eta \tag{6.11}$$

for a suitable measure $\eta \in \mathcal{M}^+_b(\Omega' \times (\partial\mathcal{B} \setminus \mathcal{T}))$ and a Borel map $a : \Omega' \times (\partial\mathcal{B} \setminus \mathcal{T}) \to \mathbb{R}^N$ with $a(x, y) \perp \nu(y)$ for η-a.e. $(x, y) \in \Omega' \times (\partial\mathcal{B} \setminus \mathcal{T})$.

In order to establish (6.11), let us consider $\mu^n \in \boldsymbol{\mathcal{X}}(\Omega')$ associated with (u_n, E_n, P_n). Up to subsequences, we may assume that

$$E_y \mu^n \llcorner_{\Omega' \times \mathcal{B}} \overset{*}{\rightharpoonup} \tilde{\lambda} \quad \text{weakly* in } \mathcal{M}_b(\Omega' \times \mathcal{Y}; \mathbb{M}^N_{\mathrm{sym}}).$$

In view of the convergences (6.8) and of the admissibility condition (6.10), the restriction of λ on $\Omega' \times \partial\mathcal{B}$ is the same as that of $\tilde{\lambda}$.

A direct computation similar to that in the proof of Proposition 4.11 shows that, upon setting

$$(E_y \mu^n \llcorner_{\Omega' \times \mathcal{B}})^i_\varepsilon(F) := \frac{1}{\varepsilon^N} E_y \mu^n (Q^i_\varepsilon \times (F \cap \mathcal{B}))$$

for every Borel set $F \subseteq \mathcal{Y}$, then, as $\varepsilon \to 0$,

$$\sum_{i \in I_\varepsilon(\Omega')} (\mathcal{L}^N_x \llcorner_{Q^i_\varepsilon}) \otimes (E_y \mu^n \llcorner_{\Omega' \times \mathcal{B}})^i_\varepsilon \overset{*}{\rightharpoonup} E_y \mu^n \llcorner_{\Omega' \times \mathcal{B}} \quad \text{weakly* in } \mathcal{M}_b(\Omega' \times \mathcal{Y}; \mathbb{M}^N_{\mathrm{sym}}).$$

Since, with obvious notation,

$$(E_y\mu^n\lfloor_{\Omega'\times\mathcal{B}})^i_\varepsilon=(E_y\mu^n)^i_\varepsilon\lfloor_{\mathcal{B}},$$

a diagonalization process yields the existence of a sequence $\{\varepsilon_n\searrow 0\}_{n\in\mathbb{N}}$ such that

$$\sum_{i\in I_{\varepsilon_n}(\Omega')}(\mathcal{L}^N_x\lfloor_{Q^i_{\varepsilon_n}})\otimes(E_y\mu^n)^i_{\varepsilon_n}\lfloor_{\mathcal{B}}\overset{*}{\rightharpoonup}\tilde{\lambda}\quad\text{weakly* in }\mathcal{M}_b(\Omega'\times\mathcal{Y};\mathrm{M}^N_{\mathrm{sym}}).$$

Now,

$$(\mu^n)^i_{\varepsilon_n}\in BD(\mathcal{Y})\quad\text{and}\quad E_y(\mu^n)^i_{\varepsilon_n}=(E_y\mu^n)^i_{\varepsilon_n}. \tag{6.12}$$

Indeed, in view of Lemma 5.4,

$$\mu^n=\mu^n_x(y)(\eta_n\otimes\mathcal{L}^N_y)$$

where $\eta_n:=\mathcal{L}^N_x+(\mathrm{proj}_\#|P_n|)^s$, and $(x,y)\mapsto\mu^n_x(y)\in\mathbb{R}^N$ is a Borel map with $\mu^n_x\in BD(\mathcal{Y})$ for η-a.e. $x\in\Omega$. Moreover, $x\mapsto E_y\mu^n_x$ is η_n-measurable and $E_y\mu^n\overset{\text{gen.}}{=}\eta_n\otimes E_y\mu^n_x$.

For every $\varepsilon>0$, $i\in I_\varepsilon(\Omega)$ and $g\in C^1(\mathcal{Y};\mathrm{M}^N_{\mathrm{sym}})$,

$$\begin{aligned}(\mu^n)^i_\varepsilon(\mathrm{div}_y\,g)&=\int_{Q^i_\varepsilon\times\mathcal{Y}}\mu^n_x(y)\cdot\mathrm{div}_y\,g(y)\,d\eta_n(x)\,dy\\&=\int_{Q^i_\varepsilon}\left(\int_{\mathcal{Y}}\mu^n_x(y)\cdot\mathrm{div}_y\,g(y)\,dy\right)d\eta_n(x)\\&=-\int_{Q^i_\varepsilon}\left(\int_{\mathcal{Y}}g(y)\,dE_y\mu^n_x(y)\right)d\eta_n(x)=-\int_{Q^i_\varepsilon\times\mathcal{Y}}g(y)\,dE_y\mu^n\\&=-(E_y\mu^n)^i_\varepsilon(g),\end{aligned}$$

where all integrals above are meaningful, hence (6.12).

Then, for every $\chi\in C^1_c(\Omega'\times\mathcal{Y};\mathrm{M}^N_{\mathrm{sym}})$ with $\mathrm{div}_y\,\chi=0$,

$$\begin{aligned}\int_{\Omega'\times\mathcal{Y}}\chi(x,y)\tilde{\lambda}(x,y)&=\lim_n\sum_{i\in I_{\varepsilon_n}(\Omega')}\int_{Q^i_{\varepsilon_n}}\left(\int_{\mathcal{B}}\chi(x,y)\,d(E_y\mu^n)^i_{\varepsilon_n}\right)dx\\&=\lim_n\sum_{i\in I_{\varepsilon_n}(\Omega')}\int_{Q^i_{\varepsilon_n}}\left(\int_{\mathcal{B}}\chi(x,y)\,dE_y(\mu^n)^i_{\varepsilon_n}\right)dx\\&=\lim_n\sum_{i\in I_{\varepsilon_n}(\Omega')}\int_{Q^i_{\varepsilon_n}}\left(\int_{\partial\mathcal{B}}\chi(x,y)\cdot[(\mu^n)^i_{\varepsilon_n}(y)\odot\nu(y)]\,d\mathcal{H}^{N-1}(y)\right)dx.\end{aligned} \tag{6.13}$$

At the expense of subtracting infinitesimal rigid body motions on $\mathcal{B}$, we may assume that

$$\int_{\partial\mathcal{B}}|(\mu^n)^i_{\varepsilon_n}|\,d\mathcal{H}^{N-1}\le C|E_y(\mu^n)^i_{\varepsilon_n}|(\mathcal{B})\le\frac{C}{\varepsilon_n^N}|E_y\mu^n|(Q^i_{\varepsilon_n}\times\mathcal{B})$$

for some constant $C>0$ independent of n and i. Since $\{E_y\mu^n\}_{n\in\mathbb{N}}$ is a bounded sequence in $\mathcal{M}_b(\Omega' \times \mathcal{Y}; \mathrm{M}^N_{\mathrm{sym}})$, the measures

$$\sum_{i \in I_{\varepsilon_n}(\Omega')} (\mathcal{L}^N_x \lfloor_{Q^i_{\varepsilon_n}}) \otimes (\mu^n)^i_{\varepsilon_n} \mathcal{H}^{N-1} \lfloor_{\partial\mathcal{B}} \in \mathcal{M}_b(\Omega' \times \partial\mathcal{B}; \mathbb{R}^N)$$

and

$$\sum_{i \in I_{\varepsilon_n}(\Omega')} (\mathcal{L}^N_x \lfloor_{Q^i_{\varepsilon_n}}) \otimes [(\mu^n)^i_{\varepsilon_n} \odot \nu \mathcal{H}^{N-1} \lfloor_{\partial\mathcal{B}}] \in \mathcal{M}_b(\Omega' \times \partial\mathcal{B}; \mathrm{M}^N_{\mathrm{sym}})$$

form bounded sequences, so that, up to subsequences, we may assume that

$$\sum_{i \in I_{\varepsilon_n}(\Omega')} (\mathcal{L}^N_x \lfloor_{Q^i_{\varepsilon_n}}) \otimes (\mu^n)^i_{\varepsilon_n} \mathcal{H}^{N-1} \lfloor_{\partial\mathcal{B}} \overset{*}{\rightharpoonup} \zeta \in \mathcal{M}_b(\Omega' \times \partial\mathcal{B}; \mathbb{R}^N),$$

and

$$\sum_{i \in I_{\varepsilon_n}(\Omega')} (\mathcal{L}^N_x \lfloor_{Q^i_{\varepsilon_n}}) \otimes [(\mu^n)^i_{\varepsilon_n} \odot \nu \mathcal{H}^{N-1} \lfloor_{\partial\mathcal{B}}] \overset{*}{\rightharpoonup} \pi \quad \text{weakly* in } \mathcal{M}_b(\Omega' \times \partial\mathcal{B}; \mathrm{M}^N_{\mathrm{sym}}).$$

In view of Lemma 4.9 and (6.13), there exists $\mu \in \boldsymbol{\mathcal{X}}(\Omega')$ such that

$$\tilde{\lambda} = \pi + E_y \mu.$$

Since ν is continuous on $\partial\mathcal{B} \setminus \mathcal{T}$, we immediately deduce that

$$\pi \lfloor_{\partial\mathcal{B}\setminus\mathcal{T}} = \frac{\zeta}{|\zeta|} \odot \nu\, |\zeta| \lfloor_{\partial\mathcal{B}\setminus\mathcal{T}},$$

so that by appealing to Proposition 4.7(b), (6.11) follows. □

The following result finally yields the flow rule for two-scale quasi-static evolutions.

Theorem 6.6 (Two-scale flow rule). *Assume that $\mathcal{Y}$ is a C^2-admissible multiphase torus and that $\partial \lfloor_{\partial\Omega} \Gamma_d$ is admissible in the sense of Definition 2.1. Let $t \mapsto (u(t), E(t), P(t)) \in \mathcal{A}^{\mathrm{hom}}(w(t))$ be a two-scale quasi-static evolution. Then, for a.e. $t \in [0, T]$,*

(a) $(\dot{u}(t), \dot{E}(t), \dot{P}(t)) \in \mathcal{A}^{\mathrm{hom}}(\dot{w}(t))$;

(b) *for $\mathcal{L}^N_x$-a.e. $x \in \Omega$,*

$$\frac{\dot{P}_x(t)}{|\dot{P}_x(t)|}(y) \in N_{K(y)}(\Sigma_D(t, x, y)) \quad \text{for } \mathcal{L}^N_y\text{-a.e. } y \in \{|\dot{P}_x(t)| > 0\},$$

where $\dot{P}_x$ results from the decomposition (5.8) of Lemma 5.4;

(c) *letting $\dot{\mu}(t) \in \boldsymbol{\mathcal{X}}(\Omega')$ be the measure associated with $(\dot{u}(t), \dot{E}(t), \dot{P}(t)) \in \mathcal{A}^{\mathrm{hom}}(\dot{w}(t))$, for $\mathcal{L}^N_x$-a.e. $x \in \Omega$ and for every $i \neq j$,*

$$\frac{\dot{\mu}^i_x(t, y) - \dot{\mu}^j_x(t, y)}{|\dot{\mu}^i_x(t, y) - \dot{\mu}^j_x(t, y)|} \in \vec{N}_{K_\Gamma(y)}((\Sigma_D(t, x, \cdot)\nu)_\tau(y)) \qquad \text{for } \mathcal{H}^{N-1}\text{-a.e. } y \in \{\dot{\mu}^i_x(t) \neq \dot{\mu}^j_x(t)\},$$

where $\dot{\mu}_x(t)$ results from the disintegration (5.9) of $\dot{\mu}(t)$, $\dot{\mu}^i_x(t)$ and $\dot{\mu}^j_x(t)$ are the traces on Γ_{ij} of the restrictions of $\dot{\mu}_x(t)$ on $\mathcal{Y}_i$ and $\mathcal{Y}_j$ respectively, assuming that ν points from $\mathcal{Y}_j$ to $\mathcal{Y}_i$, and $\vec{N}_{K_\Gamma(y)}(\tau)$ denotes the normal cone (a cone of vectors) to $K_\Gamma(y)$ at a vector $\tau \perp \nu(y)$.

Proof. Let $t \in [0,T]$ be a time such that $\dot{w}(t)$ exists in $H^1(\mathbb{R}^N;\mathbb{R}^N)$, and $\dot{u}(t)$, $\dot{E}(t)$, $\dot{P}(t)$, $\dot{\mathcal{D}}^{\mathrm{hom}}(0,t;P)$ all exist in the sense of Proposition 6.4 with

$$\dot{\mathcal{D}}^{\mathrm{hom}}(0,t;P) = -\int_{\Omega\times\mathcal{Y}} \Sigma(t) : \dot{E}(t)\,dx\,dy + \int_{\Omega} \sigma(t) : E\dot{w}(t)\,dx.$$

By Proposition 6.5 we deduce that $(\dot{u}(t), \dot{E}(t), \dot{P}(t)) \in \mathcal{A}^{\mathrm{hom}}(\dot{w}(t))$. Since $\mathcal{D}^{\mathrm{hom}}$ is a total variation, and since $\mathcal{H}^{\mathrm{hom}}$ is positively one-homogeneous, for $t_1 > t$ we have

$$\mathcal{H}^{\mathrm{hom}}\left(\frac{P(t_1)-P(t)}{t_1-t}\right) \le \frac{\mathcal{D}^{\mathrm{hom}}(0,t_1;P)-\mathcal{D}^{\mathrm{hom}}(0,t;P)}{t_1-t}.$$

Hence, taking the limit for $t_1 \to t$, and appealing to Proposition 6.5, we infer that

$$\mathcal{H}^{\mathrm{hom}}(\dot{P}(t)) \le -\int_{\Omega\times\mathcal{Y}} \Sigma(t) : \dot{E}(t)\,dx\,dy + \int_{\Omega} \sigma(t) : E\dot{w}(t)\,dx.$$

But $\Sigma(t) \in \mathcal{K}^{\mathrm{hom}}$ by Lemma 6.3, so that the opposite inequality holds true in view of Remark 5.13, and we obtain

$$\mathcal{H}^{\mathrm{hom}}(\dot{P}(t)) = -\int_{\Omega\times\mathcal{Y}} \Sigma(t) : \dot{E}(t)\,dx\,dy + \int_{\Omega} \sigma(t) : E\dot{w}(t)\,dx\,dy.$$

The result then immediately follows from Theorem 5.12. □

Remark 6.7. The disintegrations of $P(t)$ and $\dot{P}(t)$ do not imply that $\dot{P}_x(t)$ is the derivative of $P_x(t)$ in the weak* (or strict) sense of Proposition 6.4. Consequently, the flow rule of Theorem 6.6 cannot be construed as completely vindicating the two-scale evolution as that corresponding to a generalized standard material in the sense of [13].

But worse still, our flow rules view P_x (as functions of y,t) as the internal variables, whereas a consistent thermodynamical model would freeze the variable $y \in \mathcal{Y}$ and seek a flow rule in the variable x. In truth, we do not have enough structure on the measures P and on the functions Σ to switch the disintegration around, that is, to write $P = \kappa \overset{\text{gen.}}{\otimes} P_y$ with $\kappa \in \mathcal{M}_b^+(\mathcal{Y})$ and $P_y \in \mathcal{M}_b(\Omega'; \mathrm{M}^N_D)$, and to hope for a flow rule in x for $\mathcal{L}^N_y$-a.e. $y \in \mathcal{Y}$.

These discrepancies will hopefully be resolved in future investigations.

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