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Complex Variables Functions — *Inequality for entire functions involving their maximum modulus and maximum term*, by TATYANA SHAPOSHNIKOVA, communicated on 9 March 2012.

Dedicated to the memory of Gaetano Fichera

ABSTRACT. — An estimate of the Wiman-Valiron type for a maximum modulus on a polydisk of an entire function of several complex variables is obtained. The estimate contains a weight function involved also in the calculation of the radius of the admissible ball.

KEY WORDS: Entire function, several complex variables, maximal modulus and maximum term.

2010 MATHEMATICS SUBJECT CLASSIFICATION: Primary: 35G15, 35J55; Secondary: 35J67, 35E05.

INTRODUCTION

Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a multi-index with $\alpha_j \geq 0$ and let $z = (z_1, \dots, z_n)$ be a point of the n-dimensional complex space $\mathbb{C}^n$. We consider the entire function

$$f(z) = \sum_{\alpha} a_\alpha z^\alpha. \tag{0.1}$$

For any $r = (r_1, \dots, r_n)$ with $r_j \geq 0$ we introduce the maximum modulus and the maximum term of f:

$$M_f(r) = \max_{\{z:|z_j|=r_j\}} |f(z)|, \quad m_f(r) = \max_{\alpha} |a_\alpha| r^\alpha. \tag{0.2}$$

Clearly, by the Cauchy formula for the coefficients a_α,

$$m_f(r) \leq M_f(r).$$

For $n = 1$ the first result on the comparison between $M_f(r)$ and $m_f(r)$ is due to A. Wiman [W] who showed that for any r_0 there is an $r > r_0$ such that

$$M_f(r) < (\log m_f(r))^{1/2+\varepsilon} m_f(r) \tag{0.3}$$

with any $\varepsilon > 0$. Moreover, if A is the set of $r > 0$ such that (0.3) holds, then

$$\int_A \frac{dr}{r} < \infty.$$

Wiman's proof was essentially simplified by G. Valiron [V].

The following generalization of inequality (0.3) is due to P. C. Rosenbloom [R].

Theorem 1. *Let φ be a positive nondecreasing function on $(0, \infty)$ such that*

$$H(\tau) := \int_\tau^\infty \Big(\int_\tau^t \varphi(s)\, ds\Big)^{-1/2} dt < \infty \tag{0.4}$$

for all $\tau > 0$. Given any point r_0 with nonnegative coordinates, the ball

$$\{r : |\log r - \log r_0| < \sqrt{n} H(\log M_f(r_0))\} \tag{0.5}$$

contains at least one point r such that

$$M_f(r) \le C m_f(r)(\varphi(\log M_f(r)))^{n/2}, \tag{0.6}$$

where C is a positive constant and $\log r = (\log r_1, \dots, \log r_n)$.

The article [R] contains a proof of this assertion for functions of one variable and outlines the argument in the general case.

Remark. An example of a function φ for which the condition (0.4) holds is a function given for large positive s by

$$\varphi(s) = s(\log s)^2 (\log\log s)^2 \dots (\log \dots \log s)^{2+\varepsilon}$$

with an $\varepsilon > 0$.

1. Main result

The aim of this note is the proof of the following generalization of Theorem 1.

Theorem 2. *Let φ and H be the same function as in Theorem 1. Further, let h be a positive continuous increasing function such that*

$$h(2x) \le c h(x) \tag{1.1}$$

with a constant $c > 1$. For any point r_0 with nonnegative coordinates, far away from the origin, the ball

$$\{r : |\log r - \log r_0| < R\} \tag{1.2}$$

with the radius defined by the equation

$$\int_0^R \Big(h\Big(\frac{1}{2}|\log r_0| + r\Big)\Big)^{1/2} dr = c\sqrt{2n}H(\log M_f(r)) \tag{1.3}$$

contains at least one point r such that

$$M_f(r) \leq Cm_f(r)(h(|\log r|)\varphi(\log M_f(r)))^{n/2}, \tag{1.4}$$

where C is a positive constant.

Proof. We may replace a_α by $|a_\alpha|$ in the definition of f since $m_f(r)$ won't change and $M_f(r)$ does not decrease. Following [R], we introduce a random vector $\xi = (\xi_1, \dots, \xi_n)$ with probability distribution

$$P(\xi = \alpha) = \frac{a_\alpha e^{\alpha x}}{f(e^x)}, \quad x \in \mathbb{R}^n.$$

Let $E_x\xi$ and $D_x\xi$ stand for the mathematical expectation and the dispersion of ξ. We introduce the notation

$$F(x) = \log f(e^x).$$

Clearly

$$\nabla F(x) = \frac{1}{f(e^x)} \nabla_x f(e^x). \tag{1.5}$$

Furthermore,

$$\Delta F(x) = \operatorname{div} \nabla F(x) = \frac{1}{f(e^x)} \Delta_x f(e^x) - \frac{(\nabla_x f(e^x))^2}{(f(e^x))^2}. \tag{1.6}$$

By the definition of the expectation,

$$E_x(\xi) = \sum_\alpha \alpha P(\xi = \alpha)$$

and

$$E_x(\xi^2) = \sum_\alpha \alpha^2 P(\xi = \alpha).$$

This together with (1.5) gives

$$E_x(\xi) = \frac{1}{f(e^x)} \sum_\alpha \alpha a_\alpha e^{\alpha x} = \nabla F(x).$$

Further, the definition of the dispersion

$$D_x\xi = E_x(\xi^2) - (E_x(\xi))^2$$

gives

$$D_x\xi = \frac{1}{f(e^x)}\sum_{\alpha}\alpha^2 a_\alpha e^{\alpha x} - \frac{1}{(f(e^x))^2}\Big(\sum_{\alpha}\alpha a_\alpha e^{\alpha x}\Big)^2.$$

Combining this with

$$\Delta f(e^x) = \sum_{\alpha}\alpha^2 a_\alpha e^{\alpha x}$$

we see that

$$D_x\xi = \frac{1}{f(e^x)}\Delta f(e^x) - \frac{(\nabla_x f(e^x))^2}{(f(e^x))^2}.$$

Now, by (1.6)

$$D_x\xi = \Delta F(x).$$

By the Chebyshev inequality,

$$P\{|\xi - E_x\xi| \geq \lambda(D_x\xi)^{1/2}\} \leq \lambda^{-2}, \quad \lambda > 1,$$

we have

$$P\{|\xi - \nabla F(x)| \geq \lambda(\Delta F(x))^{1/2}\} \leq \lambda^{-2}, \quad \lambda > 1. \tag{1.7}$$

The probability on the left-hand side is equal to

$$\frac{1}{f(e^x)}\sum{}' a_\alpha e^{\alpha x} \tag{1.8}$$

with the sum taken over all multiindices α such that

$$|\alpha - \nabla F(x)| \geq \lambda(\Delta F(x))^{1/2}.$$

Let $\sum''$ stand for summation over α for which

$$|\alpha - \nabla F(x)| < \lambda(\Delta F(x))^{1/2}.$$

By inequalities (1.7) and (1.8) we have

$$1 - \frac{1}{f(e^x)}\sum{}'' a_\alpha e^{\alpha x} \leq \lambda^{-2}.$$

Hence

$$f(e^x) \le (1-\lambda^{-2})^{-1} \sum{}'' a_\alpha e^{\alpha x}.$$

Therefore,

$$f(e^x) \le \frac{2^n \lambda^{n+2}}{\lambda^2 - 1} m_f(e^x)(\Delta F(x))^{n/2}. \tag{1.9}$$

Let R be defined by (1.3) and let $|x_0| > 4R$. If $\Delta F(x) \le h(|x|)H(F(x))$ in the ball

$$B_R = \{x : |x - x_0| < R\},$$

the result follows.

Suppose that the opposite inequality

$$\Delta F(x) > h(|x|)H(F(x))$$

holds in the ball B_R. Since, clearly,

$$|x| \ge |x_0| - |x - x_0| \ge |x - x_0| + |x_0|/2,$$

the following inequality holds in the ball B_R as well

$$\Delta F(x) > h(|x - x_0| + |x_0|/2)H(F(x)).$$

Consider the equation

$$\Delta u(x) = h(|x - x_0| + |x_0|/2)H(u(x)) \tag{1.10}$$

in B_R. By the maximum principle for the Laplace operator, the inequality

$$F(x) \le u(x)$$

with $x \in \partial B_R$ implies the same inequality on the whole B_R.

Suppose that

$$u|_{\partial B_R} = \max_{x \in \partial B_R} F(x). \tag{1.11}$$

The solution of the Dirichlet problem (1.10)–(1.11) is unique, therefore, u depends only on $\rho = |x - x_0|$. Hence u satisfies the boundary value problem

$$(\rho^{n-1} u'_\rho)'_\rho = \rho^{n-1} h(\rho + |x_0|/2)H(u(\rho)), \tag{1.12}$$

$$u(R) = \max_{x \in \partial B_R} F(x), \quad u'(0) = 0. \tag{1.13}$$

This implies

$$u'(\rho) = \rho^{1-n} \int_0^\rho h(s + |x_0|/2)s^{n-1} H(u(s))\, ds. \tag{1.14}$$

Clearly, the function H, given by (0.4) is decreasing. Combining this with the monotonicity of the functions h, we obtain

$$u'(\rho) \le h(\rho + |x_0|/2) H(u(\rho)) n^{-1} \rho. \tag{1.15}$$

By the equation (1.12),

$$u''(\rho) = h(\rho + |x_0|/2) H(u(\rho)) - \frac{n-1}{\rho} u'(\rho).$$

This and (1.15) lead to the ordinary differential unequality

$$u''(\rho) \ge n^{-1} h(\rho + |x_0|/2) H(u(\rho)) \tag{1.16}$$

for all $\rho \in [0, R]$.

Let us show that the number R satisfies the inequality

$$\int_0^R (h(\rho + |x_0|/2))^{1/2} \, d\rho < c\sqrt{2n} H(u(|x_0|)). \tag{1.17}$$

Having proved (1.17), the result follows from the estimate $u(|x_0|) \ge F(x_0)$ and the monotonicity of H.

Since h increases, by (1.16) we have

$$u''(\rho) \ge n^{-1} h\Big(\frac{R + |x_0|}{2}\Big) H(u(\rho)) \tag{1.18}$$

for $\rho \in \Big[\frac{R+|x_0|}{2}, R\Big]$. We multiply (1.18) by $u'(\rho) > 0$ and integrate the result over $\Big[\frac{R+|x_0|}{2}, R\Big]$ to obtain

$$(u'(\rho))^2 - \Big(u'\Big(\frac{R + |x_0|}{2}\Big)\Big)^2 \ge \frac{2}{n} h\Big(\frac{R + |x_0|}{2}\Big) \int_{u((R+|x_0|)/2)}^{u(\rho)} H(s) \, ds.$$

Using $u'(\rho) > 0$ once more, we arrive at

$$\Big(\int_{u((R+|x_0|)/2)}^{u(\rho)} H(s) \, ds\Big)^{-1/2} u'(\rho) \ge \sqrt{\frac{2}{n}} \Big(h\Big(\frac{R + |x_0|}{2}\Big)\Big)^{1/2}.$$

Integrating this inequality over the interval $[R/2, R]$, we see that

$$H\Big(u\Big(\frac{R + |x_0|}{2}\Big)\Big) \ge \frac{1}{\sqrt{2n}} R \Big(h\Big(\frac{R + |x_0|}{2}\Big)\Big)^{1/2}. \tag{1.19}$$

By (1.1),

$$\int_0^R \Big(h\Big(\rho + \frac{|x_0|}{2}\Big)\Big)^{1/2} d\rho \le c \int_0^R \Big(h\Big(\frac{\rho + |x_0|}{2}\Big)\Big)^{1/2} d\rho \le cR \Big(h\Big(\frac{R + |x_0|}{2}\Big)\Big)^{1/2}.$$

Combining this with (1.19), we find

$$H\Big(u\Big(\frac{R+|x_0|}{2}\Big)\Big) \geq \frac{1}{c\sqrt{2n}}\int_0^R \Big(h\Big(\rho+\frac{|x_0|}{2}\Big)\Big)^{1/2}\,d\rho.$$

Since u is nondecreasing (see (1.14)), we have

$$u(|x_0|) \leq u\Big(\frac{R+|x_0|}{2}\Big).$$

By (0.5) the function H does not increase which implies

$$H(u(|x_0|)) \geq \frac{1}{c\sqrt{2n}}\int_0^R \Big(h\Big(\rho+\frac{|x_0|}{2}\Big)\Big)^{1/2}\,d\rho.$$

The proof is complete.

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