

## Index of elliptic operators for diffeomorphisms of manifolds

Anton Savin and Boris Sternin

**Abstract.** We develop an elliptic theory for operators associated with a diffeomorphism of a closed smooth manifold. The aim of the present paper is to obtain an index formula for such operators in terms of topological invariants of the manifold and the symbol of the operator. The symbol in this situation is an element of a certain crossed product. We express the index as the pairing of the class in K-theory defined by the symbol and the Todd class in periodic cyclic cohomology of the crossed product.

*Mathematics Subject Classification* (2010). 58J05; 19K56 46L85.

*Keywords.* Noncommutative elliptic theory, index, cyclic cohomology, crossed product, Haefliger cohomology, Todd class.

### Introduction

Let  $M$  be a smooth manifold and  $g: M \rightarrow M$  be a diffeomorphism. We develop an elliptic theory for operators of the form

$$D = \sum_k D_k T^k: C^\infty(M) \rightarrow C^\infty(M). \quad (0.1)$$

Here  $T$  is the shift operator  $Tu(x) = u(g(x))$  along the orbits of  $g$ ,  $D_k$  are pseudodifferential operators ( $\psi$ DO) on  $M$ , and the sum is assumed to be finite.

The aim of the present paper is to obtain an index formula for the operator (0.1) in terms of topological invariants of the manifold and of the symbol of the operator. Precise definitions of all the objects will be given below, but now let us note an important characteristic property of this theory. Namely, the algebra of symbols of operators (0.1) is *not commutative*. More precisely, an explicit computation shows that the algebra of symbols is the crossed product  $C^\infty(S^*M) \rtimes \mathbb{Z}$  of the algebra of functions on the cosphere bundle by the action of the group  $\mathbb{Z}$ . This essentially means that we consider noncommutative elliptic theory.

Special classes of operators (0.1) were considered by a number of authors (e.g., see [1], [2], [16], [27], [29], [30], [37], [38], [39]). In these papers, as a rule, certain conditions were imposed on the manifold  $M$  and on the diffeomorphism  $g$ . For example, in the book [27] it is assumed that the diffeomorphism is an isometry, in

[2] the diffeomorphism is arbitrary, but the manifold is one-dimensional, and so on. Further, we would like to mention papers [17] and [26], where the authors study elliptic operators of the form (0.1) that are associated with the Dirac operator and conformal diffeomorphisms. Let us stress that in the present paper we consider an arbitrary compact smooth manifold and an arbitrary diffeomorphism *without any restrictions*.

The main result of the paper is an explicit index formula for the elliptic operator (0.1). More explicitly, the answer is given by the formula

$$\text{ind } D = (2\pi i)^{-n} \langle [\sigma(D)], \text{Todd}(T_{\mathbb{C}}^*M) \rangle, \quad \dim M = n, \quad (0.2)$$

in terms of cyclic cohomology, where  $[\sigma(D)] \in K_0(C^\infty(S^*M \times \mathbb{S}^1) \rtimes \mathbb{Z})$  is the class of symbol in K-theory,  $\text{Todd}(T_{\mathbb{C}}^*M) \in \text{HP}^{\text{ev}}(C^\infty(S^*M \times \mathbb{S}^1) \rtimes \mathbb{Z})$  is the Todd class in cyclic cohomology and the brackets  $\langle \cdot, \cdot \rangle$  denote the pairing of K-theory and cyclic cohomology.

Let us briefly describe the methods used in the present paper. It is clear more or less that a noncommutative elliptic theory requires a noncommutative apparatus: noncommutative differential forms, noncommutative trace, etc. Moreover, since the diffeomorphism generates an action of the group  $\mathbb{Z}$ , the relevant topological invariants are naturally elements of the Haefliger cohomology group  $H^*(S^*M/\mathbb{Z})$  (see [22]). In this framework, we define the Chern character and establish an important intermediate index formula (interesting in its own right) as an integral of a Haefliger form over  $S^*M \times \mathbb{S}^1$ . After this, we reduce the obtained formula to the natural and elegant formula (0.2). The latter index formula can be considered as an analogue of the Atiyah–Singer formula in our situation.

We now describe the contents of the paper. In Section 1 we introduce a notion of ellipticity and prove finiteness theorem. In Section 2 we define Chern characters for crossed products, including twisted Chern character, and define the Chern character of an elliptic symbol. Section 3 is devoted to the solution of the equation

$$\text{ch } y = \text{Td } x, \quad (0.3)$$

where  $\text{Td } x$  is the Todd class of a complex vector bundle  $x$  and  $\text{ch}$  is the Chern character. The point here is that in our situation the Todd class is generally speaking undefined. However, it can be replaced by the Chern character of a bundle satisfying eq. (0.3). Finally, in Section 4 we formulate an index theorem (in Haefliger cohomology). The proof of the index theorem is given in Sections 5 and 6. Namely, in Section 5 we reduce our initial operator to a special boundary value problem on the cylinder  $M \times [0, 1]$  (see [32], [33]), which is then reduced to a certain pseudodifferential operator on the torus of the original manifold twisted by the diffeomorphism  $g$  (cf. [7], [11]). The index of the latter operator can be computed by the Atiyah–Singer formula. However, to give an index formula in terms of the original operator, we need to compare the index formula for the pseudodifferential operator on the torus and the formula announced in Section 4. So the proof of the index formula for the

original operator is complete, at least in the framework of Haefliger cohomology. In Section 7 we interpret the index formula in Haefliger cohomology as an Atiyah–Singer formula in cyclic cohomology. Here we use equivariant characteristic classes in cyclic cohomology (see [21]). In Section 8 we give some remarks and consider an example.

This work was supported in part by RFBR grant Nr. 12-01-00577, DFG grant 436 RUS 113/849/0-1 (K-theory and noncommutative geometry of stratified manifolds), the Simons Foundation and was done at the Institute of Analysis of Leibniz University of Hannover. We are grateful to Prof. Elmar Schrohe, in whose working group this research was done, for attention and excellent working conditions. We are also grateful to Prof. Victor Nistor for useful remarks.

### 1. Ellipticity and finiteness theorem

Let  $M$  be a smooth closed manifold and  $g : M \rightarrow M$  be a diffeomorphism. Consider an operator of the form

$$D = \sum_k D_k T^k : C^\infty(M, \mathbb{C}^N) \rightarrow C^\infty(M, \mathbb{C}^N), \tag{1.1}$$

where  $T : C^\infty(M) \rightarrow C^\infty(M)$ ,  $Tu(x) = u(g(x))$ , is the shift operator corresponding to  $g$ , the coefficients

$$D_k : C^\infty(M, \mathbb{C}^N) \rightarrow C^\infty(M, \mathbb{C}^N)$$

are pseudodifferential operators ( $\psi$ DO) of order zero and the sum in (1.1) is finite. Denote the principal symbols of the coefficients by

$$\sigma(D_k) \in C^\infty(S^*M, \text{Mat}_N(\mathbb{C})).$$

Here  $S^*M = T_0^*M/\mathbb{R}_+$  stands for the cosphere bundle with the projection  $\pi : S^*M \rightarrow M$ , where  $T_0^*M = T^*M \setminus 0$  is the cotangent bundle with the zero section deleted.

**Definition 1.1.** The *symbol* of the operator  $D$  is the collection  $\sigma(D) = \{\sigma(D_k)\}$  of symbols of its coefficients.

If

$$B = \sum B_k T^k : C^\infty(M, \mathbb{C}^N) \rightarrow C^\infty(M, \mathbb{C}^N) \tag{1.2}$$

is another operator of the form (1.1), then the symbol of the composition of (1.1) and (1.2) is determined by the formula

$$\sigma(DB)(k) = \sum_{l+m=k} \sigma(D_l)[(\partial g)^{l*} \sigma(B_m)]. \tag{1.3}$$

The product of symbols on the right-hand side of eq. (1.3) is called the crossed product of symbols. Here  $\partial g = (dg^t)^{-1} : T^*M \rightarrow T^*M$  is the codifferential of  $g$ .

**Definition 1.2.** An operator  $D$  is *elliptic* if there exists a symbol  $\sigma(D)^{-1}$  with a finite number of nonzero components such that

$$\sigma(D)\sigma(D)^{-1} = \mathbf{1}, \quad \sigma(D)^{-1}\sigma(D) = \mathbf{1},$$

where the product of symbols is defined by eq. (1.3), while the symbol of the identity operator  $Id = T^0$  is denoted by  $\mathbf{1}$ .

The composition formula (1.3) readily implies the following finiteness theorem.

**Theorem 1.1.** *An elliptic operator (1.1),*

$$D : H^s(M, \mathbb{C}^N) \rightarrow H^s(M, \mathbb{C}^N),$$

*is Fredholm in Sobolev spaces for all  $s$ , and its kernel and cokernel consist of smooth functions.*

*Proof.* Indeed, since  $D$  is elliptic, the inverse symbol  $\sigma(D)^{-1}$  has finitely many nonzero components. Denote by  $D^{-1}$  an arbitrary operator with symbol equal to  $\sigma(D)^{-1}$ . Then a direct computation shows that  $D^{-1}$  is an inverse of  $D$  modulo operators of negative order. □

**Remark 1.1.** The set of all symbols is an algebra with respect to the product (1.3). This algebra is actually the algebra of matrices whose entries are elements of the crossed product (e.g., see [44]) of the algebra  $C^\infty(S^*M)$  of smooth functions on  $S^*M$  and the group  $\mathbb{Z}$ . The latter algebra is denoted by  $C^\infty(S^*M) \rtimes \mathbb{Z}$ . In the present paper, we consider only algebraic crossed products whose elements have at most a finite number of nonzero components.

## 2. Chern characters for crossed products

**2.1. Chern character.** Let  $g : X \rightarrow X$  be a diffeomorphism of a smooth closed manifold  $X$  and  $E \in \text{Vect}(X)$  be a vector bundle.

We recall some facts of the theory of noncommutative differential forms (e.g., see [15], [16], [27]).

**Noncommutative differential forms.** Let  $\Lambda(X)$  be the algebra of differential forms on  $X$  with smooth coefficients. Following [27], we define the space  $\Lambda(X, \text{End } E, \mathbb{Z})$  of *noncommutative forms* on  $X$ . This space consists of finite sequences

$$a = \{a(k)\}, \quad a(k) \in \Lambda(X) \otimes \text{Hom}(g^{k*} E, E), \quad \deg a = \max_k \deg a(k),$$

which we represent as operators

$$a = \sum_k a(k)T^k : \Lambda(X, E) \rightarrow \Lambda(X, E),$$

where for  $\omega \in \Lambda(X, E)$  we set  $T\omega := g^*\omega \in \Lambda(X, g^*E)$ . This operator interpretation endows the space  $\Lambda(X, \text{End } E, \mathbb{Z})$  with an algebra structure. Namely, the product of two forms  $a = \sum_k a(k)T^k$  and  $b = \sum_l b(l)T^l$  is the form

$$ab = \sum_{k,l} a(k)g^{k*}(b(l))T^{k+l}.$$

The subalgebra of forms of zero degree is denoted by  $C^\infty(X, \text{End } E, \mathbb{Z})$ .

**Remark 2.1.** For a trivial bundle  $\mathbf{1}_n$  of rank  $n$ , we have

$$\Lambda(X, \text{End } \mathbf{1}_n, \mathbb{Z}) \simeq \Lambda(X, \text{Mat}_n(\mathbb{C})) \rtimes \mathbb{Z}.$$

More generally, suppose we are given a  $g$ -bundle  $E$ . This means that the mapping  $g: X \rightarrow X$  is extended to a fiberwise-linear mapping  $\tilde{g} = \alpha g^*: E \rightarrow E$ , where  $\alpha: g^*E \rightarrow E$  is an isomorphism of vector bundles. In this case we have an isomorphism of algebras

$$\Lambda(X, \text{End } E, \mathbb{Z}) \simeq \Lambda(X, \text{End } E) \rtimes \mathbb{Z}, \quad \sum_k a(k)T^k \mapsto \sum_k [a(k)T^k(\alpha T)^{-k}]\tilde{T}^k.$$

Here

$$\tilde{T} = \alpha T: \Lambda(X, E) \rightarrow \Lambda(X, E) \tag{2.1}$$

is the action of the shift operator on the sections of  $E$ , while the algebra  $\Lambda(X, \text{End } E) \rtimes \mathbb{Z}$  is the crossed product for the shift operator (2.1).

**Graded trace.** We define a graded trace on noncommutative forms taking values in Haefliger forms on the manifold. To this end, we first recall necessary facts about Haefliger forms and cohomology (see [22]). In the de Rham complex  $(\Lambda(X), d)$ , consider the subcomplex  $((1 - g^*)\Lambda(X), d)$ .

**Definition 2.1.** The space of Haefliger forms on  $X$ , denoted by  $\Lambda(X/\mathbb{Z})$ , is the quotient space  $\Lambda(X)/(1 - g^*)\Lambda(X)$ . The cohomology of the quotient complex  $(\Lambda(X)/(1 - g^*)\Lambda(X), d)$  is the Haefliger cohomology of  $X$  with respect to the diffeomorphism  $g$  and is denoted by  $H(X/\mathbb{Z})$ .

**Example 2.1.** It is clear from the definition that Haefliger forms are automatically  $g$ -invariant. Moreover, if  $g^N = Id$ , then the spectral decomposition with respect to  $g$  shows that the Haefliger complex is isomorphic to the complex  $(\Lambda(X)^g, d)$  of  $g$ -invariant forms. Therefore, Haefliger cohomology in this case is isomorphic to the cohomology of the quotient  $X/\mathbb{Z}_N$  of  $X$  by the action of the group generated by the diffeomorphism  $g$ . This gives an explanation for our notation of Haefliger cohomology.

Consider the mapping

$$\tau_E : \Lambda(X, \text{End } E, \mathbb{Z}) \rightarrow \Lambda(X/\mathbb{Z}), \quad \sum_k \omega(k)T^k \mapsto \text{tr}_E(\omega(0)), \quad (2.2)$$

where  $\Lambda(X/\mathbb{Z})$  is the space of Haefliger forms on  $X$ , while  $\text{tr}_E : \Lambda(X, \text{End } E) \rightarrow \Lambda(X)$  is the trace of an endomorphism of  $E$ .

The mapping (2.2) for the trivial bundle  $E = X \times \mathbb{C}^n$  will be denoted by  $\tau$ .

**Lemma 2.1.** *The mapping (2.2) is a graded trace on the algebra  $\Lambda(X, \text{End } E, \mathbb{Z})$ , i.e.,*

$$\tau_E([\omega_1, \omega_2]) = 0 \quad \text{whenever } \omega_1, \omega_2 \in \Lambda(X, \text{End } E, \mathbb{Z}),$$

where  $[\cdot, \cdot]$  stands for the supercommutator

$$[\omega_1, \omega_2] = \omega_1 \omega_2 - (-1)^{\text{deg } \omega_1 \text{ deg } \omega_2} \omega_2 \omega_1.$$

*Proof.* The proof is straightforward:

$$\begin{aligned} \tau_E(\omega_1 T^k \omega_2 T^{-k}) &= \text{tr}_E(\omega_1 g^{k*}(\omega_2)) \\ &= \text{tr}_{g^{-k*}E}(g^{-k*}(\omega_1 g^{k*}(\omega_2))) \\ &= \text{tr}_{g^{-k*}E}((g^{-k*} \omega_1) \omega_2) \\ &= (-1)^{\text{deg } \omega_1 \text{ deg } \omega_2} \tau_E(\omega_2 T^{-k} \omega_1 T^k). \end{aligned}$$

Here  $\omega_1 \in \Lambda(X, \text{Hom}(g^{k*}E, E))$  and  $\omega_2 \in \Lambda(X, \text{Hom}(g^{-k*}E, E))$ . The first and the last equalities follow from the definition of the product of noncommutative forms; the second equality follows from the properties of Haefliger forms; the third equality follows from the properties of the induced mapping.  $\square$

**Noncommutative connection and curvature form.** We choose a connection in  $E$

$$\nabla_E : \Lambda(X, E) \rightarrow \Lambda(X, E).$$

Given a projection  $p \in C^\infty(X, \text{End } E, \mathbb{Z})$ , we define a differential operator

$$\nabla := p \nabla_E p : \Lambda(X, E) \rightarrow \Lambda(X, E). \quad (2.3)$$

of order one. A *noncommutative connection* for  $p$  is the sum

$$p(\nabla_E + \omega)p$$

of the operator (2.3) and multiplication by  $p\omega p$  with  $\omega \in \Lambda^1(X, \text{End } E, \mathbb{Z})$ .

**Lemma 2.2.** *For a noncommutative connection  $\nabla$  one has*

$$d \tau_E(A) = \tau_E([\nabla, A])$$

whenever  $A \in \Lambda(X, \text{End } E, \mathbb{Z})$  is such that  $pA = A = Ap$ .

*Proof.* 1. Since  $\tau_E$  is a graded trace, we see that the right-hand side of the equality does not depend on the choice of  $\nabla$ . Therefore, below we assume that  $\nabla$  is defined as in (2.3).

2. For a trivial bundle with the trivial connection  $\nabla = p \cdot d \cdot p$  we obtain

$$\tau([p \cdot d \cdot p, A]) = \tau(pd p A + p d A - d p A) = \tau(p d A) = \tau(d A) = d \tau(A).$$

(Here we use the identities  $(dA)p + A(dp) = dA$  and  $(dp)A + p d A = dA$ , which are obtained by differentiation of  $A p = A = p A$ .)

3. Let us realize  $E$  as a subbundle in the trivial bundle  $X \times \mathbb{C}^n$ . Then the left-hand side of the desired equality is equal to

$$d\tau_E(A) = d\tau(A),$$

whereas the right-hand side is equal to

$$\tau([p \nabla_E p, A]) = \tau([p \nabla_{\mathbb{C}^n} p, A]) = \tau([p \cdot d \cdot p, A]) = d\tau(A).$$

(In the trivial bundle  $X \times \mathbb{C}^n$  we choose the connection  $\nabla_{\mathbb{C}^n}$  equal to the direct sum of the connection in  $E$  and some connection in the orthogonal complement of  $E$ .) □

**Proposition 2.1.** *For any noncommutative connection  $\nabla$  the operator*

$$\nabla^2: \Lambda(X, E) \rightarrow \Lambda(X, E)$$

*is an operator of multiplication by a 2-form. This form is denoted by*

$$\Omega \in \Lambda^2(X, \text{End } E, \mathbb{Z}) \tag{2.4}$$

*and is called the curvature form of the noncommutative connection  $\nabla$ .*

*Proof.* Let us embed  $E$  as a subbundle of the trivial bundle  $X \times \mathbb{C}^n$ . Then the direct sum of a noncommutative connection for  $p$  and some noncommutative connection for  $1 - p$  is a noncommutative connection of the form  $\nabla_{\mathbb{C}^n} = d + \omega$ , where  $\omega$  is a matrix-valued noncommutative 1-form on  $X$ .

Given a section  $u = pu$ , a direct computation enables us to compute the curvature form

$$\begin{aligned} (p \nabla p)^2 u &= (p \nabla_{\mathbb{C}^n} p)^2 u \\ &= (p(d + \omega)p)^2 u \\ &= (p(d + \omega))^2 u \\ &= (pd p p - \omega dp + \omega p \omega + dp \omega + p d \omega) u. \end{aligned} \tag{□}$$

**Chern character**

**Definition 2.2.** The Chern character form of a projection

$$p \in C^\infty(X, \text{End } E, \mathbb{Z})$$

is the Haefliger form

$$\text{ch } p := \tau_E \left( p \exp \left( - \frac{\Omega}{2\pi i} \right) \right) \in \Lambda^{\text{ev}}(X/\mathbb{Z}),$$

where  $\Omega$  is the curvature form (2.4).

**Proposition 2.2.** The form  $\text{ch } p$  is closed and its Haefliger cohomology class does not depend on the choice of a noncommutative connection and is determined by the class of projection  $p$  in the group  $K_0(C^\infty(X, \text{End } E, \mathbb{Z}))$ .

**Remark 2.2.** Here the  $K_0$ -group of  $C^\infty(X, \text{End } E, \mathbb{Z})$  is by definition the Grothendieck group of homotopy classes of matrix projections with entries in this algebra.

**Remark 2.3.** Strictly speaking, to define the Chern character on the  $K$ -group, we need to consider arbitrary matrix projections over  $C^\infty(X, \text{End } E, \mathbb{Z})$ , while we considered only scalar projections. However, matrix projections can be considered as elements of the algebra  $C^\infty(X, \text{End}(E \otimes \mathbb{C}^k), \mathbb{Z})$ . Therefore, we do not consider the matrix case to avoid excessively complicated notation.

*Proof of Proposition 2.2.* 1. By Lemma 2.2, the form  $\text{ch } p$  is closed. Indeed, we have

$$d \tau_E(\Omega^k) = \tau_E([\nabla, \Omega^k]) = 0$$

since  $[\nabla, \nabla^{2k}] = 0$ .

2. Let us show that the cohomology class of  $\text{ch } p$  does not depend on the choice of the noncommutative connection  $\nabla$ . Let  $\nabla_0, \nabla_1$  be two noncommutative connections for  $p$ . Then their difference is an operator of multiplication by a noncommutative 1-form  $\alpha := \nabla_1 - \nabla_0$ . Consider the homotopy of noncommutative connections

$$\nabla_t = (1 - t)\nabla_0 + t\nabla_1.$$

Then we have  $\frac{d}{dt}\nabla_t = p(\nabla_1 - \nabla_0)p = p\alpha p$ . Hence,

$$\begin{aligned} \frac{d}{dt} \tau_E(\nabla_t^{2k}) &= k \tau_E \left( \left( \frac{d}{dt} \nabla_t^2 \right) \nabla_t^{2k-2} \right) \\ &= k \tau_E \left( \left[ \nabla_t, \frac{d}{dt} \nabla_t \right] \nabla_t^{2k-2} \right) \\ &= k \tau_E \left( \left[ \nabla_t, \left( \frac{d}{dt} \nabla_t \right) \nabla_t^{2k-2} \right] \right) \\ &= d \tau_E \left( k \left( \frac{d}{dt} \nabla_t \right) \nabla_t^{2k-2} \right). \end{aligned}$$



Integrating this expression over  $t \in [0, 1]$ , we obtain

$$\tau_E(\nabla_1^{2k}) - \tau_E(\nabla_0^{2k}) = d\omega', \tag{2.5}$$

where  $\omega'$  is some differential form. Eq. (2.5) means that the forms  $\text{ch } p$  defined in terms of  $\nabla_1$  and  $\nabla_0$  are cohomologous.

3. Let  $p_t, t \in [0, 1]$  be a smooth homotopy of projections connecting  $p_0$  to  $p_1$ . We want to show that the difference of the corresponding Chern character forms is an exact form. By item 2 of the present proof, it suffices to consider the case, where  $p$  acts in a trivial bundle with the trivial connection  $\nabla_E = d$ . In this case,  $\tau_E$  is a differential graded trace. Hence, homotopy invariance of the Chern form in Haefliger cohomology follows from the standard computations (e.g., see [27]).  $\square$

By this proposition, we obtain a well-defined mapping (Chern character)

$$K_0(C^\infty(X, \text{End } E, \mathbb{Z})) \xrightarrow{\text{ch}} H^{\text{ev}}(X/\mathbb{Z}).$$

**Example 2.2.** Let  $E = X \times \mathbb{C}^N$  be the trivial bundle and  $\nabla_E = d$ . Then the noncommutative connection is equal to  $\nabla_p = pdp$ , its curvature form is equal to  $\Omega_p = (\nabla_p)^2 = pdpdp$ . Hence, the Chern character form is given by the standard formula

$$\text{ch } p = \text{tr}[p \exp(-dpdp/2\pi i)]_0,$$

where  $\omega_0$  denotes the coefficient at  $T^0 = 1$  and  $\text{tr}$  is the matrix trace.

**2.2. Twisted Chern character (Chern character with coefficients in a vector bundle).** Given a diffeomorphism  $g: X \rightarrow X$  as above, we defined Chern character on the  $K$ -group  $K_0(C^\infty(X) \rtimes \mathbb{Z})$ . Suppose now, we are also given a  $g$ -bundle  $E \in \text{Vect}(X)$  (i.e., there is an extension of the diffeomorphism  $g: X \rightarrow X$  to a fiberwise isomorphism  $E \rightarrow E$ ). Then on the same  $K$ -group we can define Chern character twisted by  $E$ . To define this twisted Chern character, consider the algebra homomorphism

$$\beta: C^\infty(X) \rtimes \mathbb{Z} \rightarrow C^\infty(X, \text{End } E) \rtimes \mathbb{Z}, \quad \sum_k a(k)T^k \mapsto \sum_k (a(k) \otimes 1_E)\tilde{T}^k,$$

where  $\tilde{T}: C^\infty(X, E) \rightarrow C^\infty(X, E)$  is the shift operator (see (2.1)).

**Definition 2.3.** The *twisted Chern character* is the composition of mappings

$$\text{ch}_E: K_0(C^\infty(X) \rtimes \mathbb{Z}) \xrightarrow{\beta_*} K_0(C^\infty(X, \text{End } E) \rtimes \mathbb{Z}) \xrightarrow{\text{ch}} H^{\text{ev}}(X/\mathbb{Z}).$$

If the element of  $K$ -theory comes from the commutative subalgebra  $C^\infty(X) \subset C^\infty(X) \rtimes \mathbb{Z}$ , then the twisted Chern character is computed in terms of the classical Chern character as shown in the following proposition.

**Proposition 2.3.** *The composition of mappings*

$$K_0(C^\infty(X)) \rightarrow K_0(C^\infty(X) \rtimes \mathbb{Z}) \xrightarrow{\text{ch}_E} H^{\text{ev}}(X/\mathbb{Z})$$

is equal to  $[p] \mapsto (\text{ch Im } p)(\text{ch } E)$ , where  $\text{Im } p \in \text{Vect}(X)$  is the vector bundle defined by projection  $p$ , while  $\text{ch}$  is the Chern character of a vector bundle.

*Proof.* Take  $[p] \in K_0(C^\infty(X))$ . Then we have  $\text{ch}_E(p) = \text{ch}(\beta(p))$ . But  $\beta(p) = p \otimes 1_E$  is a matrix over the algebra  $C^\infty(X, \text{End } E)$ . Hence, the class  $\text{ch}(\beta(p))$  coincides with the classical Chern character of the vector bundle  $\text{Im}(p \otimes 1_E) \simeq \text{Im } p \otimes E$ . Hence, by the multiplicative property of the classical Chern character, we obtain the desired formula  $\text{ch}_E(p) = (\text{ch Im } p)(\text{ch } E)$ .  $\square$

### 2.3. The Chern character of an elliptic symbol

**Class of symbol in K-theory.** Let  $D$  be an elliptic operator of the form (1.1). To its symbol  $\sigma(D)$  we now assign an element in K-theory. To this end, we extend the diffeomorphism  $\partial g: S^*M \rightarrow S^*M$  to a diffeomorphism  $S^*M \times \mathbb{S}^1 \rightarrow S^*M \times \mathbb{S}^1$  that acts as identity along  $\mathbb{S}^1$ . This action defines a crossed product denoted by  $C^\infty(S^*M \times \mathbb{S}^1) \rtimes \mathbb{Z}$ . Consider the projection  $\mathcal{P} = \{\mathcal{P}(\varphi)\}$

$$\mathcal{P}(\varphi) = \begin{cases} \begin{pmatrix} I_N \cos^2 \varphi & \sigma(D) \sin \varphi \cos \varphi \\ \sigma(D)^{-1} \sin \varphi \cos \varphi & I_N \sin^2 \varphi \end{pmatrix} & \text{for } \varphi \in [0, \pi/2], \\ \begin{pmatrix} I_N \cos^2 \varphi & I_N \sin \varphi \cos \varphi \\ I_N \sin \varphi \cos \varphi & I_N \sin^2 \varphi \end{pmatrix} & \text{for } \varphi \in [\pi/2, 2\pi], \end{cases}$$

where  $\varphi$  is the coordinate on the circle. Note that the coefficients of  $\mathcal{P}$  are piecewise smooth functions of  $\varphi$ .

Let us define the element

$$[\sigma(D)] = [\mathcal{P}] \in K_0(C^\infty(S^*M \times \mathbb{S}^1) \rtimes \mathbb{Z}), \tag{2.6}$$

where  $[\mathcal{P}]$  is the equivalence class of the smoothed family of projections  $\{\mathcal{P}(\varphi)\}$  in a neighborhood of submanifolds  $\varphi = 0$  and  $\varphi = \pi/2$ .

**Chern character.** Given an elliptic symbol and a  $g$ -bundle  $E \in \text{Vect}(X)$ , it follows from the constructions of the previous subsection that we have the twisted Chern character

$$\text{ch}_E[\sigma(D)] \in H^{\text{ev}}(S^*M/\mathbb{Z}) \tag{2.7}$$

in Haefliger cohomology. In Section 3, we define a special twisting bundle that is useful for the index formula. Let us now obtain a property of the Chern character (2.7) that simplifies its computation.

Let  $t = \varphi/2\pi$  be the coordinate along the generator of the torus  $S^*M \times \mathbb{S}^1$ .

**Lemma 2.3.** *One has*

$$\text{ch}_E[\sigma(D)] = \text{tr}_E \exp\left(-\frac{\nabla_{\text{tor}}^2}{2\pi i}\right) \in H^*((S^*M \times \mathbb{S}^1)/\mathbb{Z}), \tag{2.8}$$

where the operator

$$\nabla_{\text{tor}} = dt \frac{\partial}{\partial t} + t\nabla + (1-t)\sigma^{-1}\nabla\sigma, \quad \sigma = \beta(\sigma(D)), \tag{2.9}$$

is defined by an arbitrary noncommutative connection  $\nabla$  in  $E$  and  $\nabla_{\text{tor}}^2$  is the curvature form.

*Proof.* We have  $[\sigma(D)] = [\mathcal{P}]$  (see (2.6)). Consider the isomorphism

$$U_\varphi : \text{Im } \mathcal{P}(0) \rightarrow \text{Im } \mathcal{P}(\varphi), \quad \varphi \in [0, 2\pi],$$

$$U_\varphi = \begin{cases} \begin{pmatrix} I_N \cos \varphi & \sigma(-\sin \varphi) \\ \sigma^{-1} \sin \varphi & I_N \cos \varphi \end{pmatrix} & \text{if } \varphi \in [0, \pi/2], \\ \begin{pmatrix} \cos(\varphi - \pi/2) & -\sin(\varphi - \pi/2) \\ \sin(\varphi - \pi/2) & \cos(\varphi - \pi/2) \end{pmatrix} \begin{pmatrix} 0 & -\sigma \\ \sigma^{-1} & 0 \end{pmatrix} & \text{if } \varphi \in [\pi/2, 2\pi]. \end{cases}$$

We use this isomorphism and operator (2.9) to define the noncommutative connection

$$\nabla' = (\mathcal{P}(\varphi)U_\varphi)\nabla_{\text{tor}}(U_\varphi^{-1}\mathcal{P}(\varphi))$$

for the projection  $\mathcal{P} = \{\mathcal{P}(\varphi)\}$  on the cylinder  $S^*M \times [0, 2\pi]$ . We claim that this expression defines a connection on the torus  $S^*M \times \mathbb{S}^1$ . To prove this, we need to check that the coefficients of the connection at  $\varphi = 0$  and  $\varphi = 2\pi$  are compatible. At  $\varphi = 0$  we have

$$\nabla'|_{\varphi=0} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \sigma^{-1}\nabla\sigma \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \sigma^{-1}\nabla\sigma & 0 \\ 0 & 0 \end{pmatrix},$$

while at  $\varphi = 2\pi$  we obtain

$$\nabla'|_{\varphi=2\pi} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sigma^{-1} & 0 \\ 0 & \sigma \end{pmatrix} \nabla \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \sigma^{-1}\nabla\sigma & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, we obtain the equality  $\nabla'|_{\varphi=0} = \nabla'|_{\varphi=2\pi}$ , i.e., the coefficients of the connection are compatible and therefore  $\nabla'$  is a well-defined connection on the torus  $S^*M \times \mathbb{S}^1$ .

The powers of the curvature form of this connection are equal to

$$(\nabla')^{2N} = (\mathcal{P}(\varphi)U_\varphi)\nabla_{\text{tor}}^{2N}(U_\varphi^{-1}\mathcal{P}(\varphi)), \quad N \geq 1.$$

Hence we have  $\tau_E(\nabla'^{2N}) = \tau_E(\nabla_{\text{tor}}^{2N})$ , i.e., we obtain the desired equality (2.8). □

### 3. The equation $\text{ch } y = \text{Td } x$

Let  $x$  be a complex vector bundle over some space  $Z$ . Consider the equation

$$\text{ch } y = \text{Td } x, \tag{3.1}$$

where  $\text{Td } x$  is the Todd class of  $x$ . Since the Chern character defines a rational isomorphism  $K^0(Z) \otimes \mathbb{Q} \simeq H^{\text{ev}}(Z) \otimes \mathbb{Q}$ , eq. (3.1) has a unique solution  $y \in K^0(Z) \otimes \mathbb{Q}$ , which we denote for brevity by  $\psi(x)$ . Moreover, the mapping  $x \mapsto \psi(x)$  defines the operation

$$\psi : K^0(Z) \otimes \mathbb{Q} \rightarrow K^0(Z) \otimes \mathbb{Q}$$

in K-theory with rational coefficients. This operation is multiplicative,

$$\psi(a + b) = \psi(a)\psi(b)$$

(this follows from the multiplicative property of the Todd class), and stable,

$$\psi(a + 1) = \psi(a).$$

By a theorem of Atiyah [5] any stable operation in K-theory is a formal power series in Grothendieck operations  $\gamma_j, j = 1, 2, \dots$ , with rational coefficients. In addition, any multiplicative operation is determined by a formal power series

$$f(x) = 1 + \sum_{k \geq 1} a_k x^k$$

as follows:

- 1) The infinite product  $f(x_1)f(x_2) \dots$  is represented as a symmetric formal power series in variables  $x_1, x_2, \dots$ . Hence, this product is expressed as a formal power series

$$\prod_j f(x_j) = P(\sigma_1, \sigma_2, \dots)$$

in terms of elementary symmetric functions  $\sigma_1(x_1, x_2, \dots), \sigma_2(x_1, x_2, \dots), \dots$

- 2) A multiplicative operation for  $f$  is obtained if we replace the elementary symmetric functions by Grothendieck operations

$$P(\gamma_1, \gamma_2, \dots).$$

For the operation  $\psi$ , the corresponding formal power series is computed in the following proposition.

**Proposition 3.1.** *The multiplicative operation  $\psi$  is defined by the series*

$$\psi(x) = \frac{\ln(1+x)}{x}(1+x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} x^n.$$

*Proof.* To compute the coefficients of the desired series  $f$ , let us take  $Z = \mathbb{C}\mathbb{P}^N$  as  $N \rightarrow \infty$ . We have  $K(\mathbb{C}\mathbb{P}^N) = \mathbb{Z}[x]/\{x^{N+1} = 0\}$ ,  $x = [\varepsilon] - [\mathbf{1}]$ , where  $\varepsilon$  is the tautological line bundle over the projective space and  $\mathbf{1}$  is the trivial line bundle. We have  $\text{ch } x = e^u - 1$ , where  $u = [\mathbb{C}\mathbb{P}^1] \subset H^2(\mathbb{C}\mathbb{P}^N) = \mathbb{Z}$  is the generator.

Let us use the method of undetermined coefficients. Let

$$\psi(x) = \sum_{k \geq 0} c_k x^k, \quad c_0 = 1.$$

Then the equation  $\text{ch } \psi(x) = \text{Td } x$  is written as

$$\sum_k c_k (e^u - 1)^k = \frac{u}{1 - e^{-u}}.$$

Changing the variable by the rule  $e^u - 1 = t$ , this gives the desired function

$$\psi(t) = (1 + t) \frac{\ln(1 + t)}{t}. \quad \square$$

Note that Grothendieck operations can be expressed in terms of operations of direct sum, tensor product and exterior powers. Therefore, if  $E$  is a  $g$ -bundle, then  $\psi(E)$  (as a virtual bundle with rational coefficients) can also be considered as a  $g$ -bundle. A direct computation gives the following explicit expressions for the operation  $\psi$  on spaces  $Z$  of small dimension.

**Proposition 3.2.** *The operation  $\psi$  is equal to (here  $n = \text{rk } E$ )*

$$\begin{aligned} \psi(E) &= 1 + \frac{E - n}{2} \quad \text{if } \dim Z \leq 3; \\ \psi(E) &= \frac{3n^2 - 19n + 24}{24} + \frac{(-3n + 13)}{12} E - \frac{1}{6} E \otimes E + \frac{7}{12} \Lambda^2 E \quad \text{if } \dim Z \leq 5. \end{aligned}$$

*Proof.* The series  $\psi(x)$  defines the symmetric formal power series

$$\prod_j (1 + x_j) \prod_j \frac{\ln(1 + x_j)}{x_j} \equiv AB, \quad A = 1 + \sigma_1 + \sigma_2 + \dots$$

Let us express the term  $B$  in terms of elementary symmetric functions. We have

$$\begin{aligned} B &= \prod \left( 1 - \frac{x_j}{2} + \frac{x_j^2}{3} - \frac{x_j^3}{4} + \dots \right) \\ &= 1 - \frac{1}{2} \sum x_j + \frac{1}{3} \sum x_j^2 + \frac{1}{4} \sum_{i < j} x_i x_j - \frac{1}{4} \sum x_j^3 \\ &\quad - \frac{1}{6} \sum_{i \neq j} x_i x_j^2 - \frac{1}{8} \sum_{i < j < k} x_i x_j x_k + \dots, \end{aligned}$$

where dots stand for terms of orders  $\geq 4$ . Let  $p_k = \sum_j x_j^k$  and continue the computation

$$\begin{aligned} B &= 1 - \frac{p_1}{2} + \frac{p_2}{3} + \frac{\sigma_2}{4} - \frac{p_3}{12} - \frac{p_1 p_2}{6} - \frac{\sigma_3}{8} + \dots \\ &= 1 - \frac{\sigma_1}{2} + \frac{\sigma_1^2 - 2\sigma_2}{3} + \frac{\sigma_2}{4} - \frac{\sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3}{12} - \frac{\sigma_1(\sigma_1^2 - 2\sigma_2)}{6} - \frac{\sigma_3}{8} + \dots \\ &= 1 - \frac{\sigma_1}{2} + \frac{4\sigma_1^2 - 5\sigma_2}{12} + \frac{-6\sigma_1^3 + 14\sigma_1\sigma_2 - 9\sigma_3}{24} + \dots \end{aligned}$$

Here we used Newton’s formulas

$$p_1 = \sigma_1, \quad p_2 = \sigma_1^2 - 2\sigma_2, \quad p_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3.$$

This implies the following expression for  $\psi$  in terms of Grothendieck operations:

$$\begin{aligned} \psi &= (1 + \gamma_1 + \gamma_2 + \gamma_3 + \dots) \\ &\quad \left( 1 - \frac{\gamma_1}{2} + \frac{4\gamma_1^2 - 5\gamma_2}{12} + \frac{-6\gamma_1^3 + 14\gamma_1\gamma_2 - 9\gamma_3}{24} + \dots \right) \tag{3.2} \\ &= 1 + \frac{\gamma_1}{2} + \frac{-2\gamma_1^2 + 7\gamma_2}{12} + \frac{2\gamma_1^3 - 8\gamma_1\gamma_2 + 15\gamma_3}{24} + \dots \end{aligned}$$

The operations  $\gamma_j$  can be expressed in terms of exterior powers (see [5])

$$\begin{aligned} \gamma_t &\equiv \sum \gamma_j t^j \\ &= \sum_{k \geq 0} \frac{t^k}{(1-t)^k} \Lambda^k (1-t)^n \\ &= (1-t)^n \left( 1 + \frac{t}{1-t} \Lambda^1 + \frac{t^2}{(1-t)^2} \Lambda^2 + \frac{t^3}{(1-t)^3} \Lambda^3 + \dots \right) \\ &= (1-t)^n (1 + t\Lambda^1 + t^2(\Lambda^1 + \Lambda^2) + t^3(\Lambda^1 + 2\Lambda^2 + \Lambda^3) + \dots) \\ &= 1 + t(\Lambda^1 - n) + t^2 \left( \Lambda^1 + \Lambda^2 - n\Lambda^1 + \frac{n(n-1)}{2} \right) + t^3 \left( \Lambda^1 + 2\Lambda^2 + \Lambda^3 \right. \\ &\quad \left. - n\Lambda^1 - n\Lambda^2 + \frac{n(n-1)}{2} \Lambda^1 - \frac{n(n-1)(n-2)}{6} \right) + \dots \end{aligned}$$

Substituting these expressions in (3.2), we obtain the following formula

$$\begin{aligned} \psi &= 1 + \frac{\Lambda^1 - n}{2} + \frac{-2(\Lambda^1 - n)^2 + 7(\Lambda^1 + \Lambda^2 - n\Lambda^1 + n(n-1)/2)}{12} \\ &\quad + \frac{2(\Lambda^1 - n)^3 - 8(\Lambda^1 - n)(\Lambda^1 + \Lambda^2 - n\Lambda^1 + n(n-1)/2)}{24} \\ &\quad + \frac{15(\Lambda^1 + 2\Lambda^2 + \Lambda^3 - n\Lambda^1 - n\Lambda^2 + \frac{n(n-1)\Lambda^1}{2} - \frac{n(n-1)(n-2)}{6})}{24} + \dots \tag{3.3} \end{aligned}$$

In Borel–Hirzebruch formalism [12] we have  $\text{ch} = \sum e^{x_j}$ . Hence

$$\text{ch } \gamma_t = \prod (1 + t(e^{x_j} - 1)).$$

It follows from this expression that for a vector bundle  $E \in \text{Vect}(Z)$  we get  $\text{ch } \gamma_k(E) \in H^{\geq 2k}(Z)$ . Now consider the class  $\text{ch } \gamma_K(E)$ , where  $K = (k_1, \dots, k_p)$  is a multi-index, and  $\gamma_K(E) = \gamma_{k_1}(E) \dots \gamma_{k_p}(E)$ . Then this expression is identically zero whenever  $2|K| = 2 \sum k_j > \dim Z$ . This implies that the terms denoted by dots in eq. (3.3) are actually equal to zero provided that  $\dim Z \leq 7$ . Using this remark, we obtain the following expressions for the operation  $\psi$ :

$$\dim Z \leq 3:$$

$$\psi(E) = 1 + (E - n)/2.$$

$$\dim Z \leq 5:$$

$$\begin{aligned} \psi(E) &= 1 + \frac{E - n}{2} + \frac{-2(E^2 - 2nE + n^2) + 7(E + \Lambda^2 E - nE + n(n - 1)/2)}{12} \\ &= \frac{3n^2 - 19n + 24}{24} + \frac{(-3n + 13)}{12} E - \frac{1}{6} E \otimes E + \frac{7}{12} \Lambda^2 E, \end{aligned}$$

and so on. □

#### 4. Index theorem

The complexification of the cotangent bundle will be denoted by  $T_{\mathbb{C}}^*M = T^*M \otimes \mathbb{C}$ .

**Theorem 4.1.** *Let  $D$  be an elliptic operator. Then its index is equal to*

$$\text{ind } D = \int_{S^*M \times \mathbb{S}^1} \text{ch}_{\psi(T_{\mathbb{C}}^*M)}[\sigma(D)], \tag{4.1}$$

where the operation  $\psi$  was defined in Proposition 3.1 and  $\text{ch}$  stands for the twisted Chern character (see Definition 2.3).

The right-hand side of eq. (4.1) will be referred to as the *topological index* of  $D$  and denoted by  $\text{ind}_{\text{top}} D$ .

**Proposition 4.1.** *For an elliptic  $\psi$ DO  $D$ , the topological index is equal to the topological index of Atiyah and Singer (see [8]).*

*Proof.* In our situation we have

$$\text{ch}_{\psi(T_{\mathbb{C}}^*M)}[\sigma(D)] = \text{ch}[\sigma(D)] \text{ch } \psi(T_{\mathbb{C}}^*M) = \text{ch}[\sigma(D)] \text{Td}(T_{\mathbb{C}}^*M).$$

Here the first equality follows from Proposition 2.3 and the fact that  $D$  is a  $\psi$ DO. The second equality follows from the definition of  $\psi$ . These equalities show that the topological index in this case is equal to

$$\text{ind}_{\text{top}} D = \int_{S^*M \times \mathbb{S}^1} \text{ch}[\sigma(D)] \text{Td}(T_{\mathbb{C}}^*M).$$

The last expression is actually the Atiyah–Singer index formula for the index of a pseudodifferential operator  $D$ .  $\square$

Theorem 4.1 will be proved in subsequent sections. Here we give the scheme of the proof.

- (1) First in Section 5 we give a reduction of the operator (1.1) to some special operator, whose index is equal to the index of a certain elliptic  $\psi$ DO on a special smooth closed manifold: the mapping torus of the diffeomorphism  $g : M \rightarrow M$ .
- (2) Then we prove in Section 6 that the index of this  $\psi$ DO on the torus (computed using the Atiyah–Singer index formula) is equal to the topological index of the operator (1.1) on  $M$ .

## 5. Reduction of a noncommutative operator to a $\psi$ DO on a closed manifold

**5.1. Reduction to a special operator.** In this subsection we obtain a reduction (stable homotopy) of the operator (1.1) to an operator of the same type, but of a simpler form.

Given matrix projections

$$p, q \in C^\infty(S^*M, \text{Mat}_N(\mathbb{C})), \quad p^2 = p, q^2 = q,$$

over  $S^*M$ , we choose some  $\psi$ DOs with symbols equal to  $p$  and  $q$  and denote them by  $P$  and  $Q$ .

**Definition 5.1.** A *special operator* is an operator of the form

$$D = QD_0TP + (1 - Q)D_1(1 - P) : H^s(M, \mathbb{C}^N) \rightarrow H^s(M, \mathbb{C}^N), \quad (5.1)$$

where  $H^s$  is the Sobolev space, while

$$D_0 : H^s(M, \mathbb{C}^N) \rightarrow H^s(M, \mathbb{C}^N), \quad D_1 : H^s(M, \mathbb{C}^N) \rightarrow H^s(M, \mathbb{C}^N),$$

are  $\psi$ DOs of order zero such that their symbols define vector bundle isomorphisms

$$\sigma(D_0) : (\partial g)^* \text{Im } p \rightarrow \text{Im } q, \quad \sigma(D_1) : \text{Im}(1 - p) \rightarrow \text{Im}(1 - q) \quad (5.2)$$

over  $S^*M$ . These vector bundles are defined as the ranges of projections  $p, q, 1 - p, 1 - q$ .



A special operator is elliptic in the sense of Definition 1.2. Moreover, an almost-inverse operator can be defined by the formula

$$D^{-1} = PT^{-1}D_0^{-1}Q + (1 - P)D_1^{-1}(1 - Q).$$

A homotopy of elliptic operators is a family  $\{D_t\}$ ,  $t \in [0, 1]$ , of elliptic operators such that the families of their coefficients are piecewise smooth functions of the parameter  $t$  and the number of nonzero components of the family and its almost inverse family are uniformly bounded. Two elliptic operators are *stably homotopic* if there exists a homotopy between their direct sums with identity operators acting in sections of some bundles.

**Proposition 5.1** (cf. [3], [31]). *The following statements hold.*

- (1) *An arbitrary elliptic operator is stably homotopic to some special operator.*
- (2) *An arbitrary special operator can be reduced by a stable homotopy and direct sum with operator  $T \oplus T \oplus \dots \oplus T$  to a direct sum of an elliptic  $\psi$ DO and a special operator of the form*

$$D = PD_0TP + (1 - P): H^s(M, \mathbb{C}^N) \rightarrow H^s(M, \mathbb{C}^N), \tag{5.3}$$

*i.e., in (5.1) one can suppose that  $Q = P$  and  $D_1 = 1$ .*

*Proof.* (1) Indeed, a direct computation shows that the homotopy defined in the paper [39] gives the desired result, i.e., the homotopy preserves ellipticity, and we obtain a special operator at the end of the homotopy.

(2) By (5.2), we have the vector bundle isomorphism

$$\text{Im}(1 - p) \simeq \text{Im}(1 - q).$$

This implies that  $[\text{Im } p] = [\text{Im } q] \in K(S^*M)$ . If the ranks of the projections are large enough (this can be achieved by taking a direct sum of the special operator and some operator of the form  $T \oplus T \oplus \dots \oplus T$ ), then there exists a vector bundle isomorphism  $a: \text{Im } q \rightarrow \text{Im } p$ . Consider an elliptic  $\psi$ DO  $D'$  with the symbol

$$\sigma(D') = a \oplus (\sigma(D_1))^{-1}: \text{Im } q \oplus \text{Im}(1 - q) \rightarrow \text{Im } p \oplus \text{Im}(1 - p).$$

Then we obtain the factorization

$$D = D_0TP + D_1(1 - P) = (D')^{-1}(D'D_0TP + (1 - P))$$

modulo compact operators. This proves the proposition since a composition of operators is stably homotopic to their direct sum. □

**5.2. Reduction to a boundary value problem.** Let us consider the elliptic special operator

$$D = D_0TP + (1_N - P): C^\infty(M, \mathbb{C}^N) \rightarrow C^\infty(M, \mathbb{C}^N). \tag{5.4}$$

Recall that the ellipticity condition in this case means that the symbol of  $D_0$  defines an isomorphism

$$\sigma(D_0): (\partial g)^* \text{Im } \sigma(P) \rightarrow \text{Im } \sigma(P) \tag{5.5}$$

of vector bundles over  $S^*M$ . Here the vector bundles are defined by the symbol of  $P$ .

On the cylinder  $M \times [0, 1]$  with coordinates  $x$  and  $t$  consider the boundary value problem (see [32], [33])

$$\begin{cases} \left( \frac{\partial}{\partial t} + (2P - 1_N)\sqrt{\Delta_M} \right) u = f_1, & u \in H^s(M \times [0, 1], \mathbb{C}^N), \\ D_0TPu|_{t=0} - u|_{t=1} = f_2, & f_1 \in H^{s-1}(M \times [0, 1], \mathbb{C}^N), \\ & f_2 \in H^{s-1/2}(M, \mathbb{C}^N). \end{cases} \tag{5.6}$$

Here  $\Delta_M$  is the Laplace operator defined by a metric on  $M$ . This boundary value problem, denoted for brevity by  $(\mathcal{D}, B)$ , is elliptic and one has (see op. cit.)

$$\text{ind } D = \text{ind}(\mathcal{D}, B). \tag{5.7}$$

**5.3. Homotopy of the boundary condition.** Methods of the theory of boundary value problems (e.g., see [23], [40]) enable one to simplify the boundary operator in eq. (5.6) using homotopies of elliptic boundary value problems. Namely, we start with the *rotation homotopy*

$$P(\varphi) = \begin{pmatrix} (\cos^2 \varphi)P & (\cos \varphi \sin \varphi)g^{-1*}(D_0^{-1}P) \\ (\cos \varphi \sin \varphi)g^{-1*}(PD_0) & (\sin^2 \varphi)g^{-1*}(P) \end{pmatrix},$$

$\varphi \in [0, \pi/2]$ , connecting the almost projections  $P \oplus 0$  and  $0 \oplus g^{-1*}(P)$ . For all  $\varphi \in [0, \pi/2]$  the operator  $P(\varphi)$  is an almost-projection, i.e., its symbol is a projection. To check this property, it is useful to represent this homotopy in the form

$$P(\varphi) = U_\varphi \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} U_\varphi^{-1} \tag{5.8}$$

in terms of the family of almost-invertible operators

$$U_\varphi = \begin{pmatrix} (\cos \varphi)P & (-\sin \varphi)g^{-1*}(D_0^{-1}P) \\ (\sin \varphi)g^{-1*}(PD_0) & (\cos \varphi)g^{-1*}(P) \end{pmatrix} + \begin{pmatrix} 1_N - P & 0 \\ 0 & 1_N - g^{-1*}(P) \end{pmatrix}.$$

Here we have an equality  $U_\varphi^{-1} = U_{-\varphi}$  modulo compact operators.

Then we define the homotopy of operators

$$D(\varphi) = \begin{pmatrix} (\cos \varphi)D_0g^*(P) & (\sin \varphi)P \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} D_0g^*(P) & 0 \\ 0 & 0 \end{pmatrix} g^*(U_\varphi^{-1}).$$

Finally, we define the homotopy of boundary value problems on the cylinder

$$\begin{cases} \left(\frac{\partial}{\partial t} + [2P(\varphi\chi(t)) - 1_{2N}]\sqrt{\Delta_M}\right)U = F_1, \\ D(\varphi)TU|_{t=0} - U|_{t=1} = F_2 \end{cases} \tag{5.9}$$

and denote this homotopy by  $(\mathcal{D}_\varphi, B_\varphi)$ . Here the unknown function and the right-hand side belong to the spaces  $U \in H^s(M \times [0, 1], \mathbb{C}^{2N})$ ,  $F_1 \in H^{s-1}(M \times [0, 1], \mathbb{C}^{2N})$ ,  $F_2 \in H^{s-1/2}(M, \mathbb{C}^{2N})$ , and  $\chi(t)$  is a smooth nonincreasing function equal to 1 if  $t \leq 1/3$  and equal to 0 if  $t \geq 2/3$ .

**Lemma 5.1.** *The homotopy (5.9) consists of elliptic boundary value problems.*

*Proof* (cf. [33]). The boundary condition in (5.9) relates the values of  $U$  at  $t = 0$  and  $t = 1$ , i.e., it is a nonlocal condition. Nonlocal boundary value problems of this type were considered in [36]. Let us show that this problem is elliptic in the sense of the cited paper. Indeed, let us reduce the nonlocal problem to a local problem in a neighborhood of the boundary of the cylinder. To this end we introduce new unknown functions  $V$  and  $W$ :

$$V(t) = TU(t), \quad W(t) = U(1 - t).$$

In a neighborhood of the boundary the system (5.9) is written in the following equivalent form

$$\begin{cases} T\left(\frac{\partial}{\partial t} + [2P(\varphi\chi(t)) - 1_{2N}]\sqrt{\Delta_M}\right)T^{-1}V(t) = TF_1(t), \\ \left(-\frac{\partial}{\partial t} + [2P(0) - 1_{2N}]\sqrt{\Delta_M}\right)W(t) = F_1(1 - t), \\ D(\varphi)V|_{t=0} - W|_{t=0} = F_2, \end{cases} \quad 0 \leq t < 1/2. \tag{5.10}$$

Note that the system (5.10) is already local. The first two equations of the system are elliptic. Let us show that the boundary value problem (5.10) is elliptic, i.e., it satisfies the Shapiro–Lopatinskii condition (e.g., see [23]). To prove this, we consider the Calderón bundle [23] (see also [36])

$$L_+ \subset S^*M \times \mathbb{C}^{4N}$$

of the main operator in (5.10). A direct computation shows that this bundle is equal to

$$\begin{aligned} L_+ &= \text{Im}[(\partial g)^*\sigma(P(\varphi))] \oplus \text{Im}[1_{2N} - \sigma(P(0))] \\ &= \text{Im}[(\partial g)^*\sigma(U_\varphi P(0))] \oplus \text{Im}[1_{2N} - \sigma(P(0))]. \end{aligned} \tag{5.11}$$

Here the second equality follows from (5.8).

The Shapiro–Lopatinskii condition for the problem (5.10) is equivalent to the requirement that the symbol of the boundary operator, i.e., the vector bundle homomorphism

$$L_+ \rightarrow S^*M \times \mathbb{C}^{2N}, \quad (V, W) \mapsto \sigma(D(\varphi))V - W, \tag{5.12}$$

is an isomorphism. This requirement is satisfied in our case, since (5.11) implies that

$$W \in \text{Im}[1_{2N} - \sigma(P(0))],$$

and the mapping

$$\sigma(D(\varphi)) = \sigma \left[ \begin{pmatrix} D_0 g^*(P) & 0 \\ 0 & 0 \end{pmatrix} g^*(U_\varphi^{-1}) \right] : \text{Im}[(\partial g)^* \sigma(U_\varphi P(0))] \rightarrow \text{Im}[\sigma(P(0))]$$

is an isomorphism of vector bundles by (5.8) and (5.5).

So the Shapiro–Lopatinskii condition is satisfied and the problem (5.10) is elliptic. Hence, (5.9) defines a Fredholm operator.  $\square$

It follows from Lemma 5.1 and eq. (5.7) that

$$\text{ind } D = \text{ind}(\mathcal{D}_0, B_0) = \text{ind}(\mathcal{D}_{\pi/2}, B_{\pi/2}).$$

Let us now consider the boundary value problem

$$\begin{cases} \left( \frac{\partial}{\partial t} + [2P(\frac{\pi}{2}\chi(t)) - 1_{2N}] \sqrt{\Delta_{M,h(t)}} \right) U = F_1, \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} T U|_{t=0} - U|_{t=1} = F_2, \end{cases} \tag{5.13}$$

where  $\Delta_{M,h(t)}$  is the Laplace operator on  $M$  for a family of metrics  $h(t)$  smoothly depending on  $t$ .

**Lemma 5.2.** *The problem (5.13) is elliptic and its index is equal to the index of the problem  $(\mathcal{D}_{\pi/2}, B_{\pi/2})$ .*

*Proof.* It suffices to show that the linear homotopy connecting these two boundary value problems preserves ellipticity.

First, the linear homotopy between the main operators of (5.13) and (5.9) consists of elliptic operators. This follows from the fact that the operators differ only by metrics defining the Laplace operators.

Second, for the linear homotopy between the problems (5.13) and (5.9), the Calderón bundle  $L_+$  is constant. Moreover, for the boundary value problems in this homotopy the corresponding family of vector bundle homomorphisms (5.12) also does not change. Hence the linear homotopy consists of elliptic problems.  $\square$

**5.4. Reduction to a  $\psi$ DO on the torus.** Let  $M \times_g \mathbb{S}^1$  be the *torus of the diffeomorphism*  $g$ . Recall that the torus of a diffeomorphism is a closed smooth manifold obtained from the cylinder  $M \times [0, 1]$  by identifying its bases with a “twist” defined by  $g$ :

$$M \times_g \mathbb{S}^1 = M \times [0, 1] / \{(x, 0) \sim (g^{-1}(x), 1)\}.$$

Consider a family of metrics  $h(t)$  in (5.13) such that

$$h(t) = \begin{cases} h & \text{if } t < 1/3, \\ g^*h & \text{if } t > 2/3, \end{cases}$$

where  $h$  is some fixed metric. This family is a smooth family of metrics in the fibers of the bundle  $M \times_g \mathbb{S}^1$ .

The problem (5.13) defined by this family of metrics is denoted by  $(\mathcal{D}', \mathcal{B}')$ . The operator  $\mathcal{D}'$  defines an elliptic  $\psi$ DO on the torus  $M \times_g \mathbb{S}^1$ :

$$\mathcal{D}'_0 = \frac{\partial}{\partial t} + [2P(\frac{\pi}{2}\chi(t)) - 1_{2N}] \sqrt{\Delta_{M,h(t)}} : C^\infty(M \times_g \mathbb{S}^1, \mathcal{E}) \rightarrow C^\infty(M \times_g \mathbb{S}^1, \mathcal{E}),$$

where  $\mathcal{E} \in \text{Vect}(M \times_g \mathbb{S}^1)$  stands for the vector bundle with the total space

$$\mathcal{E} = (M \times [0, 1] \times \mathbb{C}^{2N}) / \{(x, 0, v_1, v_2) \sim (g^{-1}(x), 1, v_2, v_1)\}. \tag{5.14}$$

**Proposition 5.2.** *One has  $\text{ind}(\mathcal{D}', \mathcal{B}') = \text{ind } \mathcal{D}'_0$ .*

*Proof.* 1. Since the operator of boundary condition in  $(\mathcal{D}', \mathcal{B}')$  is surjective, we see that the index of the boundary value problem is equal to the index of the same boundary value problem but with homogeneous boundary condition. This condition has the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} TU|_{t=0} - U|_{t=1} = 0,$$

i.e., it coincides with the continuity condition of  $U(t)$  considered as a section of the bundle  $\mathcal{E}$  over the torus  $M \times_g \mathbb{S}^1$ .

2. Since the boundary condition is actually the continuity condition, the remaining part of the proof is standard and we omit it (e.g., see [33]). □

## 6. Comparison of topological indices

### 6.1. Computation of the index of $\psi$ DO on the torus using Atiyah–Singer formula.

The operator  $\mathcal{D}'_0$  is an operator of the form

$$\mathcal{D}'_0 = \frac{\partial}{\partial t} + A(t),$$

i.e., it is defined by a family  $\mathcal{A} = \{A(t)\}$ ,  $t \in [0, 1]$ , of operators on the sections  $M \times \{t\}$  of the torus. Moreover, the family consists of elliptic operators and the corresponding family of symbols

$$\sigma(A(t))(x, \xi) = [2\sigma(P(t))(x, \xi) - 1]|\xi|_t$$

(here  $|\xi|_t$  is the norm of a covector with respect to a family of metrics  $h(t)$ ) has a real spectrum at each point  $(x, \xi)$ . It follows that the positive spectral subspace of the symbol  $\sigma(A(t))(x, \xi)$  (by definition this subspace is generated by the eigenvectors with positive eigenvalues) is just the space  $\text{Im } \sigma(P(t))(x, \xi)$ . Hence, the family of positive spectral subspaces defines a smooth vector bundle over the torus  $S^*M \times_g \mathbb{S}^1$ . Denote this vector bundle by

$$\sigma_+(\mathcal{A}) \in \text{Vect}(S^*M \times_g \mathbb{S}^1). \tag{6.1}$$

The following lemma (cf. Theorem 7.4 in [7]) expresses the index of operator  $\mathcal{D}'_0$  in terms of the bundle (6.1).

**Lemma 6.1.** *One has*

$$\text{ind } \mathcal{D}'_0 = \int_{S^*M \times_g \mathbb{S}^1} \text{ch}[\sigma_+(\mathcal{A})] \text{Td}(T_{\mathbb{C}}^*M \times_g \mathbb{S}^1).$$

*Proof.* To make the paper self-contained, we give the proof of this fact.

1. The Atiyah–Singer index formula for  $\mathcal{D}'_0$  has the form

$$\text{ind } \mathcal{D}'_0 = \int_{T(M \times_g \mathbb{S}^1)} \text{ch}[\sigma(\mathcal{D}'_0)] \text{Td}(T_{\mathbb{C}}(M \times_g \mathbb{S}^1)). \tag{6.2}$$

Here and below in the proof we identify the tangent and cotangent bundles using some metric.

2. We have the decomposition  $T(M \times_g \mathbb{S}^1) = (TM \times_g \mathbb{S}^1) \oplus \mathbf{1}$  into directions perpendicular and parallel to the generator of the torus. Thus, we get

$$\text{Td}(T_{\mathbb{C}}(M \times_g \mathbb{S}^1)) = \text{Td}(T_{\mathbb{C}}M \times_g \mathbb{S}^1).$$

3. Denote the composition of embeddings

$$SM \times_g \mathbb{S}^1 \subset TM \times_g \mathbb{S}^1 \subset T(M \times_g \mathbb{S}^1)$$

by  $i$ . The normal bundle of this embedding is, obviously, a direct sum of two one-dimensional trivial bundles. Moreover, one has

$$[\sigma(\mathcal{D}'_0)] = i_![\sigma_+(\mathcal{A})] \in K^0(T(M \times_g \mathbb{S}^1)), \tag{6.3}$$

where

$$i_!: K^0(SM \times_g \mathbb{S}^1) \rightarrow K^0(T(M \times_g \mathbb{S}^1))$$

is the direct image mapping corresponding to the embedding  $i$  (see [5]). Applying the Riemann–Roch–Atiyah–Hirzebruch formula [6] to (6.3), we obtain

$$\text{ch}[\sigma(\mathcal{D}'_0)] = \text{ch } i_![\sigma_+(\mathcal{A})] = i_* \text{ch}[\sigma_+(\mathcal{A})].$$

(The normal bundle of  $i$  is trivial, hence the Todd class is equal to one.)

4. Substituting the formulas obtained in items 2 and 3 of the proof in eq. (6.2), we obtain

$$\text{ind } \mathcal{D}'_0 = p_*(i_*(\text{ch } \sigma_+(\mathcal{A}) \text{Td}(T_{\mathbb{C}}M \times_g \mathbb{S}^1))) = p_{0*}(\text{ch } \sigma_+(\mathcal{A}) \text{Td}(T_{\mathbb{C}}M \times_g \mathbb{S}^1)),$$

where  $p_*$  and  $p_{0*}$  stand for Gysin maps in cohomology (integration over the fundamental cycle), induced by the projections  $p: T(M \times_g \mathbb{S}^1) \rightarrow pt$  and  $p_0: SM \times_g \mathbb{S}^1 \rightarrow pt$ .

The proof of the lemma is complete. □

On the cylinder consider the vector bundle  $\text{Im } \sigma(P) \in \text{Vect}(S^*M \times [0, 1])$  and identify the fibers of this bundle over the components of the boundary using the mapping  $\sigma(D_0)$  (see (5.4)) as follows:

$$\mathcal{V} = \{(x, \xi, t, v) \mid v \in \text{Im } \sigma(P)(x, \xi)\} / \{(x, \xi, 0, v) \sim ((\partial g)^{-1}(x, \xi), 1, \sigma(D_0)((\partial g)^{-1}(x, \xi))v)\}. \tag{6.4}$$

This space is a vector bundle  $\mathcal{V} \in \text{Vect}(S^*M \times_g \mathbb{S}^1)$  over the torus  $S^*M \times_g \mathbb{S}^1$ .

**Lemma 6.2.** *One has an isomorphism of vector bundles over  $S^*M \times_g \mathbb{S}^1$ :*

$$\sigma_+(\mathcal{A}) \simeq \mathcal{V}. \tag{6.5}$$

*Proof.* The pull-backs of the bundles  $\sigma_+(\mathcal{A})$  and  $\mathcal{V}$  to the cylinder  $S^*M \times [0, 1]$  are equal to

$$\text{Im } p\left(\frac{\pi}{2}\chi(t)\right) \quad \text{and} \quad \text{Im } p(0),$$

where  $p(\varphi) = \sigma(P(\varphi))$ . The formula

$$u_{\frac{\pi}{2}\chi(t)}^{-1}: \text{Im } p\left(\frac{\pi}{2}\chi(t)\right) \rightarrow \text{Im } p(0), \tag{6.6}$$

where  $u_\varphi = \sigma(U_\varphi)$ , defines a vector bundle isomorphism on the cylinder. This mapping is well defined by eq. (5.8).

Let us verify that the isomorphism (6.6) of vector bundles over the cylinder extends by continuity to an isomorphism of bundles on the torus. To prove this, it suffices to show that the diagram

$$\begin{array}{ccc} \text{Im } p(\pi/2) & \xrightarrow{u_{\pi/2}^{-1}} & \text{Im } p(0) \\ \downarrow & & \downarrow \\ \text{Im } p(0) & \xrightarrow{u_0^{-1}} & \text{Im } p(0) \end{array} \tag{6.7}$$

is commutative. Here the horizontal mappings are just the restrictions of the isomorphism  $u_{\frac{\pi}{2}\chi(t)}^{-1}$  at  $t = 0$  (upper row) and at  $t = 1$  (lower row), while the vertical mappings are just identifications of vector bundles on the boundary of the cylinder. Recall that these identification mappings are defined in (5.14) and (6.4) and give bundles on the torus.

Let us prove that (6.7) is a commutative diagram. We have

$$u_0 = 1_{2N}, \quad u_{\pi/2}^{-1} = \sigma \left( \begin{array}{cc} 1_N - P & g^{-1*}(D_0^{-1}P) \\ -g^{-1}(PD_0) & 1_N - g^{-1*}(P) \end{array} \right),$$

$$p(0) = \sigma \left( \begin{array}{cc} P & 0 \\ 0 & 0 \end{array} \right), \quad p(\pi/2) = \sigma \left( \begin{array}{cc} 0 & 0 \\ 0 & g^{-1*}(P) \end{array} \right).$$

Suppose that  $(x, \xi, 0, v) \in \text{Im } p(\pi/2)$ . This means that  $(x, \xi) \in S^*M$  and  $v \in \text{Im } p((\partial g)^{-1}(x, \xi))$ . Then, passing the diagram (6.7) through the lower left corner, we obtain the elements

$$\begin{array}{ccc} (x, \xi, 0, v) & & \\ \downarrow & & \\ ((\partial g)^{-1}(x, \xi), v, 0) & \longrightarrow & (((\partial g)^{-1}(x, \xi), v). \end{array} \tag{6.8}$$

If we now pass the diagram through the right upper corner, we obtain the elements

$$\begin{array}{ccc} (x, \xi, 0, v) & \longrightarrow & (x, \xi, * \sigma(D_0)^{-1}((\partial g)^{-1}(x, \xi))v) \\ & & \downarrow \\ & & ((\partial g)^{-1}(x, \xi), v). \end{array} \tag{6.9}$$

Since the elements obtained in the right lower corner in (6.8) and (6.9) are equal, the diagram (6.7) is commutative. Hence, (6.6) defines an isomorphism on the cylinder, and this isomorphism defines the desired isomorphism (6.5) on the torus.  $\square$

**6.2. Comparison of the topological indices of the  $\psi$ DO on the torus and of the original operator.** Consider the equalities

$$\begin{aligned} \text{ind } D &= \text{ind } \mathcal{D}'_0 \\ &= \int_{S^*M \times_g \mathbb{S}^1} \text{ch}[\sigma_+(\mathcal{A})] \text{Td}(T_{\mathbb{C}}^*M \times_g \mathbb{S}^1) \\ &= \int_{S^*M \times_g \mathbb{S}^1} \text{ch } \mathcal{V} \text{Td}(T_{\mathbb{C}}^*M \times_g \mathbb{S}^1) \\ &= \int_{S^*M \times_g \mathbb{S}^1} \text{ch } \mathcal{V} \text{ch } \psi(T_{\mathbb{C}}^*M \times_g \mathbb{S}^1) \\ &= \int_{S^*M \times_g \mathbb{S}^1} \text{ch}(\mathcal{V} \otimes \psi(T_{\mathbb{C}}^*M \times_g \mathbb{S}^1)). \end{aligned} \tag{6.10}$$



Here the first equality follows from the results of Section 5. The second follows from Lemma 6.1 and the third follows from Lemma 6.2. The fourth equality is just the definition of operation  $\psi$ .

To complete the proof of the Index Theorem 4.1, it suffices to show that the topological index  $\text{ind}_{\text{top}} D$  of the operator (5.4) is equal to the right-hand side in (6.10). Note that, generally speaking, the element  $\psi(T_{\mathbb{C}}^*M \times_g \mathbb{S}^1) \in K(M \times_g \mathbb{S}^1) \otimes \mathbb{Q}$  is a virtual bundle, i.e., a linear combination of vector bundles with rational coefficients (see Section 3). To simplify the notation, we shall assume that  $\psi(T_{\mathbb{C}}^*M \times_g \mathbb{S}^1)$  is a vector bundle, thus omitting the corresponding sum and coefficients.

Let  $\mathcal{F} = \mathcal{V} \otimes \psi(T_{\mathbb{C}}^*M \times_g \mathbb{S}^1)$  for brevity. The sections of  $\mathcal{F}$  are just sections  $F = \{F(x, \xi, t)\}$  of the bundle  $\text{Im } p \otimes \psi(T_{\mathbb{C}}^*M)$  (here  $p = \sigma(P)$ ) on the cylinder  $S^*M \times [0, 1]$  such that

$$\sigma(D_0)(\partial g)^* F|_{t=0} = F|_{t=1}. \tag{6.11}$$

This statement is verified by a direct computation using eq. (6.4).

Let us compute the Chern character of  $\mathcal{F}$  using the formalism of connections. Let  $\nabla_p = p\nabla_\psi p$  be a connection in the bundle  $(\text{Im } p) \otimes \psi$ . Here  $\psi$  stands for the bundle  $\psi(T_{\mathbb{C}}^*M)$  for brevity, while  $\nabla_\psi$  is a connection in this bundle. Then the formula

$$\nabla'_{\text{tor}} = dt \frac{\partial}{\partial t} + t\nabla_p + (1-t)[\sigma(D_0)(\partial g)^*]^{-1} \nabla_p [\sigma(D_0)(\partial g)^*]$$

defines the connection in  $\mathcal{F}$ :

$$\nabla'_{\text{tor}} : \Lambda(S^*M \times_g \mathbb{S}^1, \mathcal{F}) \rightarrow \Lambda(S^*M \times_g \mathbb{S}^1, \mathcal{F}).$$

A direct computation using (6.11) shows that this connection is well defined. The Chern character form of  $\mathcal{F}$  is defined by the classical formula

$$\text{ch } \mathcal{F} = \text{tr} \left( p \exp \left( - \frac{\nabla'^2_{\text{tor}}}{2\pi i} \right) \right).$$

Let us now consider the original operator (5.4). By Lemma 2.3 we have for this operator (and  $E = \psi(T_{\mathbb{C}}^*M)$ )

$$\text{ch}_{\psi(T_{\mathbb{C}}^*M)}[\sigma(D)] = \text{tr} \left( \exp \left( - \frac{\nabla^2_{\text{tor}}}{2\pi i} \right) \right),$$

where the noncommutative connection is equal to

$$\nabla_{\text{tor}} = dt \frac{\partial}{\partial t} + t\nabla + (1-t)(\sigma(D)^{-1})\nabla(\sigma(D)) \tag{6.12}$$

and is expressed in terms of some connection  $\nabla$  in the bundle  $\mathbb{C}^N \otimes \psi$  over  $S^*M$ . Let us now define  $\nabla$  by

$$\nabla = p\nabla_\psi p + (1-p)\nabla_\psi(1-p) \tag{6.13}$$

and recall that

$$\sigma(D) = \sigma(D_0)Tp + (1 - p), \quad \sigma(D)^{-1} = (\sigma(D_0)T)^{-1}p + (1 - p) \quad (6.14)$$

(see (5.4)). Substituting the expressions (6.13) and (6.14) in eq. (6.12), we obtain

$$\begin{aligned} \nabla_{\text{tor}} &= dt \frac{\partial}{\partial t} + tp \nabla_{\psi} p + (1 - p) \nabla_{\psi} (1 - p) \\ &\quad + (1 - t)(\sigma(D_0)T)^{-1} p \nabla_{\psi} p (\sigma(D_0)T) \\ &= p \nabla'_{\text{tor}} p + (1 - p) (dt \frac{\partial}{\partial t} + \nabla_{\psi}) (1 - p). \end{aligned}$$

This implies that the curvature form is equal to

$$(\nabla_{\text{tor}})^2 = p \nabla'_{\text{tor}}{}^2 p + [(1 - p) \nabla_{\psi} (1 - p)]^2.$$

Hence the Chern character forms for the connections  $\nabla'_{\text{tor}}$  and  $\nabla_{\text{tor}}$  differ by a form that does not contain  $dt$ . Hence, the integrals of these Chern character forms are equal,

$$\int_{S^*M \times [0,1]} \text{tr} \left( p \exp \left( - \frac{(\nabla_{\text{tor}})^2}{2\pi i} \right) \right) = \int_{S^*M \times [0,1]} \text{tr} \left( p \exp \left( - \frac{(\nabla'_{\text{tor}})^2}{2\pi i} \right) \right),$$

i.e., we obtain the desired equality

$$\int_{S^*M \times_g \mathbb{S}^1} \text{ch}(\mathcal{V} \otimes \psi(T_{\mathbb{C}}^*M \times_g \mathbb{S}^1)) = \int_{S^*M \times [0,1]} \text{ch}_{\psi(T_{\mathbb{C}}^*M)}[\sigma(D)] = \text{ind}_{\text{top}} D.$$

The proof of the Index Theorem 4.1 is now complete.

## 7. Index formula in cyclic cohomology

In this section we give an interpretation of the index formula (4.1) in terms of cyclic cohomology (see [16], [43] and for cyclic cohomology of crossed products [14], [19], [28]).

### 7.1. Equivariant Chern character

**1. Chern character in cyclic cohomology.** Let  $E \in \text{Vect}(X)$  be a vector bundle over a smooth closed oriented manifold  $X$ ,  $\dim X = n$ . We fix a connection  $\nabla_E$  in  $E$  and define following [21] the multilinear functionals

$$\begin{aligned} \text{Char}^k(E, \nabla_E; a_0, a_1, \dots, a_k) &= \frac{(-1)^{(n-k)/2}}{\left(\frac{n+k}{2}\right)!} \sum_{i_0 + \dots + i_k = (n-k)/2} \int_X \text{tr}_E [(a_0 \theta^{i_0} \nabla(a_1) \theta^{i_1} \nabla(a_2) \dots \nabla(a_k) \theta^{i_k})_0], \end{aligned} \quad (7.1)$$

where  $a_k \in C^\infty(X, \text{End } E, \mathbb{Z})$  and  $k = n, n - 2, n - 4, \dots$  (cf. Jaffe–Lesniewski–Osterwalder formula [24]). Here for a noncommutative form  $\omega$  by  $\omega_0$  we denote the coefficient at  $T^0 = 1$ ,  $\theta = \nabla_E^2$  is the curvature of the connection, while the operator  $\nabla: \Lambda(X, \text{End } E, \mathbb{Z}) \rightarrow \Lambda(X, \text{End } E, \mathbb{Z})$  is defined as

$$\nabla(\omega) = \nabla\omega - (-1)^{\text{deg } \omega} \omega \nabla$$

or, more explicitly,

$$\nabla\left(\sum_k \omega_k T^k\right) = \sum_k [\nabla_E \omega_k - (-1)^{\text{deg } \omega_k} \omega_k g^{k*}(\nabla_E)] T^k,$$

where the expression  $\nabla_E \omega_k - (-1)^{\text{deg } \omega_k} \omega_k g^{k*}(\nabla_E)$  is an operator of multiplication by a 1-form. It follows from [21] that the collection of functionals  $\{\text{Char}^k(E, \nabla_E)\}$  defines a cyclic cocycle over the algebra  $C^\infty(X, \text{End } E, \mathbb{Z})$ , and the class of this cycle in periodic cyclic cohomology

$$\text{Char}(E) = \{[\text{Char}^k(E, \nabla_E)]\} \in \text{HP}^*(C^\infty(X, \text{End } E, \mathbb{Z}))$$

does not depend on the choice of connection  $\nabla_E$ .

**Example 7.1.** Let  $E$  be a flat bundle, i.e.,  $\theta = 0$ . Then the Chern character (7.1) has only one nonzero component  $\text{Char}^n(E)$  that is equal to  $(\dim X = n)$

$$\text{Char}^n(E, \nabla_E; a_0, \dots, a_n) = \frac{1}{n!} \int_X \text{tr}_E(a_0 \nabla(a_1) \nabla(a_2) \dots \nabla(a_n))_0.$$

## 2. Relation to the Chern character in Haefliger cohomology

**Proposition 7.1.** *One has a commutative diagram*

$$\begin{array}{ccc} & K_0(C^\infty(X, \text{End } E, \mathbb{Z})) & \\ \text{ch} \swarrow & & \searrow C_n(\cdot, \text{Char}(E)) \\ H^{\text{ev}}(X/\mathbb{Z}) & \xrightarrow{\int_X} & \mathbb{C}, \end{array} \tag{7.2}$$

where  $C_n = (2\pi i)^{-n/2}$ ,  $\text{ch}$  is the Chern character from Section 2.1,  $\int_X$  stands for the integral and  $\langle \cdot, \cdot \rangle$  is the pairing of the  $K_0$ -group with cyclic cohomology. This pairing is defined by the formula

$$\langle [p], [\varphi] \rangle = \sum_k \frac{(-1)^k (2k)!}{k!} \varphi_{2k}(p - 1/2, p, \dots, p), \tag{7.3}$$

where  $[p] \in K_0(C^\infty(X, \text{End } E, \mathbb{Z}))$ ,  $[\varphi] \in \text{HP}^{\text{ev}}(C^\infty(X, \text{End } E, \mathbb{Z}))$ , and the cyclic cocycle  $\varphi$  is extended to matrix elements in the usual way:

$$\varphi_l(m_0 \otimes a_0, m_1 \otimes a_1, \dots, m_l \otimes a_l) = \text{tr}(m_0 m_1 \dots m_l) \varphi_l(a_0, a_1, \dots, a_l).$$

*Proof.* 1. Let us carry out an additional construction. Namely, we embed the triangle (7.2) in the diagram

$$\begin{array}{ccc}
 & K_0(C^\infty(X, \text{Mat}_N(\mathbb{C}), \mathbb{Z})) & \\
 \text{ch} \swarrow & \uparrow & \searrow C_n(\cdot, \text{Char}(X \times \mathbb{C}^N)) \\
 & K_0(C^\infty(X, \text{End } E, \mathbb{Z})) & \\
 \text{ch} \swarrow & & \searrow C_n(\cdot, \text{Char}(E)) \\
 H^{\text{ev}}(X/\mathbb{Z}) & \xrightarrow{f_X} & \mathbb{C}
 \end{array} \tag{7.4}$$

Here the mapping  $K_0(C^\infty(X, \text{End } E, \mathbb{Z})) \rightarrow K_0(C^\infty(X, \text{Mat}_N(\mathbb{C}), \mathbb{Z}))$  is induced by an embedding  $E \subset X \times \mathbb{C}^N$  in the trivial bundle.

2. We claim that the left and the right triangles of the diagram (7.4) are commutative. Indeed, let us prove the commutativity of the left triangle (the commutativity of the right triangle is obtained similarly). Suppose that  $E = \text{Im } q \subset X \times \mathbb{C}^N$ , where  $q$  is a projection in the trivial bundle. Then we have an isomorphism

$$C^\infty(X, \text{End } E, \mathbb{Z}) = q[C^\infty(X, \text{Mat}_N(\mathbb{C}), \mathbb{Z})]q.$$

In particular, to a projection  $p$  over  $C^\infty(X, \text{End } E, \mathbb{Z})$  we assign a projection  $p'$  over the algebra  $C^\infty(X, \text{Mat}_N(\mathbb{C}), \mathbb{Z})$ . Thus, we have

$$\text{ch}[p] = [\text{ch}(p, \nabla_E)] = [\text{ch}(p', q\nabla_E q + (1 - q)d(1 - q))] = \text{ch}[p'].$$

Here the first and last equalities follow from the definition of the Chern character, and the equality in the middle follows from the equality of the corresponding differential forms.

3. The perimeter of the diagram (7.4) is also a commutative triangle. Indeed, in the trivial bundle, let us choose the flat connection defined by the exterior differential  $d$ . Then by Example 2.2 for a projection  $p$  over  $C^\infty(X, \text{Mat}_N(\mathbb{C}), \mathbb{Z})$  we obtain

$$\int_X \text{ch}[p] = \frac{1}{(n/2)!} \left(-\frac{1}{2\pi i}\right)^{n/2} \int_X \text{tr}[p(dpdp)^{n/2}]_0.$$

On the other hand, it follows from the formula obtained in Example 7.1 that

$$\langle [p], \text{Char}(X \times \mathbb{C}^N) \rangle = \frac{(-1)^{n/2}}{(n/2)!} \int_X \text{tr}[p(dpdp)^{n/2}]_0.$$

We see that the last two expressions differ only by the factor  $C_n = (2\pi i)^{-n/2}$ . This proves that the perimeter of the diagram (7.4) is commutative.

4. We proved the commutativity of all the triangles in the diagram (7.4) except for the lower triangle. Hence, the lower triangle is commutative.

The proof of the proposition is complete. □

**3. Equivariant Chern character [21].** Define the equivariant Chern character of a  $g$ -bundle  $E$  on  $X$

$$\text{Ch}(E) \in \text{HP}^*(C^\infty(X) \rtimes \mathbb{Z})$$

as  $\text{Ch}(E) := \beta^* \text{Char}(E)$ , where  $\beta^*: \text{HP}^*(C^\infty(X, \text{End } E) \rtimes \mathbb{Z}) \rightarrow \text{HP}^*(C^\infty(X) \rtimes \mathbb{Z})$  is the mapping induced by the homomorphism of algebras

$$\beta: C^\infty(X) \rtimes \mathbb{Z} \rightarrow C^\infty(X, \text{End } E) \rtimes \mathbb{Z}; \quad \sum_k \omega_k T^k \mapsto \sum_k (\omega_k \otimes 1_E) \tilde{T}^k.$$

There is an analogue of the commutative diagram (7.2) for the equivariant Chern character. Namely, one has

$$\begin{array}{ccc}
 & K_0(C^\infty(X) \rtimes \mathbb{Z}) & \\
 \text{ch}_E \swarrow & & \searrow C_n(\cdot, \text{Ch}(E)) \\
 H^{\text{ev}}(X/\mathbb{Z}) & \xrightarrow{f_X} & \mathbb{C}.
 \end{array} \tag{7.5}$$

**7.2. Index formula in cyclic cohomology.** Given a  $g$ -bundle  $E \in \text{Vect}(X)$  over a smooth closed oriented manifold  $X$ , we define the equivariant Todd class

$$\text{Todd}(E) \in \text{HP}^*(C^\infty(X) \rtimes \mathbb{Z})$$

as  $\text{Todd}(E) := \text{Ch}(\psi(E))$ , where  $\psi$  is the operation in rational K-theory defined in Section 3.

**1. Index formula**

**Theorem 7.1.** For an elliptic operator  $D$  one has an index formula

$$\text{ind } D = (2\pi i)^{-n} \langle [\sigma(D)], \text{Todd}(\pi^* T_{\mathbb{C}}^* M) \rangle, \quad \dim M = n, \tag{7.6}$$

where  $\pi: S^*M \times \mathbb{S}^1 \rightarrow M$  is the projection and the brackets  $\langle \cdot, \cdot \rangle$  stand for the pairing of K-theory with cyclic cohomology (see (7.3)).

*Proof.* The index formula (4.1) gives us

$$\text{ind } D = \int_{S^*M \times \mathbb{S}^1} \text{ch}_{\psi(\pi^* T_{\mathbb{C}}^* M)}[\sigma(D)]. \tag{7.7}$$

Using the commutative diagram (7.5) and the definition of the equivariant Todd class, we can rewrite the right-hand side in (7.7) in the desired form

$$\begin{aligned}
 \int_{S^*M \times \mathbb{S}^1} \text{ch}_{\psi(\pi^* T_{\mathbb{C}}^* M)}[\sigma(D)] &= (2\pi i)^{-n} \langle [\sigma(D)], \text{Ch}(\psi(\pi^* T_{\mathbb{C}}^* M)) \rangle \\
 &= (2\pi i)^{-n} \langle [\sigma(D)], \text{Todd}(\pi^* T_{\mathbb{C}}^* M) \rangle. \quad \square
 \end{aligned}$$

**2. A special case.** Suppose that the Todd class  $\text{Td}(T_{\mathbb{C}}^*(M \times_g \mathbb{S}^1))$  of the complexified cotangent bundle of the twisted torus is equal to one. Then one can replace the class  $\text{Todd}(\pi^* T_{\mathbb{C}}^* M)$  in (7.6) simply by the transverse fundamental class of the manifold  $S^*M \times \mathbb{S}^1$  in the sense of [15]. This enables one to write the index formula in the form

$$\text{ind } D = \frac{(n-1)!}{(2\pi i)^n (2n-1)!} \int_{S^*M} \text{tr}(\sigma^{-1} d\sigma)_0^{2n-1}, \quad \sigma = \sigma(D). \quad (7.8)$$

The index formula (7.8) is a corollary of the following more general statement.

**Proposition 7.2.** *Suppose that the homology class Poincaré dual to the Todd class  $\text{Td}(T_{\mathbb{C}}^*M \times_g \mathbb{S}^1)$  has a representative<sup>1</sup> of the form*

$$\omega \mapsto z \left( \int_{\mathbb{S}^1} \omega \right), \quad \omega \in \Lambda(S^*M \times_g \mathbb{S}^1),$$

where  $z$  is a closed  $g$ -invariant current on  $S^*M$ . Then the equivariant Todd class in the index formula (7.6) can be replaced by the collection of cyclic cocycles with the components

$$(a_0, \dots, a_{2k}) \mapsto \frac{(2\pi i)^{n-k}}{(2k)!} z(a_0 da_1 da_2 \dots da_{2k}), \quad k = 0, 1, \dots, n.$$

*Proof.* The proof of this proposition is similar to the proof of index formula (7.6). The main difference is that instead of equality (6.10) one uses equalities of the form

$$\text{ind } D = \text{ind } \mathcal{D}'_0 = \int_{S^*M \times_g \mathbb{S}^1} (\text{ch } \mathcal{V}) \text{Td}(T_{\mathbb{C}}^*M \times_g \mathbb{S}^1) = z \left( \int_{\mathbb{S}^1} \text{ch } \mathcal{V} \right). \quad \square$$

**Example 7.2.** Suppose that  $\text{Td}(T_{\mathbb{C}}^*M \times_g \mathbb{S}^1) = 1$ . Then we can take the current  $z$  of degree  $2n - 1$  defined by integration over  $S^*M$ . Then Proposition 7.2 gives the index formula (7.8) (after a standard integration over  $\mathbb{S}^1$ ). This remark applies, for instance, to elliptic operators for a diffeomorphism of the sphere in the connected component of the identity (see [17], [26]).

Proposition 7.2 can be applied if  $g$  is an isometry. In this case we define the current  $z$  by the formula

$$z(\omega) = \int_{S^*M} \omega \wedge \text{Td}(T_{\mathbb{C}}^*M), \quad \omega \in \Lambda(S^*M),$$

where  $\text{Td}(T_{\mathbb{C}}^*M)$  is the differential form representing the Todd class using a  $g$ -invariant metric. In this case we obtain the index formula first proved in [27].

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<sup>1</sup>Here the homology group is treated in terms of closed de Rham currents (see [18]).

**8. Examples. Remarks**

**8.1. Example. Operators on the torus  $\mathbb{T}^3$ .** The index formula (7.6), despite its compact and elegant form, often leads to serious computational difficulties, when one really needs to compute the index of a specific operator. To solve this problem, it is sometimes useful to simplify the formula so that the new formula could really be used to compute the desired number.

In this section, we exhibit a procedure of this form for a relatively simple operator related to the Dirac operator. The answer we obtain is quite suitable to get explicit numerical expression for the index of the problem.

1. Consider the torus  $\mathbb{T}^3 = \mathbb{R}^3/2\pi\mathbb{Z}^3$  with coordinates  $x = (x_1, x_2, x_3)$  and the diffeomorphism<sup>2</sup>

$$g: \mathbb{T}^3 \rightarrow \mathbb{T}^3, \quad g \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} p = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Consider the Dirac operator on  $\mathbb{T}^3$ ,

$$\sum_{j=1}^3 c_j \left( -i \frac{\partial}{\partial x_j} \right) : C^\infty(\mathbb{T}^3, \mathbb{C}^2) \rightarrow C^\infty(\mathbb{T}^3, \mathbb{C}^2), \tag{8.1}$$

where

$$c_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad c_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

stand for Pauli matrices. The Dirac operator is elliptic and self-adjoint in the space  $L^2$ . Therefore, it has a discrete real spectrum, while the eigenvalues have finite multiplicities. Consider the positive spectral projection for the Dirac operator, i.e., the orthogonal projection on the subspace generated by eigenfunctions of the Dirac operator with positive eigenvalues. This projection is denoted by  $P$  and is a  $\psi$ DO of order zero (see [42]).

**Theorem 8.1.** *Let  $f \in C^\infty(\mathbb{T}^3, \text{Mat}_N(\mathbb{C}))$  be a function taking values in invertible matrices. Then the operator*

$$D = (f \otimes 1)(1 \otimes P)T(1 \otimes P) + 1 \otimes (1 - P) : H^s(\mathbb{T}^3, \mathbb{C}^N \otimes \mathbb{C}^2) \rightarrow H^s(\mathbb{T}^3, \mathbb{C}^N \otimes \mathbb{C}^2), \tag{8.2}$$

where  $T = g^*$  is the shift operator for  $g$ , is Fredholm for all  $s$  and its index is equal to

$$\text{ind } D = \frac{1}{(2\pi i)^{2 \cdot 3!}} \int_{\mathbb{T}^3} \text{tr}(f^{-1}df)^3. \tag{8.3}$$

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<sup>2</sup>This diffeomorphism is ‘‘Arnold’s cat map’’ [4] acting along  $x_1, x_2$  and the identity map along  $x_3$ .

Let us write the operator (8.2) simply as  $D = fPTP + 1 - P$  omitting the tensor products.

2. Let us prove that  $D$  is elliptic. To this end, we first compute the symbol of  $P$ . The symbol of the Dirac operator (8.1) is equal to

$$c(\xi) = c_1\xi_1 + c_2\xi_2 + c_3\xi_3 \in \text{Mat}_2(\mathbb{C}),$$

where  $\xi = (\xi_1, \xi_2, \xi_3)$  stand for variables dual to  $x$ . In what follows, it is useful to write the following Clifford identity (e.g., see [25]):

$$c(\xi)c(\xi')v + c(\xi')c(\xi)v = 2(\xi, \xi')v, \quad \xi, \xi' \in \mathbb{R}^3, \quad v \in \mathbb{C}^2, \quad (8.4)$$

where on the right-hand side of (8.4) we have the inner product of vectors. In particular, eq. (8.4) implies that the matrix

$$p(\xi) = \frac{1 + c(\xi)}{2}, \quad |\xi| = 1$$

is a rank-one projection. Moreover, this projection is just the positive spectral projection of the symbol  $c(\xi)$  of the Dirac operator. We extend this function to a degree zero homogeneous function in  $\xi$ . Then the results of the paper [42] give the equality

$$\sigma(P) = p.$$

We are now ready to prove that  $D$  is elliptic. Consider the mapping

$$u(\xi) = p(\xi): \text{Im}(\partial g)^* p(\xi) \rightarrow \text{Im} p(\xi). \quad (8.5)$$

We claim that the mapping (8.5) is invertible (cf. [33]). Indeed, let us consider the converse, i.e., suppose that for some  $\xi$  we have a nonzero vector

$$v \in \text{Im}(\partial g)^* p(\xi) = \text{Im} p(g^{-1}\xi) \quad \text{such that } p(\xi)v = 0. \quad (8.6)$$

In terms of Clifford multiplication, condition (8.6) is written as

$$c(\xi)v = -v, \quad c\left(\frac{g^{-1}\xi}{|g^{-1}\xi|}\right)v = v.$$

Substituting these two formulas in (8.4), we get

$$-2v = 2\left(\xi, \frac{g^{-1}\xi}{|g^{-1}\xi|}\right)v \quad \text{or} \quad \cos(\xi, g^{-1}\xi) = -1,$$

i.e., the vectors  $\xi$  and  $g^{-1}\xi$  form an angle equal to  $\pi$ . But this cannot be true since the matrix  $g^{-1}$  has no negative eigenvalues. This contradiction shows that the mapping (8.5) is an isomorphism.



Denote by  $U^{-1}$  a  $\psi$ DO on  $\mathbb{T}^3$  such that the restriction of its symbol to the subspace  $\text{Im } p(\xi)$  coincides with  $u(\xi)^{-1}$ . We claim that the operator

$$B = T^{-1} f^{-1} U^{-1} P + 1 - P$$

is an almost inverse of  $D$  (i.e., inverse up to operators of negative order). Indeed, for example, let us compute the composition of symbols:

$$\begin{aligned} \sigma(D)\sigma(B) &= (fpTp + 1 - p)(T^{-1}u^{-1}pf^{-1} + 1 - p) \\ &= fpTpT^{-1}u^{-1}pf^{-1} + (1 - p) + (1 - p)T^{-1}u^{-1}pf^{-1} \\ &= fp[(\partial g)^* p]u^{-1}pf^{-1} + (1 - p) + (1 - p)T^{-1}TpT^{-1}u^{-1}pf^{-1} \\ &= fpf^{-1} + (1 - p) + (1 - p)pT^{-1}u^{-1}pf^{-1} \\ &= p + 1 - p = 1. \end{aligned} \tag{8.7}$$

The equality  $\sigma(B)\sigma(D) = 1$  is obtained similarly.

Thus,  $D$  is elliptic and Fredholm by Theorem 1.1.

3. By the Index Theorem 4.1 the analytic index of  $D$  is equal to the topological index of its symbol. Let us compute the topological index of  $\sigma(D)$ . The symbol  $\sigma(D)$  has the factorization

$$\sigma(D) = \sigma_0\sigma_1, \quad \sigma_0 = pfp + 1 - p, \quad \sigma_1 = pTp + (1 - p), \tag{8.8}$$

into two elliptic symbols, where  $\sigma_0$  does not contain the shift operator  $T$ . Let us compute the topological indices of these symbols. The index of  $\sigma_0$  coincides with the Atiyah–Singer topological index and is equal to (see [9], [10])

$$\text{ind}_{\text{top}} \sigma_0 = \int_{S^*\mathbb{T}^3} \text{ch}[f] \text{ch}(\text{Im } p) \text{Td}(T_{\mathbb{C}}^*\mathbb{T}^3),$$

where  $[f] \in K^1(\mathbb{T}^3)$  is the class of  $f$  in the odd  $K$ -group. Further, we get

$$\begin{aligned} \int_{S^*\mathbb{T}^3} \text{ch}[f] \text{ch}(\text{Im } p) \text{Td}(T_{\mathbb{C}}^*\mathbb{T}^3) &= \int_{S^*\mathbb{T}^3} \text{ch}[f] \text{ch}(\text{Im } p) \\ &= \int_{\mathbb{T}^3} \text{ch}[f] \int_{\mathbb{S}^2} \text{ch}(\text{Im } p) \\ &= C \int_{\mathbb{T}^3} \text{tr}(f^{-1}df)^3, \end{aligned}$$

where  $C = ((2\pi i)^{23})^{-1}$ . Here we first noted that the tangent bundle of the torus is trivial and replaced the Todd class by one. Then, we used the decomposition  $S^*\mathbb{T}^3 = \mathbb{T}^3 \times \mathbb{S}^2$  with the coordinates  $x, \xi$  on the factors. Moreover, since  $f$  depends only on  $x$ , and  $p$  depends only on  $\xi$ , the integral over  $\mathbb{T}^3 \times \mathbb{S}^2$  is just the product of an integral over  $\mathbb{T}^3$  and an integral over  $\mathbb{S}^2$ . In the next to the last equality,

the Chern character in the first factor is represented by a differential form and we noted that  $\text{Im } p$  is the Bott bundle on  $\mathbb{S}^2$  (see [5] and [20]) and one has

$$\int_{\mathbb{S}^2} \text{ch Im } p = 1.$$

So we obtain

$$\text{ind}_{\text{top}} \sigma_0 = \frac{1}{(2\pi i)^{23}!} \int_{\mathbb{T}^3} \text{tr}(f^{-1}df)^3. \tag{8.9}$$

**Proposition 8.1.** *One has  $\text{ind}_{\text{top}} \sigma_1 = 0$ .*

*Proof.* By eq. (7.8), the topological index of  $\sigma_1$  is equal to

$$\text{ind}_{\text{top}} \sigma_1 = \frac{2!}{(2\pi i)^{35}!} \int_{S^*M} \text{tr}(\sigma_1^{-1}d\sigma_1)_0^5. \tag{8.10}$$

Let us compute this integral. The symbol  $\sigma_1 \in C^\infty(S^*\mathbb{T}^3, \text{Mat}_2(\mathbb{C})) \rtimes \mathbb{Z}$  is constant in  $x$ . Thus, its differential  $d\sigma_1 \in \Lambda^1(S^*\mathbb{T}^3, \text{Mat}_2(\mathbb{C})) \rtimes \mathbb{Z}$  does not contain differentials  $dx_j$ . The product  $\sigma_1^{-1}d\sigma_1$  also has no differentials  $dx_j$ . This uses the fact that  $g$  is a linear diffeomorphism. The same reasoning shows that the form

$$(\sigma_1^{-1}d\sigma_1)^5$$

(see (8.10)) also does not contain differentials  $dx_j$ . On the other hand, the degree of this form is equal to five. Therefore, this form is identically zero. Thus, (8.10) implies that the topological index of  $\sigma_1$  is zero. □

The formula (8.3) now follows from eq. (8.9) and Proposition 8.1. This completes the proof of Theorem 8.1.

4. Let us give a direct proof of Theorem 8.1.

Namely, let us first prove that  $D$  is elliptic. One has an equality (cf. (8.8)) modulo compact operators

$$D = D_0D_1, \quad D_0 = fP + (1 - P), \quad D_1 = PTP + (1 - P).$$

Here  $D_0$  is an elliptic  $\psi DO$  and its index (computed by the Atiyah–Singer formula) is equal to the right-hand side in (8.9) (see the above computation). Thus, to prove Theorem 8.1, it suffices to show that  $D_1$  is a Fredholm operator of index zero.

**Proposition 8.2.** *The operator  $D_1 = PTP + (1 - P)$  is invertible.*

*Proof.* Let us treat functions on the torus as Fourier series  $\sum_k a_k e^{i(k,x)}$ , where  $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$ , and  $(k, x) = k_1x_1 + k_2x_2 + k_3x_3$ . In this notation, we have

$$P(\sum_k a_k e^{i(k,x)}) = \sum_{k \neq 0} a_k p(k) e^{i(k,x)}, \tag{8.11}$$

where  $p(k)$  is the value of the function  $p(\xi)$  at  $\xi = k$ . In this representation, the shift operator is equal to

$$T\left(\sum_k a_k e^{i(k,x)}\right) = \sum_k a_k e^{i(gk,x)},$$

since  $g$  has a symmetric matrix. Let  $U^{-1}$  be the operator defined by a formula of the form (8.11) using the symbol  $u^{-1}(\xi)$ . In this case the mappings

$$P : \text{Im } TPT^{-1} \rightarrow \text{Im } P$$

and

$$U^{-1} : \text{Im } P \rightarrow \text{Im } TPT^{-1}$$

are inverses of each other. It follows that the operator  $B = T^{-1}U^{-1}P + 1 - P$  is the inverse of  $D_1$ . The proofs of these statements are similar to the computation (8.7).  $\square$

**8.2. Remark. Special operators as operators in subspaces.** Let us give here a method of computing the index of special operators using elliptic theory in subspaces defined by pseudodifferential projections (see [34], [35]). We write a special operator (5.1) as

$$D = QD_0TP + (1 - Q)D_1(1 - P) : C^\infty(M, \mathbb{C}^N) \rightarrow C^\infty(M, \mathbb{C}^N). \quad (8.12)$$

Without loss of generality, we can assume that  $P$  and  $Q$  are projections  $P^2 = P$ ,  $Q^2 = Q$ . In this case, the operator  $D$  is a direct sum

$$D = QD_0TP \oplus (1 - Q)D_1(1 - P)$$

of operators acting in subspaces defined by the projections  $P, Q, 1 - P, 1 - Q$ . Using this decomposition, we can compute the index of  $D$ . Indeed, we get

$$\begin{aligned} \text{ind } D &= \text{ind}[QD_0T : \text{Im } P \rightarrow \text{Im } Q] \\ &\quad + \text{ind}[(1 - Q)D_1 : \text{Im}(1 - P) \rightarrow \text{Im}(1 - Q)] \\ &= \text{ind}[D_0 : \text{Im } g^*P \rightarrow \text{Im } Q] + \text{ind}[D_1 : \text{Im}(1 - P) \rightarrow \text{Im}(1 - Q)]. \end{aligned} \quad (8.13)$$

Here in the last equality we used the fact that  $T$  defines an isomorphism of the ranges of the projections  $P$  and  $g^*P = TPT^{-1}$ .

An application of the index formulas obtained in the papers [34], [35] to the operators in (8.13) gives an index formula for  $D$ . To formulate the result, consider the involution  $\alpha : T^*M \rightarrow T^*M$ ,  $\alpha(x, \xi) = (x, -\xi)$  and for a  $\psi$ DO  $A$  let  $\alpha^*A$  denote any  $\psi$ DO with the symbol  $\alpha^*\sigma(A)$ .

**Proposition 8.3.** *Let a special operator (8.12) be elliptic, the manifold  $M$  be odd-dimensional, and the projections  $P$ ,  $Q$  be even, i.e., they satisfy the condition*

$$\alpha^* \sigma(P) = \sigma(P), \quad \alpha^* \sigma(Q) = \sigma(Q).$$

Then one has the equality

$$\text{ind } D = \frac{1}{2} \text{ind}[D_0(\alpha^*(D_0)^{-1})Q + D_1(\alpha^*(D_1)^{-1})(1 - Q)], \quad (8.14)$$

where the operator in the square brackets is an elliptic  $\psi$ DO on  $M$ .

*Proof.* 1. Application of the index formula from the paper [34] to the operators  $D_0: \text{Im } g^*P \rightarrow \text{Im } Q$  and  $D_1: \text{Im}(1 - P) \rightarrow \text{Im}(1 - Q)$  gives us

$$\begin{aligned} \text{ind}(D_0: \text{Im } g^*P \rightarrow \text{Im } Q) \\ = \frac{1}{2} \text{ind}[D_0(\alpha^*(D_0)^{-1})Q + (1 - Q)] + d(g^*P) - d(Q), \end{aligned} \quad (8.15)$$

$$\begin{aligned} \text{ind}(D_1: \text{Im}(1 - P) \rightarrow \text{Im}(1 - Q)) \\ = \frac{1}{2} \text{ind}[Q + D_1(\alpha^*(D_1)^{-1})(1 - Q)] + d(1 - P) - d(1 - Q), \end{aligned} \quad (8.16)$$

where  $d$  is the homotopy invariant of even pseudodifferential projections constructed in [34]. Adding the last two expressions (8.15) and (8.16), we obtain the following expression for the index of  $D$ :

$$\begin{aligned} \text{ind } D = \frac{1}{2} \text{ind}[D_0(\alpha^*(D_0)^{-1})Q + D_1(\alpha^*(D_1)^{-1})(1 - Q)] \\ + (d(g^*P) + d(1 - P)) - (d(Q) + d(1 - Q)). \end{aligned} \quad (8.17)$$

The cited paper contains the following properties of the functional  $d$ :

$$d(Q) + d(1 - Q) = 0 \quad \text{and} \quad d(g^*P) = d(P).$$

Hence, the last two terms in eq. (8.17) are equal to zero and we obtain the desired index formula (8.14).  $\square$

There is an analog of this proposition on even-dimensional manifolds (see [35]).

**8.3. Remark. A generalization of the notion of ellipticity.** In [2], [3] a different condition of ellipticity of operators (1.1) is used. This condition does not require that the number of nonzero components of the inverse symbol is finite. In this situation, the symbol is naturally an element of the  $C^*$ -crossed product  $C(S^*M) \rtimes \mathbb{Z}$  (see [44]) of the algebra of continuous symbols on  $S^*M$  by the action of the diffeomorphism  $g$ , and the ellipticity is just the invertibility in this  $C^*$ -crossed product. On the other hand, it was shown in the papers [33], [39] that an elliptic operator in this sense is

stably homotopic to an operator elliptic in the sense of Definition 1.2. This implies that to obtain an index formula for this class of operators it suffices to extend the cyclic cocycle  $\text{Todd} \in \text{HP}^*(C^\infty(S^*M \times \mathbb{S}^1) \rtimes \mathbb{Z})$  (see (7.6)) to some local algebra  $\mathcal{A}$  such that

$$C^\infty(S^*M) \rtimes \mathbb{Z} \subset \mathcal{A} \subset C(S^*M) \rtimes \mathbb{Z}$$

or, in more invariant form, to define a class  $\overline{\text{Todd}} \in \text{HP}^*(\mathcal{A})$  that is the pull-back of the class

$$\text{Todd} \in \text{HP}^*(C^\infty(S^*M \times \mathbb{S}^1) \rtimes \mathbb{Z})$$

under the embedding

$$C^\infty(S^*M \times \mathbb{S}^1) \rtimes \mathbb{Z} \subset \mathcal{A}.$$

Such extensions are known for many interesting classes of diffeomorphisms (e.g., see [13], [15], [39], [41]). Therefore, we obtain an index formula of the type (7.6) for operators elliptic in the sense of [3] for these classes of diffeomorphisms.

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Received June 23, 2011; revised August 27, 2012

A. Savin, Peoples' Friendship University of Russia, Miklukho-Maklaya str. 6, 117198, Moscow, Russia, and Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany

E-mail: [a.yu.savin@gmail.com](mailto:a.yu.savin@gmail.com)

B. Sternin, Peoples' Friendship University of Russia, Miklukho-Maklaya str. 6, 117198, Moscow, Russia, and Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany

E-mail: [b.yu.sternin@gmail.com](mailto:b.yu.sternin@gmail.com)