

Equivariant Poincaré Series and Monodromy Zeta Functions of Quasihomogeneous Polynomials

by

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Abstract

In earlier work, the authors described a relation between the Poincaré series and the classical monodromy zeta function corresponding to a quasihomogeneous polynomial. Here we formulate an equivariant version of this relation in terms of the Burnside rings of finite abelian groups and their analogues.

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Let $f(x_1, \dots, x_n)$ be a quasihomogeneous polynomial. In [E], [EG1], a relation was described between the Poincaré series $P_X(t)$ of the coordinate ring of the hypersurface singularity $X = \{f = 0\}$ and the classical monodromy zeta function $\zeta_f(t)$ of f . The relation involved the so-called Saito duality [S1], [S2]. Namely, in [EG1] it was shown that

$$(1) \quad P_X(t) \cdot \text{Or}_X(t) = \tilde{\zeta}_f^*(t),$$

where $\text{Or}_X(t)$ is a rational function determined by the orbit types of the natural \mathbb{C}^* -action on X (see, e.g., [EG2]), $\tilde{\zeta}_f = \zeta_f(t)/(1-t)$ is the reduced monodromy zeta function of f , and $\tilde{\zeta}_f^*(t)$ is the Saito dual of $\tilde{\zeta}_f(t)$ with respect to the quasidegree of the polynomial f .

This relation had no intrinsic explanation. It was obtained by computing both sides and comparing the results. In particular, the role of the Saito duality remained unclear. In [EG3], an equivariant version of the Saito duality for a finite

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abelian group G was formulated as a transformation between the Burnside rings of G and of the group G^* of its characters. Here we use the Burnside rings and their analogues to define equivariant versions of the ingredients of the relation (1) and to give an equivariant analogue of it. This generalization can help to understand the role of the ingredients of the relation, in particular, of the Saito duality.

Let f be a quasihomogeneous polynomial in n variables x_1, \dots, x_n of degree d with weights q_1, \dots, q_n (q_i are positive integers, $\gcd(q_1, \dots, q_n) = 1$), i.e.

$$f(\lambda^{q_1} x_1, \dots, \lambda^{q_n} x_n) = \lambda^d f(x_1, \dots, x_n) \quad \text{for } \lambda \in \mathbb{C}.$$

From now on we assume that f determines the system of weights $(q_1, \dots, q_n; d)$ in a unique way. This means that in the lattice \mathbb{Z}^n of monomials in x_1, \dots, x_n (a point $(k_1, \dots, k_n) \in \mathbb{Z}^n$ corresponds to the monomial $x_1^{k_1} \cdots x_n^{k_n}$), the monomials appearing in f with non-zero coefficients generate an affine hyperplane (namely $\{\sum_i q_i k_i = d\}$). The system of weights $(q_1, \dots, q_n; d)$ defines a \mathbb{C}^* -action on the space \mathbb{C}^n :

$$(2) \quad \lambda * (x_1, \dots, x_n) = (\lambda^{q_1} x_1, \dots, \lambda^{q_n} x_n).$$

Let

$$G_f = \{\underline{\lambda} = (\lambda_1, \dots, \lambda_n) \in (\mathbb{C}^*)^n : f(\lambda_1 x_1, \dots, \lambda_n x_n) = f(x_1, \dots, x_n)\}$$

be the (abelian) *symmetry group* of f , i.e. the group of diagonal linear transformations of \mathbb{C}^n preserving f . Let

$$\overline{G}_f = \{\underline{\lambda} \in (\mathbb{C}^*)^n : f(\lambda_1 x_1, \dots, \lambda_n x_n) = \alpha(\underline{\lambda}) f(x_1, \dots, x_n)\}$$

be the *extended symmetry group* of f , i.e. the group of diagonal linear transformations of \mathbb{C}^n preserving f up to a constant factor ($\alpha : \overline{G}_f \rightarrow \mathbb{C}^*$ is a one-dimensional representation of the group \overline{G}_f). In other words, this is the group of diagonal linear transformations preserving the hypersurface $X = \{f = 0\} \subset \mathbb{C}^n$. The group \overline{G}_f contains both the symmetry group G_f and the group \mathbb{C}^* corresponding to the action (2) and is generated by these two subgroups. The intersection of G_f and \mathbb{C}^* is the cyclic subgroup of order d in \mathbb{C}^* . (It is generated by the monodromy transformation of f : see below.)

For a group G , let $R(G)$ be the ring of complex representations of G . As an abelian group, $R(G)$ is freely generated by the isomorphism classes of irreducible representations of G . For an abelian group G (say, for a subgroup of G_f or of \overline{G}_f) all irreducible representations are one-dimensional, i.e. are elements of the group of characters $G^* = \text{Hom}(G, \mathbb{C}^*)$.

Let \overline{G} be a subgroup of the extended symmetry group \overline{G}_f of the polynomial f containing the subgroup \mathbb{C}^* , and let $G = \overline{G} \cap G_f$. We shall call an irreducible

(one-dimensional) representation α of \overline{G} *non-positive* ($\alpha \leq 0$) or *negative* ($\alpha < 0$) if, for $a \in \mathbb{C}^* \subset \overline{G}$, one has $\alpha(a) = a^k$ with k non-positive or negative respectively. Let $R_-(\overline{G})$ be the subring of $R(\overline{G})$ generated by all non-positive representations. The ring $R_-(\overline{G})$ contains the ideal I generated by all negative representations of \overline{G} . Let $\widehat{R}_-(\overline{G})$ be the completion of the ring $R_-(\overline{G})$ with respect to the ideal I . Elements of the ring $\widehat{R}_-(\overline{G})$ are (formal) sums of the form $\sum_{\alpha \in \overline{G}^*, \alpha \leq 0} s_\alpha [\alpha]$ with integer coefficients s_α . Let $\widehat{I} \subset \widehat{R}_-(\overline{G})$ be the corresponding completion of the ideal I .

One has the following homomorphism (isomorphism) Exp from \widehat{I} regarded as a group with respect to addition to $1 + \widehat{I}$ regarded as a group with respect to multiplication:

$$\text{Exp}\left(\sum_{\alpha \in \overline{G}^*, \alpha < 0} s_\alpha [\alpha]\right) = \prod_{\alpha \in \overline{G}^*, \alpha < 0} (1 - [\alpha])^{-s_\alpha}.$$

The inverse is the homomorphism $\text{Log} : 1 + \widehat{I} \rightarrow \widehat{I}$ (cf. [GLM]).

Let $A_X = \mathbb{C}[x]/(f)$ be the coordinate ring of the zero level set $X = f^{-1}(0)$. The group \overline{G} acts both on $\mathbb{C}[x]$ and on A_X by $a * g(x) = g(a^{-1} * x)$ for $a \in \overline{G}$. This representation is negative on any \overline{G} -invariant one-dimensional subspace of $\mathbb{C}[x]$ or of A_X (say, on the subspace generated by a monomial) except the one consisting of the constant functions on which it is non-positive. Let A be either $A_{\mathbb{C}^n} = \mathbb{C}[x]$ or A_X . For a (one-dimensional) irreducible representation α of \overline{G} , let A^α be the corresponding subspace of A : $A^\alpha = \{g \in A : a * g = \alpha(a)g \text{ for } a \in \overline{G}\}$. For each α , the subspace A^α is finite-dimensional and is generated by monomials.

Definition 1. The \overline{G} -equivariant Poincaré series $P^{\overline{G}}$ of the ring A is the element of the completion $\widehat{R}_-(\overline{G})$ of the representation ring of the group \overline{G} defined by

$$(3) \quad P^{\overline{G}} = \sum_{\alpha \in \overline{G}^*, \alpha \leq 0} \dim A^\alpha \cdot [\alpha].$$

Remark. For $\overline{G} = \mathbb{C}^*$, an irreducible \mathbb{C}^* -representation is a power of the tautological representation. If one denotes the inverse of the tautological representation by t , one gets the Poincaré series (3) as a power series in t . In this case it coincides with the usual Poincaré series of the ring A corresponding to the (quasihomogeneous) grading defined by the weights.

For $A = A_X$ or $A_{\mathbb{C}^n}$ we shall denote the \overline{G} -equivariant Poincaré series $P^{\overline{G}}$ by $P_X^{\overline{G}}$ and $P_{\mathbb{C}^n}^{\overline{G}}$ respectively. Let $\alpha_{x_i}, i = 1, \dots, n$, and α_f be the representations of \overline{G} on the one-dimensional subspaces in $\mathbb{C}[x]$ generated by x_i and f respectively.

Proposition 2.

$$P_X^{\overline{G}} = \frac{1 - [\alpha_f]}{\prod_{i=1}^n (1 - [\alpha_{x_i}])}.$$

Proof. The proof is essentially the same as in the non-equivariant case. One has

$$P_{\mathbb{C}^n}^{\overline{G}} = \sum_{(k_1, \dots, k_n) \in \mathbb{Z}_{\geq 0}^n} [\alpha_{x_1}^{k_1} \cdots \alpha_{x_n}^{k_n}] = \frac{1}{\prod_{i=1}^n (1 - [\alpha_{x_i}])}.$$

One considers the exact sequence

$$0 \rightarrow A_{\mathbb{C}^n} \xrightarrow{f} A_{\mathbb{C}^n} \xrightarrow{\pi} A_X \rightarrow 0.$$

The homomorphism π maps $A_{\mathbb{C}^n}^\alpha$ to A_X^α . If $g \in A_{\mathbb{C}^n}^\alpha$, i.e. $a * g = \alpha(a)g$ for $a \in \overline{G}$, then $a * (fg) = \alpha(a)\alpha_f(a) \cdot fg$, i.e. $fg \in A_{\mathbb{C}^n}^{\alpha \cdot \alpha_f}$. Therefore $\dim A_X^\alpha = \dim A_{\mathbb{C}^n}^\alpha - \dim A_{\mathbb{C}^n}^{\alpha/\alpha_f}$. This yields the statement. \square

Corollary 3.

$$\text{Log } P_X^{\overline{G}} = \sum_{i=1}^n [\alpha_{x_i}] - [\alpha_f].$$

Note that $\text{Log } P_X^{\overline{G}}$ is an element of the representation ring $R(\overline{G})$ (more precisely of the subring $R_-(\overline{G}) \subset R(\overline{G})$), not only of the completion $\widehat{R}_-(\overline{G})$.

Now we recall the necessary definitions and facts about Burnside rings of finite groups (for more details see, e.g., [EG3]) and give an appropriate extension of this notion to subgroups of \overline{G}_f containing \mathbb{C}^* .

Let G be a finite group. A G -set is a set with an action of G . A G -set is *irreducible* if the action of G on it is transitive. Isomorphism classes of irreducible G -sets are in one-to-one correspondence with conjugacy classes of subgroups of G : to the conjugacy class containing a subgroup $H \subset G$ one associates the isomorphism class $[G/H]$ of the G -set G/H . The *Grothendieck ring* $K_0(\text{f.}G\text{-sets})$ of finite G -sets (also called the *Burnside ring* of G ; see, e.g., [K]) is the (abelian) group generated by the isomorphism classes of finite G -sets modulo the relation $[A \amalg B] = [A] + [B]$ for finite G -sets A and B . Multiplication in $K_0(\text{f.}G\text{-sets})$ is defined by the cartesian product. As an abelian group, $K_0(\text{f.}G\text{-sets})$ is freely generated by the isomorphism classes of irreducible G -sets. The element 1 in $K_0(\text{f.}G\text{-sets})$ is represented by the G -set consisting of one point (with the trivial G -action).

There is a natural homomorphism from $K_0(\text{f.}G\text{-sets})$ to $R(G)$ which sends a G -set X to the (vector) space of (complex valued) functions on X .

For a subgroup $H \subset G$ there are natural maps $\text{Res}_H^G : K_0(\text{f.}G\text{-sets}) \rightarrow K_0(\text{f.}H\text{-sets})$ and $\text{Ind}_H^G : K_0(\text{f.}H\text{-sets}) \rightarrow K_0(\text{f.}G\text{-sets})$. The *restriction map* Res_H^G sends a G -set X to the same set considered with the H -action. The *induction map*

Ind_H^G sends an H -set X to the product $G \times X$ factorized by the natural equivalence: $(g_1, x_1) \sim (g_2, x_2)$ if there exists $g \in H$ such that $g_2 = g_1 g$, $x_2 = g^{-1} x_1$ with the natural (left) G -action. The induction map Ind_H^G sends $[H/H']$ (H' is a subgroup of H) to $[G/H']$. Both maps are group homomorphisms, but Ind_H^G is not a ring homomorphism.

For an action of a group G on a set X and for $x \in X$, let $G_x = \{g \in G : gx = x\}$ be the isotropy group of x . For a subgroup $H \subset G$ let $X^{(H)} = \{x \in X : G_x = H\}$ be the set of points with isotropy group H .

We recall the definition of the G -equivariant zeta function of f from [EG3]. See an explanation of this notion therein.

The monodromy transformation of f can be defined as the element $h = h_f \in G_f$ given by

$$h = (\exp(2\pi i q_1/d), \dots, \exp(2\pi i q_n/d)).$$

As a map from the Milnor fibre $V_f = f^{-1}(1)$ to itself, h defines a (faithful) action of the cyclic group $\mathbb{Z}_d = \langle h \rangle$ of order d on V_f . Let

$$(4) \quad \zeta_f(t) = \prod_{q \geq 0} (\det(\text{id} - t \cdot h_*|_{H_q(V_f)}))^{(-1)^q}$$

be the (classical) monodromy zeta function of f (that is, the zeta function of the transformation h). One can show that in the above-described situation one has

$$\zeta_f(t) = \prod_{m|d} (1 - t^m)^{s_m},$$

where $s_m = \chi(V_f^{(\mathbb{Z}_d/m)})/m$ are integers. If in (4) one considers the action of h_* on the reduced homology groups of V_f , one obtains the *reduced monodromy zeta function* $\tilde{\zeta}_f(t) = \zeta_f(t)/(1 - t)$.

There is a natural one-to-one correspondence between functions of the form

$$(5) \quad \varphi(t) = \prod_{m|d} (1 - t^m)^{s_m}$$

and elements of the Burnside ring $K_0(\text{f.}\mathbb{Z}_d\text{-sets})$ of the cyclic group \mathbb{Z}_d (see [EG3]). The function $\varphi(t)$ from (5) corresponds to the element

$$\sum_{m|\bar{d}} s_m [\mathbb{Z}_d/\mathbb{Z}_d/m] \in K_0(\text{f.}\mathbb{Z}_d\text{-sets}).$$

For the monodromy transformation h_f in these terms one has

$$(6) \quad \zeta_f = \sum_{H \subset \mathbb{Z}_d} \chi(V_f^{(H)}/\mathbb{Z}_d) [\mathbb{Z}_d/H] \in K_0(\text{f.}\mathbb{Z}_d\text{-sets}).$$

The coefficient $\chi(V_f^{(H)}/\mathbb{Z}_d)$ is the Euler characteristic of the space (manifold) of orbits of type \mathbb{Z}_d/H in V_f .

Now let G be a subgroup of the symmetry group G_f of the quasihomogeneous polynomial f containing the monodromy transformation h . Equation (6) inspires the following definition (see [EG3]).

Definition 4. The G -equivariant zeta function of f is the element

$$(7) \quad \zeta_f^G = \sum_{H \subset G} \chi(V_f^{(H)}/G)[G/H]$$

of the Burnside ring $K_0(\text{f.}G\text{-sets})$.

The coefficient $\chi(V_f^{(H)}/G)$ is the Euler characteristic of the space (manifold) of orbits of type G/H in V_f .

Definition 5. The reduced G -equivariant zeta function of f is $\tilde{\zeta}_f^G = \zeta_f^G - 1$.

Let $K_0(\text{f.}\overline{G}\text{-sets})$ be the Grothendieck group of \overline{G} -sets with finite numbers of orbits and finite isotropy groups of points. This group is freely generated by the classes of the \overline{G} -sets \overline{G}/H with finite subgroups H .

Remark. There is no natural ring structure on $K_0(\text{f.}\overline{G}\text{-sets})$.

As above (for finite groups), one also has the natural induction map $\text{Ind}_{\overline{G}}^{\overline{G}} : K_0(\text{f.}G\text{-sets}) \rightarrow K_0(\text{f.}\overline{G}\text{-sets})$ which sends $[G/H]$ to $[\overline{G}/H]$ for a subgroup $H \subset G$. A left inverse to this map is the reduction map $\text{Red} : K_0(\text{f.}\overline{G}\text{-sets}) \rightarrow K_0(\text{f.}G\text{-sets})$ which sends $[\overline{G}/H]$ to $[G/H \cap G]$ ($H \subset \overline{G}$, $|H| < \infty$).

The group \overline{G} acts on the zero level set X of f . The \overline{G} -equivariant orbit invariant $\text{Or}_X^{\overline{G}}$ counts the orbits of the \overline{G} -action on X of different types.

Definition 6 (cf. the definition in [EG1], [EG2]).

$$\text{Or}_X^{\overline{G}} := \sum_{H \subset \overline{G}, |H| < \infty} \chi(X^{(H)}/\overline{G})[\overline{G}/H] \in K_0(\text{f.}\overline{G}\text{-sets}).$$

Let the tautological map Tau from the ideal $I \subset R_-(\overline{G})$ to the group $K_0(\text{f.}\overline{G}\text{-sets})$ be the (additive) group homomorphism mapping the class $[\alpha]$ of a one-dimensional representation α to the class of the punctured space \mathbb{C}^* of the space (line) \mathbb{C}^1 of the representation with the action of \overline{G} defined by α . Note that $\text{Tau}[\alpha]$ can be defined in the same way for a positive representation α and one has $\text{Tau}[\alpha] = \text{Tau}[\alpha^{-1}]$ (via the isomorphism of \mathbb{C}^* with itself which sends $z \in \mathbb{C}^*$ to z^{-1}).

Theorem 7. *One has*

$$(8) \quad \text{Tau}(\text{Log } P_X^{\overline{G}}) - \text{Or}_X^{\overline{G}} = \text{Ind}_G^{\overline{G}} \tilde{\zeta}_f^G$$

in $K_0(\text{f.}\overline{G}\text{-sets})$.

Proof. For $I \subset I_0 = \{1, \dots, n\}$, let $|I|$ be the number of elements of I , let $(\mathbb{C}^*)^I := \{(x_1, \dots, x_n) \in \mathbb{C}^n : x_i \neq 0 \text{ for } i \in I, x_i = 0 \text{ for } i \notin I\}$ be the corresponding coordinate torus of dimension $|I|$ ($(\mathbb{C}^*)^\emptyset = \emptyset$), and let $G^I \subset \overline{G}$ be the isotropy subgroup $\{a \in \overline{G} : ax = x \text{ for all } x \in (\mathbb{C}^*)^I\}$. Let $X^I = X \cap (\mathbb{C}^*)^I$ and $Y^I = X^I/\overline{G}$. One has

$$\text{Or}_X^{\overline{G}} = \sum_I \chi(Y^I) \cdot [\overline{G}/G^I].$$

The Milnor fibre $V_f = f^{-1}(1)$ is the union $\bigcup_I (V_f \cap (\mathbb{C}^*)^I)$ of G -invariant varieties. Therefore

$$\tilde{\zeta}_f^G = \sum_I \chi((V_f \cap (\mathbb{C}^*)^I)/G)[G/G^I] - 1 = \sum_I [\chi((\mathbb{C}^*)^I/\overline{G}) - \chi(Y^I)][G/G^I] - 1.$$

Note that, if $G^I \not\subset G$, then $V_f \cap (\mathbb{C}^*)^I = \emptyset$. For $|I| \neq 1$, one has $\chi((\mathbb{C}^*)^I/\overline{G}) = 0$, and for $|I| = 1$, $\chi((\mathbb{C}^*)^I/\overline{G}) = 1$. Therefore

$$\text{Or}_X^{\overline{G}} + \text{Ind}_G^{\overline{G}} \tilde{\zeta}_f^G = \sum_{i=1}^n [\overline{G}/G^{\{i\}}] - \text{Ind}_G^{\overline{G}} 1 = \text{Tau}(\text{Log } P_X^{\overline{G}}). \quad \square$$

Remark. 1. One could prefer to have an equation like (8) in the Burnside ring $K_0(\text{f.}G\text{-sets})$. One can see that (8) implies the equation

$$(9) \quad \tilde{\zeta}_f^G = \text{Red}(\text{Tau}(\text{Log } P_X^{\overline{G}}) - \text{Or}_X^{\overline{G}})$$

in $K_0(\text{f.}G\text{-sets})$ with the reduction map $\text{Red} : K_0(\text{f.}\overline{G}\text{-sets}) \rightarrow K_0(\text{f.}G\text{-sets})$. However from a formal point of view (9) is weaker than (8).

2. Looking at the relation (8), one observes that there is no Saito duality in the sense of [EG3] involved in it. It appears in (1) because of the method used to encode the \mathbb{C}^* -action on X in $\text{Or}_X(t)$.

3. One can see that, generally speaking, both $\text{Or}_X^{\overline{G}}$ and $\tilde{\zeta}_f^G$ contain much more summands than $\text{Tau}(\text{Log } P_X^{\overline{G}})$. In particular, $\text{Tau}(\text{Log } P_X^{\overline{G}})$ contains only summands represented by irreducible \overline{G} -sets isomorphic (as varieties) to \mathbb{C}^* . This gives the hint that (8) (and therefore also (1)) is essentially a relation between $\text{Or}_X^{\overline{G}}$ and $\tilde{\zeta}_f^G$, where the Poincaré series $P_X^{\overline{G}}$ plays rather the role of a correction term.

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