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Gelfand–Tsetlin bases for spherical monogenics in dimension 3

Sebastian Bock, Klaus Gürlebeck, Roman Lávička and Vladimír Souček

Abstract. The main aim of this paper is to recall the notion of Gelfand–Tsetlin bases (GT bases for short) and to use it for an explicit construction of orthogonal bases for the spaces of spherical monogenics (i.e., homogeneous solutions of the Dirac or the generalized Cauchy–Riemann equation, respectively) in dimension 3. In the paper, using the GT construction, we obtain explicit orthogonal bases for spherical monogenics in dimension 3. We compare them with those constructed by the first and the second author recently (by a direct analytic approach) and we show in addition that the GT basis has the Appell property with respect to all three variables. The last fact is quite important for future applications.

1. Introduction

The main aim of this paper is to discuss explicit constructions of orthogonal bases for the spaces of spherical monogenics (i.e., homogeneous solutions of the Dirac or the generalized Cauchy–Riemann equation, respectively), mainly in dimension 3. The theory of solutions to the Dirac or to the Cauchy–Riemann operator can be seen at the same time as a generalization of the (one-dimensional) complex function theory as well as a refinement of harmonic analysis. Both function classes share many properties with each other and are quite analogous to the complex case. The theory for the solutions of the Cauchy–Riemann operator contains the concept of hypercomplex derivability whereas in the case of the Dirac equation, due to the full rotational invariance of the solutions, more tools from harmonic analysis are directly applicable.

Constructing orthogonal bases for spaces of solutions of differential equations is, in general, a difficult problem. We show in the first part of the paper that the

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approach formulated by Gelfand and Tsetlin makes a construction of orthogonal bases easier in case of the Dirac equation.

The notion of a Gelfand–Tsetlin basis (GT basis) was formulated for irreducible (finite dimensional) modules over a general classical simple Lie algebra $\mathfrak{g}$ (see [27] for the original paper, and [37] for a review paper with many further references). The main problem solved in [27] was to write down matrices representing basis elements of $\mathfrak{g}$ with respect to the GT basis. In the case when an irreducible $\mathfrak{g}$-module is realized explicitly (usually as a subspace of the space of solutions of invariant differential equations), it is often possible to construct its GT basis in a quite algorithmic way. The main advantage of GT bases for practical applications is the fact that a GT basis is automatically orthogonal with respect to any invariant inner product on the given irreducible module.

The problem of constructing basis functions in spaces of monogenic functions has a long history. In the very beginning the task was to construct sufficiently many concrete monogenic functions. Already the work of R. Fueter contains the idea to consider a special kind of homogeneous monogenic polynomials as generalization of the complex powers z^n and to look for an analogue of the Taylor series expansion. The result was a series expansion in Fueter polynomials [26]. The important gain compared with the real Taylor series expansion for a real analytic function was the possibility to express the increment of a quaternion-valued functions by a hypercomplex increment of the arguments. Much later, in [8], these series were reinvented and in [33] connected with the problem of hypercomplex derivability. Finally, it was shown that for Clifford algebra valued functions the existence of a local Taylor series expansion in the symmetric powers [33], hypercomplex derivability and monogenicity are equivalent, which is a very comfortable situation and advantageous for the solution of more complicated differential equations by means of monogenic functions. With the needs of numerical approximations, motivated also by geometrical properties and invariance properties, a construction of simple orthogonal systems of monogenic polynomials was needed. These problems were connected with the idea of the Fischer decomposition (originally in the paper [25]) and with the so called Almansi decompositions (see the references in [35]). The main disadvantage of the Fueter polynomials for numerical purposes was that they are not orthogonal with respect to the L_2-inner product. That is why it was not possible to relate Taylor and Fourier expansions so easily as in the complex case, i.e., to relate the local and the global behaviour of the functions. The first explicit constructions of complete orthonormal polynomial systems in the important case of dimension 3 were made by I. Cação [16], and the first and the second authors and H. Malonek [14], [13], [15]. The main idea was the application of the Cauchy–Riemann operator to an orthogonal system of spherical harmonics and an explicit orthonormalization of the resulting system. These results were the basis for Fourier expansions and related applications like the definition of a continuous operator of monogenic primitivation in the L_2-space of monogenic functions.

Furthermore, in [22] (pages 254–264) and [40], [42], [32], another construction of orthogonal bases for spherical monogenics even in all dimensions is explained. In particular, in [22] (Theorem 2.2.3, p. 315), the so-called Cauchy–

Kovalevskaya (CK) method was developed. However, this method is not used in [22] for a construction of orthogonal bases, although the construction is obvious not only in dimension 3 but in an arbitrary dimension as we explain in Section 3. Actually, in this paper we use the CK method to obtain an explicit construction of the GT bases for spherical monogenics in dimension 3. In [31], the GT bases for this case are obtained in quite a different way and, in particular, simple expressions of elements of these bases in terms of the Legendre polynomials are given there. By the way, the Cauchy–Kovalevskaya method is applicable in other settings as well, see [11], [12], [9], [10] and [21]. Similar questions were also considered by R. Delanghe for the Riesz system, see [19] and also [43], [38].

Looking back at the complex case we observe that the basis functions for Taylor and Fourier expansions are principally the same; they are real multiples of each other. An important property of this basis is the so-called Appell property of the system $\{z^n\}_{n\in\mathbb{N}}$ with respect to the complex derivative. Originally, P. Appell introduced in [3] polynomials with the property that $\frac{d}{d_x}P_n(x) = n\,P_{n-1}(x)$. This property makes it possible to differentiate and integrate power series expansions easily, summand by summand, and to obtain immediately a series of the same structure. Later on, Sheffer [39] invented generating functions to construct Appell systems or Appell sequences, and depending on the interests of the authors nowadays one or the other of these approaches is preferred.

The generalization of the Appell idea to monogenic polynomials (as solutions of the Cauchy–Riemann equations) requires a correct understanding of the hypercomplex derivative (see [41], [36] and [29]). First Appell systems of paravector-valued monogenic polynomials were constructed by H. Malonek et. al., [34], [23], [24]. These systems were orthogonal but not complete with respect to the L_2-inner product and it was observed that the system coincides also with a system of "special monogenic functions" as constructed in [1] without mentioning the Appell property. In [30] it was shown that the same Appell system can be obtained by the Fueter–Sce extension of the complex Appell system $\{z^n\}_{n\in\mathbb{N}}$. In [18], I. Cação and H. Malonek constructed an orthogonal Appell basis in L_2, equipped with the real inner product, for the solutions of the Riesz system in dimension 3. Later, in a series of papers ([7], [6], [5]), the first and the second author construct an orthogonal Appell basis of monogenic polynomials for the space of square integrable solutions of the Cauchy–Riemann system in $\mathbb{R}^3$ (Moisil–Teodorescu system) with respect to the quaternion-valued inner product. In [6], this system was used to approximate solutions of the Lamé–Navier equations of linear elasticity theory.

Important for practical applications is also that this Appell system can be defined recursively (see [5] and Theorem 5.4 below) and that it is not longer necessary to start with spherical harmonics.

The question arises if this system is only one that fortunately could be constructed or if it is unique (in a certain sense). Because of the increasing number of calculations, it becomes important to understand the general principle underlying the constructions to find a way to construct bases in all dimensions. First results were obtained in [4], where a unified and explicit principle for construction monogenic Appell bases in dimension 2, 3 and 4 was described.

In low dimensions (3 or 4), it is quite common to consider quaternion valued functions instead of spinor valued ones, and to replace complex vector spaces of solutions with vector spaces over the skew field of real quaternions. Analyzing all the mentioned concrete results on Appell systems of monogenic polynomials and relating them to the case of the Dirac equation, it becomes apparent that there is some general scheme in the background – the so-called Gelfand–Tsetlin bases. It is possible to relate both pictures, and we shall do so below.

In this paper, we apply a general scheme of GT bases to the case of spherical monogenics in dimension 3, and we write down explicit formulae for the corresponding orthogonal GT bases in terms of spinor valued and quaternion valued functions. The elements of the obtained bases can be easily renormalized to have the Appell property. Actually, it turns out that such a requirement characterizes the bases uniquely (see Theorem 5.1 below). We compare then the formulae obtained for quaternion valued functions with those obtained by the first and the second author in [7] and we show that they coincide.

A substantial new information obtained in the paper is the fact that the GT basis for spinor valued monogenic polynomials has the important Appell property not only for the last variable but with respect to all three variables. Hence partial derivatives (with respect to all three variables) of a basis element are multiples of other basis elements. In Section 6 (see Definition 6.2), we introduce the Taylor series for L_2-integrable monogenic functions. As a consequence of the generalized Appell property (see Section 4, Remark 4.5), partial derivatives of the Taylor expansion have a very simple form, the corresponding coefficients being given as multiples of those of the original series. This makes it possible to compute partial derivatives of a monogenic function using only coefficients in its Taylor series.

In Section 2, we start with a short summary of the notation needed to formulate a general construction of the GT bases. In Section 3, we show that the branching rules needed to perform the construction of the GT bases explicitly can be realized using only classical tools of Clifford analysis, namely, the Fischer decomposition and the Cauchy–Kovalevskaya extension. Actually, we just apply the Cauchy–Kovalevskaya method developed already in Theorem 2.2.3, p. 315, of [22]. In the rest of the paper, we study properties of GT bases mainly in dimension 3. A detailed study of GT bases in higher dimensions will be given in a subsequent paper. An explicit construction of the GT bases in dimension 3 is written down in Section 4; see Theorem 4.2 and Corollary 4.3. To do it, we use the Fischer decomposition in dimension 2 in the same way as is done in higher dimensions. Let us remark that the Fischer decomposition in dimension 2 (see Theorem 4.1) is not usually considered in Clifford analysis and it has a slightly different form than in higher dimensions. In particular, we show that the GT bases for spinor valued spherical monogenics in dimension 3 possess a generalization of the Appell property, that is, they possess an Appell property not only with respect to the last real variable x_3 but also with respect to the remaining complex variables z and $\overline{z}$; see Corollary 4.3. Finally, in Section 5, we introduce the quaternionic formulation and we describe its relation to the spinor case. We reformulate the GT bases in quaternionic language (see Theorem 5.1 and Corollary 5.2 below) and we show that the bases having the Appell property coincide with those constructed by the first and the second author in [7] for the Cauchy–Riemann system. This system has the Appell property with respect to the hypercomplex derivative on the basis polynomials orthogonal to the hyperholomorphic constants and then with respect to a complex derivative on the remaining basis functions. At the end of the paper we present some applications of both approaches and construct new Taylor series and Fourier series expansions, respectively.

2. Preliminaries

First we introduce some notation. Let $(e_1, \ldots, e_m)$ be the standard basis of the Euclidean space $\mathbb{R}^m$ and let $\mathbb{C}_m$ be the complex Clifford algebra generated by the vectors $e_1, \ldots, e_m$ such that $e_j^2 = -1$ for $j = 1, \ldots, m$. As usual, we identify a vector $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ with the element $x_1 e_1 + \cdots + x_m e_m$ of $\mathbb{C}_m$. Recall that the Spin group $\mathrm{Spin}(m)$ is defined as the set of products of an even number of unit vectors of $\mathbb{R}^m$ endowed with the Clifford multiplication. Now we introduce the spaces of spherical monogenics. For a vector space V, we denote by $\mathcal{P}_k(\mathbb{R}^m, V)$ the space of V-valued polynomials in $\mathbb{R}^m$ which are homogeneous of degree k. Let S be a subspace of $\mathbb{C}_m$ invariant with respect to left multiplication by elements of $\mathrm{Spin}(m)$. Then put

$$\mathcal{M}_k(\mathbb{R}^m, S) = \left\{P \in \mathcal{P}_k(\mathbb{R}^m, S) : \partial P = 0\right\}, \tag{2.1}$$

where the Dirac operator ∂ in $\mathbb{R}^m$ is defined by

$$\partial = e_1 \frac{\partial}{\partial x_1} + \cdots + e_m \frac{\partial}{\partial x_m}.$$

It is well known that if S is a basic spinor representation of the group $\mathrm{Spin}(m)$ then the space $\mathcal{M}_k(\mathbb{R}^m, S)$ of spherical monogenics is an irreducible module under the so-called L-action, defined by

$$[L(s)(P)](x) = s\,P(s^{-1}xs), \quad s \in \mathrm{Spin}(m) \ \text{ and } \ x = (x_1, \ldots, x_m) \in \mathbb{R}^m.$$

In this paper, we are interested in a construction of GT bases of spherical monogenics. Let us recall briefly the concept of GT bases for the orthogonal case; see [37], [27]. In what follows, we deal with complex representations of the Lie algebra $\mathfrak{so}(m)$ of the Spin group $\mathrm{Spin}(m)$. Let us consider a general irreducible $\mathfrak{so}(m)$-module $V(\mu_m)$ with the highest weight μ_m. In the even dimensional case $m = 2n$, the highest weight μ_m is a vector

$$\mu_m = (\lambda_{m,1}, \ldots, \lambda_{m,n})$$

consisting entirely of integers or entirely of non-zero half integers which satisfy the relation

$$\lambda_{m,1} \geq \lambda_{m,2} \geq \cdots \geq \lambda_{m,n-1} \geq |\lambda_{m,n}|. \tag{2.2}$$

In the odd dimensional case $m = 2n+1$, the vector $\mu_m = (\lambda_{m,1}, \ldots, \lambda_{m,n})$ satisfies instead the condition

$$\lambda_{m,1} \geq \lambda_{m,2} \geq \cdots \geq \lambda_{m,n} \geq 0. \tag{2.3}$$

Furthermore, as is well known, the Lie algebra $\mathfrak{so}(m)$ can be realized as the space of bivectors of the Clifford algebra $\mathbb{C}_m$. In what follows, we consider a chain of Lie algebras

$$\mathfrak{so}(m) \supset \mathfrak{so}(m-1) \supset \cdots \supset \mathfrak{so}(2)\,, \tag{2.4}$$

where, for $k = 2, \ldots, m$,

$$\mathfrak{so}(k) = \big\langle \{e_{ij} : 1 \leq i < j \leq k\} \big\rangle.$$

Here $e_{ij} = e_i e_j$ and $\langle M \rangle$ stands for the span of a set M.

The key ingredient for the introduction of a GT basis is the following branching rule, well known in representation theory: As an $\mathfrak{so}(m-1)$-module, the given module $V(\mu_m)$ decomposes into a multiplicity free direct sum of irreducible $\mathfrak{so}(m-1)$-modules,

$$V(\mu_m) = \bigoplus_{\mu_{m-1}} V(\mu_m, \mu_{m-1}) \tag{2.5}$$

where the direct sum is taken over the highest weights μ_{m-1} satisfying the conditions (2.6) and (2.7) below. Moreover, it is well known that if the weight μ_m consists entirely of non-zero half integers (or integers), then so do all the highest weights μ_{m-1}. In the case when $m = 2n$, the direct sum (2.5) is taken over all the highest weights $\mu_{m-1} = (\lambda_{m-1,1}, \ldots, \lambda_{m-1,n-1})$ such that

$$\lambda_{m,1} \geq \lambda_{m-1,1} \geq \lambda_{m,2} \geq \cdots \geq \lambda_{m,n-1} \geq \lambda_{m-1,n-1} \geq |\lambda_{m,n}|. \tag{2.6}$$

In the case when $m = 2n+1$, the direct sum (2.5) is taken over all the highest weights $\mu_{m-1} = (\lambda_{m-1,1}, \ldots, \lambda_{m-1,n})$ such that

$$\lambda_{m,1} \geq \lambda_{m-1,1} \geq \lambda_{m,2} \geq \cdots \geq \lambda_{m,n-1} \geq \lambda_{m-1,n-1} \geq \lambda_{m,n} \geq |\lambda_{m-1,n}|. \tag{2.7}$$

Moreover, with respect to any given invariant inner product on the module $V(\mu_m)$, the decomposition (2.5) is even orthogonal.

Of course, we can decompose further each module $V(\mu_m, \mu_{m-1})$ of the decomposition (2.5) into irreducible $\mathfrak{so}(m-2)$-modules $V(\mu_m, \mu_{m-1}, \mu_{m-2})$ and so on. Hence we end up with the decomposition of the given $\mathfrak{so}(m)$-module $V(\mu_m)$ into irreducible $\mathfrak{so}(2)$-modules $V(\mu)$. Moreover, any such module $V(\mu)$ is uniquely determined by the so-called Gelfand–Tsetlin pattern:

$$\mu = (\mu_m, \mu_{m-1}, \ldots, \mu_2). \tag{2.8}$$

Here, μ as in (2.8) is called the Gelfand–Tsetlin pattern provided that each vector μ_j satisfies the conditions (2.2)–(2.7) (with m replaced by j) and the numbers $\lambda_{j,k}$ are either all integers or all non-zero half integers. We denote by $P(\mu_m)$

the set of the Gelfand–Tsetlin patterns whose first term is the highest weight μ_m. To summarize, we decompose the given module $V(\mu_m)$ into the direct sum of irreducible $\mathfrak{so}(2)$-modules

$$V(\mu_m) = \bigoplus_{\mu \in P(\mu_m)} V(\mu). \tag{2.9}$$

Moreover, the decomposition (2.9) is obviously orthogonal. Let us note that the decomposition (2.9) is uniquely specified by the choice of the chain of Lie subalgebras (2.4).

Since all submodules $V(\mu)$ are, in fact, one-dimensional we obtain easily an orthogonal basis of $V(\mu_m)$ by taking a non-zero vector $e(\mu)$ from each module $V(\mu)$. The orthogonal basis

$$E = \{e(\mu) : \mu \in P(\mu_m)\}$$

is then called a GT basis of the module $V(\mu_m)$. It is easily seen that, by the definition, the vector $e(\mu)$ is uniquely determined by $\mu \in P(\mu_m)$ up to a scalar multiple.

3. The Cauchy–Kovalevskaya method

To construct a GT basis for the $\mathfrak{so}(m)$-module $\mathcal{M}_k(\mathbb{R}^m, S)$ it is clear that we need to describe quite explicitly the branching rule (2.5) for this module, that is, its decomposition into irreducible $\mathfrak{so}(m-1)$-submodules. To this end we use only two basic tools from Clifford analysis, namely, the Cauchy–Kovalevskaya extension and the Fischer decomposition of spinor-valued polynomials. Actually, we just apply the Cauchy–Kovalevskaya method developed already in [22] (Theorem 2.2.3, page 315). We first state the Fischer decomposition, see [22], page 206.

Proposition 3.1. *Let $m \geq 3$ and let S be a spinor space of the Clifford algebra $\mathbb{C}_m$, that is, S is an irreducible (left) module over $\mathbb{C}_m$. Then,*

$$\mathcal{P}_k(\mathbb{R}^m, S) = \bigoplus_{j=0}^{k} x^j \mathcal{M}_{k-j}(\mathbb{R}^m, S).$$

Remark 3.2. An analogous decomposition is valid also in the dimension $m = 2$; see Theorem 4.1 below for details.

Now we recall the Cauchy–Kovalevskaya extension. Let p be a k-homogeneous polynomial in $\mathbb{R}^m$ which takes values in a spinor space S of $\mathbb{C}_m$. Such a polynomial p can be uniquely expressed as

$$p(x) = \sum_{j=0}^{k} p_j(\underline{x})\, x_m^j\,,$$

where p_j is an S-valued polynomial in $\underline{x} = (x_1, \ldots, x_{m-1}) \in \mathbb{R}^{m-1}$ which is homogeneous of degree $k - j$.

Moreover, putting

$$\underline{\partial} = e_1 \frac{\partial}{\partial x_1} + \cdots + e_{m-1} \frac{\partial}{\partial x_{m-1}},$$

it is easy to see that the Dirac equation $\partial p = 0$ holds if and only if, for each $j = 0, \ldots, k$,

$$p_j = \frac{1}{j}(e_m \underline{\partial})\, p_{j-1} = \cdots = \frac{1}{j!}(e_m \underline{\partial})^j p_0.$$

In this case, we have therefore that

$$p(x) = \sum_{j=0}^{k} \frac{1}{j!}(e_m x_m \underline{\partial})^j p_0(\underline{x}) = (e^{e_m x_m \underline{\partial}} p_0)(x).$$

Now it is easy to obtain the following result, see [22], page 152:

Proposition 3.3. *Let S be a basic spinor representation of the group* Spin(m). *Then the Cauchy–Kovalevskaya extension operator*

$$CK = e^{e_m x_m \underline{\partial}}$$

is an $\mathfrak{so}(m-1)$-invariant isomorphism of the module $\mathcal{P}_k(\mathbb{R}^{m-1}, S)$ onto the module $\mathcal{M}_k(\mathbb{R}^m, S)$.

As we explain later, to describe explicitly the branching rules in our situation we need to understand the CK extension of particular terms in the Fischer decomposition, that is, the CK extension of polynomials of the form $\underline{x}^j p(\underline{x})$, with p being a spherical monogenic. But first recall that the Gegenbauer polynomial C_j^{ν} is defined as

$$C_j^{\nu}(z) = \sum_{i=0}^{[j/2]} \frac{(-1)^i (\nu)_{j-i}}{i!(j-2i)!}(2z)^{j-2i} \quad \text{with} \quad (\nu)_j = \nu(\nu+1)\cdots(\nu+j-1); \tag{3.1}$$

see page 302 of [2].

Lemma 3.4. *Let $j \in \mathbb{N}_0$ and $p \in \mathcal{M}_k(\mathbb{R}^{m-1}, S)$. Then we have that*

$$CK((\underline{x} e_m)^j p(\underline{x})) = X^{(j)} p(\underline{x})\,,$$

where $X^{(0)} = 1$ and, for $j \in \mathbb{N}$, the polynomial $X^{(j)} = X_k^{(j)}$ is given by

$$X_k^{(j)}(\underline{x}, x_m) = \mu_k^j r^j \left(C_j^{m/2+k-1}(\frac{x_m}{r}) + \frac{m+2k-2}{m+2k+j-2} C_{j-1}^{m/2+k}(\frac{x_m}{r}) \frac{\underline{x} e_m}{r} \right)$$

with $r = (x_1^2 + x_2^2 + \cdots + x_m^2)^{1/2}$, $\mu_k^{2l} = (-1)^l \left(C_{2l}^{m/2+k-1}(0)\right)^{-1}$, and

$$\mu_k^{2l+1} = (-1)^l \, \frac{m+2k+2l-1}{m+2k-2} \left(C_{2l}^{m/2+k}(0)\right)^{-1}.$$

Proof. In [22] (Theorem 2.2.1, page 312), the corresponding polynomial we denote here by $\tilde{X}_k^{(j)}$ is computed for the Cauchy–Riemann operator. Fortunately, there is an obvious relation between these two polynomials. Namely, we have that

$$X_k^{(j)}(\underline{x}, x_m) = \begin{cases} \tilde{X}_k^{(j)}(\underline{x}e_m, x_m), & j \text{ even}, \\ -\tilde{X}_k^{(j)}(\underline{x}e_m, x_m)e_m, & j \text{ odd}. \end{cases}$$

To complete the proof it is sufficient to use the explicit formula for the polynomial $\tilde{X}_k^{(j)}$. □

At this moment we are ready to describe the decomposition of the $\mathfrak{so}(m)$-module $\mathcal{M}_k(\mathbb{R}^m, S)$ into irreducible $\mathfrak{so}(m-1)$-submodules. We start with the even dimensional case.

The even dimensional case. In the case when $m = 2n$, there is a unique (up to equivalence) irreducible module S_m over $\mathbb{C}_m$. As a Spin(m)-module, S_m is reducible and decomposes into two inequivalent irreducible submodules

$$S_m = S_m^+ \oplus S_m^-.$$

Actually, $S_{2n}^{\pm}$ are the unique basic spinor representations of the group Spin$(2n)$ and, putting $\theta_{2n} = (-i)^n e_1 e_2 \cdots e_{2n}$, we have that

$$S_{2n}^{\pm} = \{u \in S_{2n} : \theta_{2n} u = \pm u\}. \tag{3.2}$$

Furthermore, as Spin$(2n-1)$-modules, S_{2n}^+ and S_{2n}^- remain irreducible but become equivalent to each other.

Let S be a basic spinor representation for Spin$(2n)$, that is, $S \simeq S_{2n}^+$ or $S \simeq S_{2n}^-$. In either case, it is easy to see that Proposition 3.3 implies that

$$\mathcal{M}_k(\mathbb{R}^{2n}, S) = CK(\mathcal{P}_k(\mathbb{R}^{2n-1}, S)).$$

Moreover, using Proposition 3.1, we get the following decomposition of the spaces $\mathcal{P}_k(\mathbb{R}^{2n-1}, S)$ into inequivalent irreducible $\mathfrak{so}(2n-1)$-submodules:

$$\mathcal{P}_k(\mathbb{R}^{2n-1}, S) = \bigoplus_{j=0}^{k} (\underline{x}e_{2n})^j \mathcal{M}_{k-j}(\mathbb{R}^{2n-1}, S).$$

Finally, applying the CK extension to this decomposition and using Lemma 3.4, we get obviously the next result, cf. [22], Theorem 2.2.3, page 315:

Theorem 3.5. *Let $n \geq 2$ and let S be a basic spinor representation for* Spin$(2n)$. *Then the $\mathfrak{so}(2n)$-module $\mathcal{M}_k(\mathbb{R}^{2n}, S)$ decomposes into mutually inequivalent irreducible $\mathfrak{so}(2n-1)$-submodules as*

$$\mathcal{M}_k(\mathbb{R}^{2n}, S) = \bigoplus_{j=0}^{k} X^{(j)} \mathcal{M}_{k-j}(\mathbb{R}^{2n-1}, S).$$

Of course, using Theorem 3.5, it is easy to construct GT bases in dimension $2n$ once we know GT bases in dimension $2n-1$.

Corollary 3.6. *Let $\mathcal{B}_j^{2n-1}(S)$ be* GT *bases of the modules $\mathcal{M}_j(\mathbb{R}^{2n-1}, S)$ for all $j = 0, \ldots, k$. Then we have that the set*

$$\mathcal{B}_k^{2n}(S) = \bigcup_{j=0}^{k} X^{(j)} \mathcal{B}_{k-j}^{2n-1}(S)$$

is a GT *basis of the module $\mathcal{M}_k(\mathbb{R}^{2n}, S)$. Here the polynomial $X^{(j)}$ is defined as in Lemma 3.4 and, of course, we put*

$$X^{(j)} \mathcal{B}_{k-j}^{2n-1}(S) = \left\{ X^{(j)} p \mid p \in \mathcal{B}_{k-j}^{2n-1}(S) \right\}.$$

Now we are going to deal with the odd dimensional case.

The odd dimensional case. In the case when $m = 2n+1$, there are just two different irreducible $\mathbb{C}_m$-modules (equivalent to) $S_{m+1}^{\pm}$. On the other hand, there exists only a unique basic spinor representation S of the group $\mathrm{Spin}(m)$. In particular, as $\mathrm{Spin}(m)$-modules, the modules $S_{m+1}^{\pm}$ are both equivalent to S. Moreover, S can be viewed also as an irreducible $\mathbb{C}_{2n}$-module, that is, $S \simeq S_{2n}$. As we know (see (3.2)), we have therefore that $S = S^+ \oplus S^-$, where

$$S^{\pm} = \left\{ u \in S : \theta_{2n} u = \pm u \right\}$$

are both irreducible $\mathrm{Spin}(2n)$-modules.

Furthermore, according to Proposition 3.3, we have that

$$\mathcal{M}_k(\mathbb{R}^m, S) = CK(\mathcal{P}_k(\mathbb{R}^{m-1}, S)).$$

By Proposition 3.1, we can easily obtain the following decomposition of the space $\mathcal{P}_k(\mathbb{R}^{m-1}, S)$ into inequivalent irreducible $\mathfrak{so}(m-1)$-submodules:

$$\mathcal{P}_k(\mathbb{R}^{m-1}, S) = \bigoplus_{j=0}^{k} (\underline{x} e_m)^j \mathcal{M}_{k-j}(\mathbb{R}^{m-1}, S^+) \oplus (\underline{x} e_m)^j \mathcal{M}_{k-j}(\mathbb{R}^{m-1}, S^-).$$

Applying the CK extension to this decomposition together with Lemma 3.4 gives the following result, cf. [22], Theorem 2.2.3, page 315:

Theorem 3.7. *Let $n \geq 2$ and let S stand for a basic spinor representation of* $\mathrm{Spin}(2n+1)$. *Then the $\mathfrak{so}(2n+1)$-module $\mathcal{M}_k(\mathbb{R}^{2n+1}, S)$ decomposes into inequivalent irreducible $\mathfrak{so}(2n)$-submodules as follows:*

$$\mathcal{M}_k(\mathbb{R}^{2n+1}, S) = \bigoplus_{j=0}^{k} X^{(j)} \mathcal{M}_{k-j}(\mathbb{R}^{2n}, S^+) \oplus X^{(j)} \mathcal{M}_{k-j}(\mathbb{R}^{2n}, S^-).$$

Corollary 3.8. *Let* $\mathcal{B}_j^{2n}(S^\pm)$ *be* GT *bases of the modules* $\mathcal{M}_j(\mathbb{R}^{2n}, S^\pm)$ *for all* $j = 0, \ldots, k$. *Then we have that the set*

$$\mathcal{B}_k^{2n+1}(S) = \bigcup_{j=0}^{k} X^{(j)}\mathcal{B}_{k-j}^{2n}(S^+) \cup X^{(j)}\mathcal{B}_{k-j}^{2n}(S^-)$$

is a GT *basis of the module* $\mathcal{M}_k(\mathbb{R}^{2n+1}, S)$. *Here the polynomial* $X^{(j)}$ *is defined as in Lemma* 3.4.

To summarize, Corollaries 3.6 and 3.8 tell us that GT bases for spherical monogenics can be obtained inductively. Indeed, whenever we know GT bases in dimension $m-1$ we can easily construct GT bases in dimension m.

4. Gelfand–Tsetlin bases in dimension 3

In this section, we construct explicitly GT bases for spinor valued spherical monogenics in dimension 3. First we recall a realization of basic spinor representations $S_{2n}^\pm$.

Basic spinor representations $\boldsymbol{S_{2n}^\pm}$**.** For $j = 1, \ldots, n$, put

$$w_j = \frac{1}{2}(e_{2j-1} + ie_{2j}), \quad \overline{w}_j = \frac{1}{2}(-e_{2j-1} + ie_{2j}), \quad \text{and} \quad I_j = \overline{w}_j w_j.$$

Then $I_1, \ldots, I_n$ are mutually commuting idempotent elements in $\mathbb{C}_{2n}$. Moreover, $I = I_1 I_2 \cdots I_n$ is a primitive idempotent in $\mathbb{C}_{2n}$, and

$$S_{2n} = \mathbb{C}_{2n} I$$

is a minimal left ideal in $\mathbb{C}_{2n}$. Putting $W = \langle w_1, \ldots, w_n \rangle$, we have that

$$S_{2n} = \Lambda(W)I, \quad S_{2n}^+ = \Lambda^+(W)I \ \text{ and } \ S_{2n}^- = \Lambda^-(W)\,, I$$

where $\Lambda(W)$ is the exterior algebra over W with the even part $\Lambda^+(W)$ and the odd part $\Lambda^-(W)$. See pages 114–118 of [22] for details.

Furthermore, it is well known that, for each $u \in \mathbb{C}_{2n}$, there is a unique complex number $[u]_0$ such that $IuI = [u]_0 I$ and that an inner product on S_{2n} is given by

$$(s,t) = [\overline{u}v]_0 \quad \text{for } s = uI, \text{ and } t = vI, \text{ with } u, v \in \mathbb{C}_{2n}. \tag{4.1}$$

Here, for each Clifford number $u \in \mathbb{C}_m$, $\overline{u}$ stands for its Clifford conjugate. See pages 120–125 of [22] for details.

In the next paragraph, we introduce invariant inner products on the spin modules of spherical monogenics.

Invariant inner products. Let us remark that on each (finite-dimensional) irreducible representation of $\mathrm{Spin}(m)$ there exists an invariant inner product and, in addition, that the invariant inner product is determined uniquely up to a positive multiple. In what follows, we recall two well known realizations of the invariant inner product on the module $\mathcal{M}_k(\mathbb{R}^m, S)$, namely, the L_2-inner product and the Fischer inner product. For $P, Q \in \mathcal{M}_k(\mathbb{R}^m, S)$, we define the L_2-inner product of P and Q as

$$(P,Q)_1 = \int_{\mathbb{B}_m} (P,Q)\, d\lambda^m\,, \tag{4.2}$$

where $\mathbb{B}_m$ is the unit ball in $\mathbb{R}^m$ and $d\lambda^m$ is the Lebesgue measure in $\mathbb{R}^m$.

Now we introduce the Fischer inner product. Each $P \in \mathcal{P}_k(\mathbb{R}^m, S)$ is of the form

$$P(x) = \sum_{|\alpha|=k} a_\alpha x^\alpha\,,$$

where the sum is taken over all multi-indexes $\alpha = (\alpha_1, \dots, \alpha_m)$ of $\mathbb{N}_0^m$ with $|\alpha| = \alpha_1 + \cdots + \alpha_m = k$, all coefficients a_α belong to S and $x^\alpha = x_1^{\alpha_1} \cdots x_m^{\alpha_m}$. For $P, Q \in \mathcal{P}_k(\mathbb{R}^m, S)$, we define the Fischer inner product of P and Q as

$$(P,Q)_2 = \sum_{|\alpha|=k} \alpha!\ (a_\alpha, b_\alpha)\,, \tag{4.3}$$

where $\alpha! = \alpha_1! \cdots \alpha_m!$, $P(x) = \sum a_\alpha x^\alpha$, and $Q(x) = \sum b_\alpha x^\alpha$. It is easily seen that

$$(P,Q)_2 = \Big[\Big(\overline{P}\Big(\frac{\partial}{\partial x}\Big) Q\Big)(0)\Big]_0 \quad \text{with} \quad \overline{P}\Big(\frac{\partial}{\partial x}\Big) = \sum_{|\alpha|=k} \overline{a}_\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha}.$$

Here $\partial^{|\alpha|}/\partial x^\alpha = (\partial^{\alpha_1}/\partial x_1^{\alpha_1}) \cdots (\partial^{\alpha_m}/\partial x_m^{\alpha_m})$ as usual.

Fischer decompositions in the dimension $m = 2$. As we have remarked in the introduction, the Fischer decomposition in dimension 2 is not usually considered in Clifford analysis and it has a slightly different form than in higher dimensions. In this case, we have that $\mathfrak{so}(2) = \langle e_{12} \rangle$, $S = S_2 = \langle I_1, w_1 I_1 \rangle$, $S^+ = \langle I_1 \rangle$, and $S^- = \langle w_1 I_1 \rangle$, with

$$I_1 = \frac{1}{2}(1 - i e_{12}) \quad \text{and} \quad w_1 I_1 = \frac{1}{2}(e_1 + i e_2).$$

Each $s \in S$ is of the form $s = s^+ I_1 + s^- w_1 I_1$ for some complex numbers $s^\pm$. We write $s = (s^+, s^-)$. Let us remark that each $P \in \mathcal{P}_k(\mathbb{R}^2, S)$ can be expressed as $P = (P^+, P^-)$ for some complex valued k-homogeneous polynomials $P^\pm$ in variables $z = x_1 + i x_2$ and $\overline{z} = x_1 - i x_2$. Furthermore, the action of $\mathfrak{so}(2)$ on the space $\mathcal{P}_k(\mathbb{R}^2, S)$ is given by

$$dL(e_{12}/2) = \frac{d}{dt} L(\exp(t e_{12}/2))|_{t=0} = \frac{e_{12}}{2} + x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}.$$

Put $L_{12} = dL(e_{12}/2)$. Now it is easy to show the next result.

Theorem 4.1. *Let $\mathcal{M}_j^{2,\pm} = \mathcal{M}_j(\mathbb{R}^2, S^{\pm})$ for each $j = 0, \dots, k$. Then we have that $\mathcal{M}_j^{2,+} = \langle(\overline{z}^j, 0)\rangle$, $\mathcal{M}_j^{2,-} = \langle(0, z^j)\rangle$,*

$$\mathcal{P}_k(\mathbb{R}^2, S^+) = \bigoplus_{j=0}^{k} z^j \mathcal{M}_{k-j}^{2,+} \quad \textit{and} \quad \mathcal{P}_k(\mathbb{R}^2, S^-) = \bigoplus_{j=0}^{k} \overline{z}^j \mathcal{M}_{k-j}^{2,-}.$$

In addition, for each $j = 0, \dots, k$, the $\mathfrak{so}(2)$-modules $z^j \mathcal{M}_{k-j}^{2,+}$ and $\overline{z}^j \mathcal{M}_{k-j}^{2,-}$ are both irreducible with the highest weights $k + \frac{1}{2} - 2j$ and $-k - \frac{1}{2} + 2j$, respectively.

Proof. Let $P \in \mathcal{P}_k(\mathbb{R}^2, S)$ and $P = (P^+, P^-)$. Write

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\Big) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\Big).$$

Since $e_1 P = (-P^-, P^+)$, $e_{12} P = (iP^+, -iP^-)$ and $\partial = e_1(\frac{\partial}{\partial x_1} - e_{12}\frac{\partial}{\partial x_2})$ we have that

$$\partial P = 2\Big(-\frac{\partial P^-}{\partial \overline{z}}, \frac{\partial P^+}{\partial z}\Big).$$

Assume now that P is S^+-valued, that is, $P = (P^+, 0)$ and

$$P^+(z, \overline{z}) = \sum_{j=0}^{k} a_j z^j \overline{z}^{k-j} \quad (a_j \in \mathbb{C}).$$

Obviously, $\partial P = 0$ if and only if $P^+ = a_k \overline{z}^k$. Hence it remains to show that the module $z^j \mathcal{M}_{k-j}^{2,+}$ has the highest weight $k + \frac{1}{2} - 2j$. This follows from the fact that weights are just eigenvalues of the operator $H = -iL_{12}$, and

$$H((z^j \overline{z}^{k-j}, 0)) = (k + \frac{1}{2} - 2j)(z^j \overline{z}^{k-j}, 0).$$

For S^--valued polynomials, an analogous proof works. □

The decompositions of the spaces $\mathcal{P}_k^+ = \mathcal{P}_k(\mathbb{R}^2, S^+)$ are depicted in columns of Figure 1. In this diagram, we write $z^j \overline{z}^k$ for $(z^j \overline{z}^k, 0)$. Moreover, all irreducible submodules with the same highest weight are contained in the row labeled by this highest weight.

Of course, an analogous diagram can be created for S^--valued polynomials. However, in this case, the labels of the rows of the diagram are shifted. In particular, the row beginning with $\langle 1\rangle$ is labeled by $-1/2$.

GT bases for the dimension $m = 3$. In this paragraph, we obtain explicit formulae for the GT bases of spinor valued spherical monogenics in dimension 3. In this case, we have that $S \simeq S_4^{\pm}$, $\mathfrak{so}(3) = \langle e_{12}, e_{23}, e_{31}\rangle$ and $\mathfrak{so}(2) = \langle e_{12}\rangle$. Furthermore, the action of $\mathfrak{so}(3)$ on the space $\mathcal{P}_k(\mathbb{R}^3, S)$ is given by

$$L_{ij} = dL(e_{ij}/2) = \frac{e_{ij}}{2} + x_j \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_j} \quad (i \neq j).$$

	$\mathcal{P}_0^+$	$\mathcal{P}_1^+$	$\mathcal{P}_2^+$	$\mathcal{P}_3^+$	$\mathcal{P}_4^+$	
$\frac{7}{2}$				$\langle \overline{z}^3 \rangle$		$\dots$
$\frac{5}{2}$			$\langle \overline{z}^2 \rangle$		$\langle z\overline{z}^3 \rangle$	
$\frac{3}{2}$		$\langle \overline{z} \rangle$		$\langle z\overline{z}^2 \rangle$		$\dots$
$\frac{1}{2}$	$\langle 1 \rangle$		$\langle z\overline{z} \rangle$		$\langle z^2\overline{z}^2 \rangle$	
$-\frac{1}{2}$		$\langle z \rangle$		$\langle z^2\overline{z} \rangle$		$\dots$
$-\frac{3}{2}$			$\langle z^2 \rangle$		$\langle z^3\overline{z} \rangle$	
$-\frac{5}{2}$				$\langle z^3 \rangle$		$\dots$

FIGURE 1. The decomposition of the modules $\mathcal{P}_k^+ = \mathcal{P}_k(\mathbb{R}^2, S^+)$.

As an $\mathfrak{so}(2)$-module, the module S is reducible and decomposes into two inequivalent irreducible submodules $S = S^+ \oplus S^-$, with

$$S^{\pm} = \big\{u \in S : -ie_{12}\, u = \pm u\big\}.$$

Let $v^{\pm}$ be generators of $S^{\pm}$, that is, $S^{\pm} = \langle v^{\pm} \rangle$. We can construct a GT basis in this case using Proposition 3.3 and Theorem 4.1.

Theorem 4.2. *For each $k \in \mathbb{N}_0$, the polynomials*

$$f_{2j}^k = e^{x_3 e_3 \underline{\partial}} \left(\frac{z^j \overline{z}^{k-j}}{j!(k-j)!}\, v^+ \right)$$

and

$$f_{2j+1}^k = e^{x_3 e_3 \underline{\partial}} \left(\frac{z^j \overline{z}^{k-j}}{j!(k-j)!}\, v^- \right), \quad j = 0, \dots, k$$

form a GT *basis of the irreducible $\mathfrak{so}(3)$-module $\mathcal{M}_k(\mathbb{R}^3, S)$. Moreover, for each $j = 0, \dots, 2k+1$, the polynomial f_j^k is a weight vector with the weight $k + \frac{1}{2} - j$. That is, putting $H = -iL_{12}$, we have that $Hf_j^k = (k + \frac{1}{2} - j) f_j^k$.*

It is not difficult to express the GT bases from Theorem 4.2 even more explicitly. To do this we identify the space S with $\mathbb{C}^2$. Indeed, each $s \in S$ is of the form

$$s = s^+ v^+ + s^- v^-$$

for some complex numbers s^+ and s^-. We write $s = (s^+, s^-)$ for short. For the sake of explicitness, we limit ourselves to the cases $S = S_4^+$ and $S = S_4^-$. In the former case, we put $v^+ = I$ and $v^- = w_1 w_2 I$. In the latter case, we put $v^+ = w_2 I$ and $v^- = w_1 I$. In these cases, explicit formulae for GT-bases are given in Corollary 4.3 below.

Corollary 4.3. *Let $\{f_0^{k,\pm}, \dots, f_{2k+1}^{k,\pm}\}$ be the* GT *bases of $\mathcal{M}_k(\mathbb{R}^3, S_4^\pm)$ defined in Theorem 4.2.*

(a) *For each $k \in \mathbb{N}_0$ and $j = 0, \dots, k$, we have that*

$$f_{2j}^{k,\pm} = (p_j^k, \mp q_j^k) \quad \text{and} \quad f_{2j+1}^{k,\pm} = (\pm q_{j+1}^k,\, p_j^k)$$

where

$$p_j^k(z, \overline{z}, x_3) = \sum_{s=0}^{\min(j,k-j)} (-1)^s \frac{(2x_3)^{2s}\, z^{j-s}\, \overline{z}^{k-j-s}}{(2s)!(j-s)!(k-j-s)!} \qquad \text{and}$$

$$q_j^k(z, \overline{z}, x_3) = \sum_{s=0}^{\min(j-1,k-j)} (-1)^s \frac{(2x_3)^{2s+1}\, z^{j-1-s}\, \overline{z}^{k-j-s}}{(2s+1)!(j-1-s)!(k-j-s)!}.$$

Here $q_0^k = 0 = q_{k+1}^k$.

(b) *Moreover, for each $k \in \mathbb{N}$, we have that*

$$\frac{\partial f_j^{k,\pm}}{\partial x_3} = \begin{cases} \mp(-1)^j 2\, f_{j-1}^{k-1,\pm}, & j = 1, \dots, 2k; \\ 0, & j = 0, 2k+1; \end{cases}$$

$$\frac{\partial f_j^{k,\pm}}{\partial z} = \begin{cases} f_{j-2}^{k-1,\pm}, & j = 2, \dots, 2k+1; \\ 0, & j = 0, 1; \end{cases}$$

$$\frac{\partial f_j^{k,\pm}}{\partial \overline{z}} = \begin{cases} f_j^{k-1,\pm}, & j = 0, \dots, 2k-1; \\ 0, & j = 2k, 2k+1. \end{cases}$$

(c) *Finally, for $k \in \mathbb{N}_0$ and $j = 0, \dots, 2k+1$, we have that*

$$f_{2k+1-j}^{k,\pm} = (-1)^j (f_j^{k,\pm})^*$$

where $s^* = (-\overline{s_2}, \overline{s_1})$ for each $s = (s_1, s_2) \in S$.

Proof. Let $S = S_4^\pm$. Obviously, we have that

$$e_3 \underline{\partial} P = e_{31} \frac{\partial P}{\partial x_1} + e_{32} \frac{\partial P}{\partial x_2} = \pm 2 \Big(\frac{\partial P_2}{\partial \overline{z}}, -\frac{\partial P_1}{\partial z} \Big).$$

Putting $P_j^k = \big(\frac{z^j\overline{z}^{k-j}}{j!(k-j)!}, 0\big)$ and $Q_j^k = \big(0, \frac{z^j\overline{z}^{k-j}}{j!(k-j)!}\big)$, we get thus that

$$(e_3\underline{\partial})^{2s} P_j^k = (-1)^s 2^{2s} P_{j-s}^{k-2s}, \quad (e_3\underline{\partial})^{2s} Q_j^k = (-1)^s 2^{2s} Q_{j-s}^{k-2s},$$
$$(e_3\underline{\partial})^{2s+1} P_j^k = \mp(-1)^s 2^{2s+1} Q_{j-s-1}^{k-(2s+1)}, \quad (e_3\underline{\partial})^{2s+1} Q_j^k = \pm(-1)^s 2^{2s+1} P_{j-s}^{k-(2s+1)}.$$

Using these relations it is easy to obtain the explicit formulae for $f_j^{k,\pm}$. Obviously, the statements (b) and (c) can be verified directly using these explicit formulæ. On the other hand, the property (b) follows also from the following formula:

$$\frac{\partial}{\partial x_3}(e^{x_3 e_3 \underline{\partial}} P) = e^{x_3 e_3 \underline{\partial}}(e_3 \underline{\partial} P)$$

and from the fact that the derivatives $\partial/\partial z$ and $\partial/\partial\overline{z}$ both commute with the CK extension operator $e^{x_3 e_3 \underline{\partial}}$. □

Remark 4.4. It is easy to express the elements $f_j^{k,\pm}$ of the GT bases from Corollary 4.3 in terms of hypergeometric series ${}_2F_1$ or Jacobi polynomials, see pages 64 and 99 of [2]. Indeed, we have that

$$p_j^k = {}_2F_1\Big(-j, -k+j, \frac{1}{2}; -\frac{x_3^2}{|z|^2}\Big)\,\frac{z^j\overline{z}^{k-j}}{j!(k-j)!},$$
$$q_j^k = {}_2F_1\Big(-j+1, -k+j, \frac{3}{2}; -\frac{x_3^2}{|z|^2}\Big)\,\frac{2x_3\, z^{j-1}\overline{z}^{k-j}}{(j-1)!(k-j)!}.$$

Here $|z|^2 = z\overline{z}$ and the hypergeometric series ${}_2F_1(a, b, c; y)$ is given by

$${}_2F_1(a, b, c; y) = \sum_{s=0}^{\infty} \frac{(a)_s(b)_s}{(c)_s s!} y^s.$$

In Figure 2, structural properties of the GT basis in this case are shown. In the k-th column of Figure 2, the decomposition of the $\mathfrak{so}(3)$-module

$$\mathcal{M}_k = \mathcal{M}_k(\mathbb{R}^3, S)$$

into irreducible $\mathfrak{so}(2)$-submodules can be found. Moreover, all irreducible $\mathfrak{so}(2)$-submodules with the same highest weight are contained in the row labeled by this highest weight. By Theorem 4.2, it is easy to see that Figure 2 is, in an obvious sense, composed of the diagrams for S^+ and S^--valued polynomials in $\mathbb{R}^2$ (see Figure 1). By Corollary 4.3, we know that the application of the derivative $\partial/\partial x_3$ to the elements of the GT basis shifts the given row to the left, the derivative $\partial/\partial\overline{z}$ moves them diagonally downward, and $\partial/\partial z$, diagonally upward. In other words, the GT bases in this case possess an Appell property not only with respect to the last real variable x_3 but also with respect to the complex variables z and $\overline{z}$. Moreover, the upper triangle in Figure 2 is mapped onto the lower one by the transformation $(\cdot)^*$.

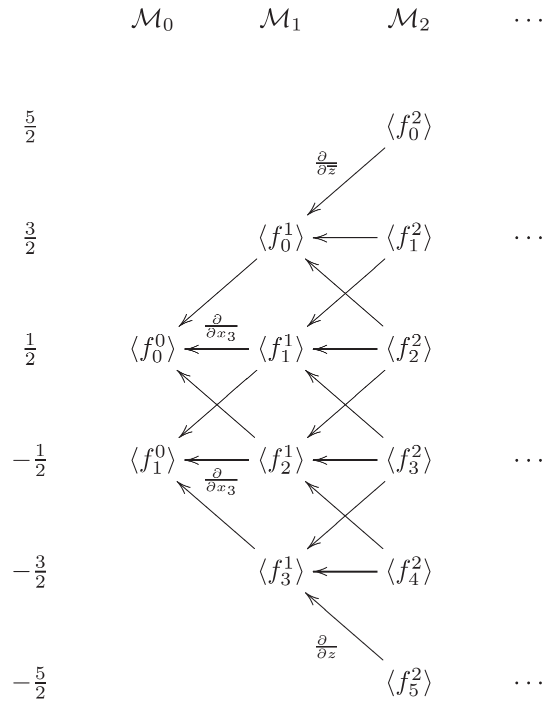

FIGURE 2. The decomposition of the modules $\mathcal{M}_k = \mathcal{M}_k(\mathbb{R}^3, S)$

Remark 4.5. Let $S = S_4^{\pm}$. It is not difficult to find non-zero constants $d_j^{k,\pm}$ such that the polynomials $\hat{f}_j^k = d_j^{k,\pm} f_j^{k,\pm}$ satisfy the following properties:

$$\hat{f}_0^k = \overline{z}^k v^+,\ \hat{f}_{2k+1}^k = z^k v^- \quad \text{and} \quad \frac{\partial \hat{f}_j^k}{\partial x_3} = \begin{cases} k\ \hat{f}_{j-1}^{k-1}, & j = 1, \dots, 2k; \\ 0, & j = 0, 2k+1. \end{cases} \tag{4.4}$$

Indeed, it is necessary and sufficient to put, for each $j = 0, \dots, k$,

$$d_j^{k,\pm} = (\mp 1)^j (-1)^{(j+1)j/2}\ 2^{-j}\ k! \quad \text{and} \quad d_{2k+1-j}^{k,\pm} = (-1)^j d_j^k.$$

Moreover, we have obviously that

$$\hat{f}_{2k+1-j}^k = (\hat{f}_j^k)^*, \quad \frac{\partial \hat{f}_j^k}{\partial z} = a_j^{k,\pm} \hat{f}_{j-2}^{k-1} \quad \text{and} \quad \frac{\partial \hat{f}_j^k}{\partial \overline{z}} = b_j^{k,\pm} \hat{f}_j^{k-1}\,, \tag{4.5}$$

where the constants $a_j^{k,\pm}$ and $b_j^{k,\pm}$ are given by

$$a_j^{k,\pm} = \begin{cases} 0, & j = 0, 1; \\ -\frac{1}{4}k, & 2 \le j \le k; \\ \mp \frac{1}{2}k, & j = k+1; \\ k, & k+2 \le j \le 2k+1; \end{cases} \qquad b_j^{k,\pm} = \begin{cases} k, & 0 \le j \le k-1; \\ \pm \frac{1}{2}k, & j = k; \\ -\frac{1}{4}k, & k+1 \le j \le 2k-1; \\ 0, & j = 2k, 2k+1 \end{cases}$$

Furthermore, by the definition of GT bases and their structural properties shown in Figure 2, it is clear that, for $k \in \mathbb{N}_0$, the sets

$$\{\hat{f}_j^k |\ j = 0, \dots, 2k+1\}$$

are the GT bases of the modules $\mathcal{M}_k(\mathbb{R}^3, S)$, uniquely determined by the property (4.4) and the condition that, for $j = 0, \dots, 2k+1$,

$$H \hat{f}_j^k = \left(k + \frac{1}{2} - j\right) \hat{f}_j^k, \quad \text{with } H = -iL_{12}.$$

5. Quaternion valued polynomials in $\mathbb{R}^3$

In this section, we reformulate the GT bases obtained in the previous section for quaternion valued spherical monogenics.

Quaternionic formulation. In what follows, $\mathbb{H}$ stands for the skew field of real quaternions q with the imaginary units i_1, i_2 and i_3, that is,

$$i_1^2 = i_2^2 = i_3^2 = i_1 i_2 i_3 = -1 \ \text{ and } \ q = q_0 + q_1 i_1 + q_2 i_2 + q_3 i_3, (q_0, q_1, q_2, q_3) \in \mathbb{R}^4.$$

For a quaternion q, put $\overline{q} = q_0 - q_1 i_1 - q_2 i_2 - q_3 i_3$. We realize $\mathbb{H}$ as the subalgebra of complex 2×2 matrices of the form

$$q = \begin{pmatrix} q_0 + iq_3 & -q_2 + iq_1 \\ q_2 + iq_1 & q_0 - iq_3 \end{pmatrix}. \tag{5.1}$$

In particular, we have that

$$i_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad i_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad i_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

If $s = (q_0 + iq_3, q_2 + iq_1) \in \mathbb{C}^2$, then we write $q(s)$ for the quaternion q as in (5.1). For $s = (s_1, s_2) \in \mathbb{C}^2$, $q(s)$ is thus the 2×2 matrix which has s as the first column and $s^* = (-\overline{s_2}, \overline{s_1})$ as the second one. It is easy to see that $q(s)\, i_2 = q(s^*)$ and that

$$q(s) = \operatorname{Re} s_1 + i_1 \operatorname{Im} s_2 + i_2 \operatorname{Re} s_2 + i_3 \operatorname{Im} s_1\,,$$

where, for a complex number z, we write $\operatorname{Re} z$ for its real part and $\operatorname{Im} z$ for its imaginary part.

Furthermore, we identify $\mathfrak{so}(3)$ with $\langle i_1, i_2, i_3 \rangle$ as follows: $e_{12} \simeq i_3$, $e_{23} \simeq i_1$ and $e_{31} \simeq i_2$. Then we realize the basic spinor representation S as the space $\mathbb{C}^2$ of column vectors

$$s = \begin{pmatrix} q_0 + iq_3 \\ q_2 + iq_1 \end{pmatrix}.$$

Here the action of $\mathfrak{so}(3)$ on S is given by the matrix multiplication on the left.

Now we are interested in quaternion valued polynomials $Q = Q(y)$ in the variable $y = (y_0, y_1, y_2)$ of $\mathbb{R}^3$. Let us denote by $\mathcal{M}_k(\mathbb{R}^3, \mathbb{H})$ the space of $\mathbb{H}$-valued k-homogeneous polynomials Q satisfying the Cauchy–Riemann equation $DQ = 0$ with

$$D = \frac{\partial}{\partial y_0} + i_1 \frac{\partial}{\partial y_1} + i_2 \frac{\partial}{\partial y_2}.$$

We can consider naturally $\mathcal{M}_k(\mathbb{R}^3, \mathbb{H})$ as a right $\mathbb{H}$-linear Hilbert space with the $\mathbb{H}$-valued inner product

$$(Q, R)_{\mathbb{H}} = \int_{S^2} \overline{Q} R \, d\sigma.$$

Moreover, we can identify $\mathcal{M}_k(\mathbb{R}^3, \mathbb{H})$ with the $\mathfrak{so}(3)$-module $\mathcal{M}_k(\mathbb{R}^3, S)$ we have studied in the previous paragraph as follows. Let $P = P(x)$ be an S-valued polynomial in the variable $x = (x_1, x_2, x_3)$ of $\mathbb{R}^3$. We define a corresponding $\mathbb{H}$-valued polynomial $Q(P)$ in $\mathbb{R}^3$ by

$$Q(P)(y_0, y_1, y_2) = q(P)(-y_2, y_1, y_0). \tag{5.2}$$

Then it is easy to see that $Q(P) \in \mathcal{M}_k(\mathbb{R}^3, \mathbb{H})$ if and only if

$$i_1 \frac{\partial P}{\partial x_1} + i_2 \frac{\partial P}{\partial x_2} + i_3 \frac{\partial P}{\partial x_3} = 0,$$

that is, $P \in \mathcal{M}_k(\mathbb{R}^3, S)$. In addition, for each $P, R \in \mathcal{M}_k(\mathbb{R}^3, S)$, we have that

$$(Q(P), Q(R))_{\mathbb{H}} = q((P, R)_1, (P^*, R)_1)\,, \tag{5.3}$$

where $(\cdot, \cdot)_1$ is the complex valued inner product defined as in (4.2). Using the identification (5.2) and Theorem 4.2, we obtain easily orthogonal bases of quaternion valued spherical monogenics.

Theorem 5.1. *For each $k \in \mathbb{N}_0$, there exists an orthogonal basis*

$$\{g^k_j \mid j = 0, \dots, k\} \tag{5.4}$$

of the right $\mathbb{H}$-linear Hilbert space $\mathcal{M}_k(\mathbb{R}^3, \mathbb{H})$ such that:

(i) *For $j = 0, \dots, k$, let h^k_j and h^k_{2k+1-j} be the first and the second column of the (matrix valued) polynomial g^k_j, respectively. Then, for each $j = 0, \dots 2k+1$, we have that*

$$H h^k_j = \Big(k + \frac{1}{2} - j\Big) h^k_j\,, \quad \text{with} \quad H = -i\left(\frac{i_3}{2} + y_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_2}\right). \tag{5.5}$$

(ii) *We have that*

$$\frac{\partial g^k_j}{\partial y_0} = \begin{cases} k g^{k-1}_{j-1}, & j = 1, \dots, k; \\ 0, & j = 0. \end{cases}$$

(iii) *For each $k \in \mathbb{N}_0$, we have that $g^k_0 = (y_1 - i_3 y_2)^k$.*

Moreover, the polynomials g^k_j are determined uniquely by the conditions (i), (ii) *and* (iii).

In addition, for each $k \in \mathbb{N}_0$, the polynomials

$$h_0^k,\ h_1^k, \dots, h_{2k+1}^k$$

form a GT *basis of the $\mathfrak{so}(3)$-module $\tilde{\mathcal{M}}_k(\mathbb{R}^3, S)$ of S-valued k-homogeneous polynomials h in $\mathbb{R}^3$ satisfying the Cauchy–Riemann equation $Dh = 0$. Moreover, the polynomials h_j^k are determined uniquely by the condition (5.5), by the Appell property*

$$\frac{\partial h_j^k}{\partial y_0} = \begin{cases} k h_{j-1}^{k-1}, & j = 1, \dots, 2k; \\ 0, & j = 0, 2k+1; \end{cases} \tag{5.6}$$

and by the condition that $h_0^k = (\overline{u}^k, 0)$ and $h_{2k+1}^k = (0, u^k)$, with $u = y_1 + iy_2$ and $\overline{u} = y_1 - iy_2$.

Proof. (a) We first construct GT bases of S-valued monogenic polynomials in $\mathbb{R}^3$ by applying Theorem 4.2. Indeed, for $P \in \mathcal{M}_k(\mathbb{R}^3, S)$, we have that

$$i_2 \frac{\partial P}{\partial x_1} - i_1 \frac{\partial P}{\partial x_2} = 2\Big(-\frac{\partial P_2}{\partial \overline{z}}, \frac{\partial P_1}{\partial z}\Big).$$

As in the proof of Corollary 4.3, we get easily that the set

$$\{f_0^{k,-}, \dots, f_{2k+1}^{k,-}\}$$

is a GT basis of $\mathcal{M}_k(\mathbb{R}^3, S)$.

(b) For each $k \in \mathbb{N}_0$ and $j = 0, \dots, 2k+1$, put

$$\hat{h}_j^k(y_0, y_1, y_2) = (f_j^{k,-})(-y_2, y_1, y_0).$$

Obviously, the set

$$\{\hat{h}_j^k |\ j = 0, \dots, 2k+1\}$$

is a GT basis of the module $\tilde{\mathcal{M}}_k(\mathbb{R}^3, S)$. It is easy to see that

$$\hat{h}_{2j}^k = (-1)^{k-j} i^k (p_j^k, -iq_j^k) \quad \text{and} \quad \hat{h}_{2j+1}^k = (-1)^{k-j} i^k (-iq_{j+1}^k, p_j^k)\,,$$

where $p_j^k = p_j^k(u, \overline{u}, y_0)$ and $q_j^k = q_j^k(u, \overline{u}, y_0)$ are defined as in Corollary 4.3.

(c) We can find non-zero complex numbers $c_j^k \in \mathbb{C}$ such that the polynomials $h_j^k = c_j^k \hat{h}_j^k$ satisfy, in addition, condition (5.6), $h_0^k = (\overline{u}^k, 0)$, $h_{2k+1}^k = (0, u^k)$ and $h_{2k+1-j}^k = (h_j^k)^*$. Indeed, for each $k \in \mathbb{N}_0$, put $c_0^k = i^k k!$. Moreover, it is easy to see that

$$\frac{\partial \hat{h}_j^k}{\partial y_0} = (-1)^j\ 2\hat{h}_{j-1}^{k-1}.$$

This implies that we need to have $c_j^k = (-1)^j 2^{-1} k c_{j-1}^{k-1}$. Hence we are forced to put, for each $j = 0, \dots, k$,

$$c_j^k = (-1)^{(j+1)j/2}\ 2^{-j}\ k!\ i^{k-j} \quad \text{and} \quad c_{2k+1-j}^k = (-1)^j \overline{c}_j^k.$$

(d) Finally, for each $k \in \mathbb{N}_0$ and $j = 0, \dots, k$, define an $\mathbb{H}$-valued polynomial g_j^k corresponding to the S-valued polynomial h_j^k by

$$g_j^k = q(h_j^k).$$

By (c) and (5.3), we have that the set

$$\{g_j^k |\ j = 0, \dots, k\}$$

is orthogonal with respect to the $\mathbb{H}$-valued inner product $(\cdot,\cdot)_{\mathbb{H}}$. Actually, this set is, in fact, a basis of the right $\mathbb{H}$-linear Hilbert space $\mathcal{M}_k(\mathbb{R}^3, \mathbb{H})$ because

$$g_j^k\ i_2 = q(h_j^k)\ i_2 = q((h_j^k)^*) = q(h_{2k+1-j}^k).$$

Obviously, the conditions (i), (ii) and (iii) are satisfied.

(e) Since weight vectors of the operator H are determined uniquely up to non-zero multiples the construction gives also the uniqueness of the bases satisfying the conditions (i), (ii) and (iii). □

From the proof of Theorem 5.1 we get easily the next result:

Corollary 5.2. *Let the set $\{g_j^k |\ j = 0, \dots, k\}$ be the orthogonal basis of the right $\mathbb{H}$-linear Hilbert space $\mathcal{M}_k(\mathbb{R}^3, \mathbb{H})$ as in Theorem 5.1. Then, for each $j = 0, \dots, k$, we have that*

$$g_j^k = \begin{cases} (-1)^l\ k!\ 2^{-j}(\operatorname{Re} p_l^k - i_1 \operatorname{Re} q_l^k + i_2 \operatorname{Im} q_l^k + i_3 \operatorname{Im} p_l^k), & j = 2l; \\ (-1)^l\ k!\ 2^{-j}(\operatorname{Re} q_{l+1}^k + i_1 \operatorname{Re} p_l^k - i_2 \operatorname{Im} p_l^k + i_3 \operatorname{Im} q_{l+1}^k), & j = 2l+1. \end{cases}$$

Here $u = y_1 + iy_2$, $\overline{u} = y_1 - iy_2$ and $p_j^k = p_j^k(u, \overline{u}, y_0)$, $q_j^k = q_j^k(u, \overline{u}, y_0)$ are complex polynomials defined as in Corollary 4.3.

Remark 5.3. In [31], the GT bases for this case are obtained in quite a different way. In particular, the elements g_j^k of these bases are expressed in terms of the Legendre polynomials as follows. Using spherical coordinates,

$$y_0 = r\cos\theta, \quad y_1 = r\sin\theta\cos\varphi, \quad y_2 = r\sin\theta\sin\varphi\,,$$

with $0 \le r$, $-\pi \le \varphi \le \pi$ and $0 \le \theta \le \pi$, we have that

$$g_j^k(r,\theta,\varphi) = (k!/j!)(-2)^{k-j} r^k\ \left(g_{j,0}^k + g_{j,1}^k\ i_1 + g_{j,2}^k\ i_2 + g_{j,3}^k\ i_3\right),$$

where

$$g_{j,0}^k = P_k^{j-k}(\cos\theta)\cos(j-k)\varphi, \qquad g_{j,1}^k = -jP_k^{j-k-1}(\cos\theta)\cos(j-k-1)\varphi,$$
$$g_{j,2}^k = jP_k^{j-k-1}(\cos\theta)\sin(j-k-1)\varphi, \qquad g_{j,3}^k = P_k^{j-k}(\cos\theta)\sin(j-k)\varphi.$$

Here, P_k^0 is the k-th Legendre polynomial and P_k^l are its associated Legendre functions.

In the last paragraph, we show that the GT bases obtained for quaternion valued spherical monogenics coincide with those constructed by the first and the second author in [7].

Identification of the bases. Condition (ii) of Theorem 5.1 tells us that the monogenic polynomials g_j^k form an Appell system. In [6] and in Theorem 7.2 of [7], an orthogonal Appell system of quaternion valued spherical monogenics has been constructed quite explicitly from an orthogonal system of real valued spherical harmonics. Further, in [6] and [5], very compact recursion formulae have been obtained for the elements of the Appell basis. From these recursion formulae it is already apparent that the wanted Appell system can be constructed without starting with spherical harmonics. These results are summarized in the following theorem:

Theorem 5.4 ([6], [7], [5]). *The system of inner solid spherical monogenics*

$$\left\{A_n^l\, :\, l=0,\dots,n\right\}_{n\in\mathbb{N}_0},$$

where, for each $n\in\mathbb{N}$ and $l=0,\dots,n$, the elements are given by the two-step recurrence formula

$$A_{n+1}^l=\frac{n+1}{2(n-l+1)(n+l+2)}\left[\Big((2n+3)y+(2n+1)\bar{y}\Big)A_n^l-2n\,y\overline{y}\,A_{n-1}^l\right], \tag{5.7}$$

with

$$A_{l+1}^l\;=\;\frac{1}{4}\big[(2l+3)y+(2l+1)\bar{y}\big]\,A_l^l\quad \text{and}\quad A_l^l\;=\;(y_1-i_3y_2)^l\,,$$

is an orthogonal Appell basis in $L^2(\mathbb{B}_3,\mathbb{H})\cap\ker D$ such that, for each $n\in\mathbb{N}$,

$$\overline{D}_0A_n^l=\begin{cases} n\,A_{n-1}^l, & l=0,\dots,n-1\\ 0, & l=n\end{cases}$$

and

$$D_{\mathbb{C}}A_n^n=n\,A_{n-1}^{n-1}$$

hold. Here, $y:=y_0+i_1y_1+i_2y_2$ denotes the reduced quaternion. The Cauchy–Riemann operators used are defined by

$$\overline{D}_0:=\frac{1}{2}\Big(\frac{\partial}{\partial y_0}-i_1\frac{\partial}{\partial y_1}-i_2\frac{\partial}{\partial y_2}\Big)\quad\text{and}\quad D_{\mathbb{C}}:=\frac{1}{2}\Big(\frac{\partial}{\partial y_1}+i_3\frac{\partial}{\partial y_2}\Big).$$

At this point, let us comment on some structural properties of the Appell system (5.7) coming from an analytical point of view. Firstly, the two-step recurrence formulae relate Appell polynomials of different degree n; however, the index l is fixed. Referring to Figure 3, this structurally means that the elements of the $(l+1)$-th column are recursively generated by the initial elements A_l^l, which in fact belong to the subset of the so-called hyperholomorphic constants. Such generalized constants are characterized in a quite natural way: A function f is called a hyperholomorphic constant if f belongs to the function space $f\in\ker D$ (the space of monogenic solutions to the Moisil–Teodorescu system), and vanishes after (hypercomplex) derivation. In this context, we refer again to [33] and [29], wherein the authors have proved that the operator $\overline{D}_0=\frac{1}{2}\overline{D}$ corresponds to the concept

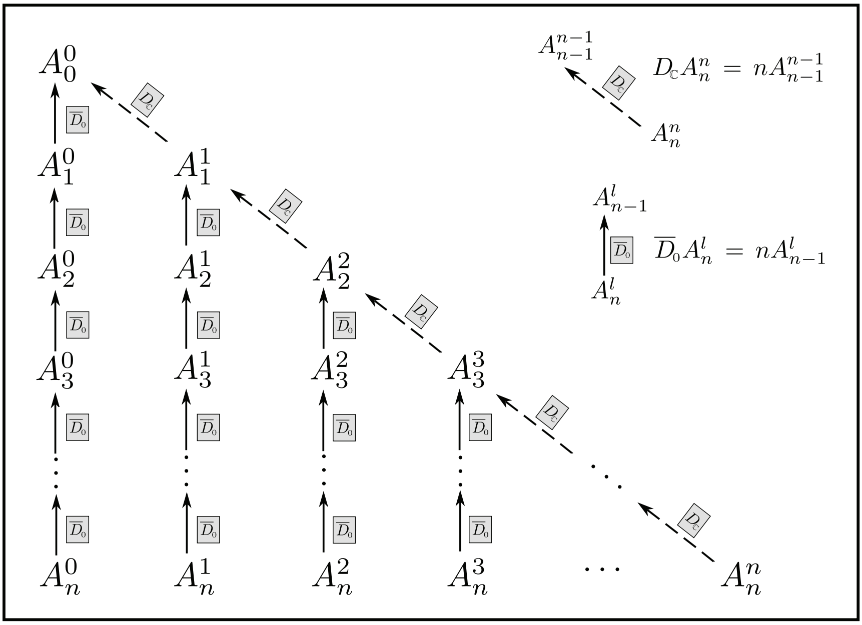

FIGURE 3. Structural properties of the orthogonal Appell basis A_n^l.

of the hypercomplex derivative. Thus a hyperholomorphic constant is analogously characterized as in the complex one-dimensional case by $f \in \ker \overline{D}_0 \cap \ker D$.

Secondly, Figure 3 further illustrates the action of the differential operators on the Appell basis (5.7). Precisely, the application of the hypercomplex derivative $\overline{D}_0$ to an arbitrary Appell polynomial A_n^l causes a shift of the degree in a fixed column l whereas the application of the lower dimensional (complex) derivative $D_{\mathbb{C}}$ causes a shift of the degree as well as a shift of the column. Here, it should be emphasized that the action of the differential operator $D_{\mathbb{C}}$ is restricted to the set of hyperholomorphic constants and thus, referring to Figure 3, maps along the upper diagonal. As a consequence, one can conclude that for an arbitrary Appell polynomial A_n^l, $l = 0, \dots, n$, $n \in \mathbb{N}_0$ of the system (5.7), first the $(n-l)$-fold application of $\overline{D}_0$ and afterwards the l-fold application of $D_{\mathbb{C}}$ yields

$$D_{\mathbb{C}}^{\,l}\, \overline{D}_0^{\,n-l}\, A_n^l = n!\,.$$

This property essentially enables the definition of a new Taylor series expansion (see Section 6) in terms of the Appell set (5.7) first introduced in [6], [7]. Finally, it is easy to see that the system from Theorem 5.4 satisfies the conditions (i), (ii) and (iii) of Theorem 5.1. Hence using the GT approach and Theorem 5.1 based on it, it is possible to show that $g_j^k = A_k^{k-j}$ for all k, j.

6. Orthogonal power series expansions

In view of practical applications of the basis, in [6], [7], the latter basis in particular was used to define a new Taylor series expansion which is a direct consequence of the Appell property of the basis:

Definition 6.1 (Taylor series in $L_2(\mathbb{B}_3,\mathbb{H})\cap\ker D$). Let $f\in L_2(\mathbb{B}_3,\mathbb{H})\cap\ker D$. The series representation

$$f := \sum_{n=0}^{\infty}\sum_{l=0}^{n} A_n^l \mathbf{t}_{n,l}, \quad \text{with} \quad \mathbf{t}_{n,l} = \frac{1}{n!} D_{\mathbb{C}}^{\,l}\, \overline{D}_0^{\,n-l} f(y)\Big|_{y=0} \tag{6.1}$$

is called the generalized Taylor series in $L_2(\mathbb{B}_3,\mathbb{H})\cap\ker D$. The notations $\overline{D}_0^{\,k}$ and $D_{\mathbb{C}}^{\,k}$ indicate the k-fold application of the corresponding differential operators ($k\in\mathbb{N}$) and the corresponding identity operator ($k=0$), respectively.

We observe that the Taylor coefficients are given by successive applications of the hypercomplex derivative $\overline{D}_0$ to the principal part of the monogenic function and the "complex" derivative $D_{\mathbb{C}}$ to the "constant" part (the subset of hyperholomorphic constants) of the monogenic function. This Taylor series expansion meets exactly the concept of hypercomplex derivability and improves Fueter's approach which is based on partial derivatives with respect to the real variables x_1 and x_2.

Similarly, in case of spinor valued functions, using again the Appell property of the corresponding GT basis (see Remark 4.5 at the end of Section 4), we can define the following Taylor series expansion:

Definition 6.2 (Taylor series in $L_2(\mathbb{B}_3,S)\cap\ker\partial$). Let $f\in L_2(\mathbb{B}_3,S)\cap\ker\partial$. The series representation

$$f = \sum_{k=0}^{\infty}\sum_{j=0}^{2k+1} \mathbf{t}_j^k\, \hat{f}_j^k \tag{6.2}$$

with the complex coefficients $\mathbf{t}_j^k$ such that

$$\mathbf{t}_j^k\, v^+ = \frac{1}{k!}\,\frac{\partial^k f(x)}{\partial x_3^j\,\partial\overline{z}^{k-j}}\Big|_{x=0} \quad \text{for} \quad j=0,\dots,k;$$

$$\mathbf{t}_j^k\, v^- = \frac{1}{k!}\,\frac{\partial^k f(x)}{\partial x_3^{2k+1-j}\,\partial z^{j-k-1}}\Big|_{x=0} \quad \text{for} \quad j=k+1,\dots,2k+1.$$

is called the generalized Taylor series in $L_2(\mathbb{B}_3,S)\cap\ker\partial$.

Let us note that the partial derivatives $\partial/\partial x_3$, $\partial/\partial z$ and $\partial/\partial\overline{z}$ commute with each other.

It is interesting to compare the Taylor series from Definitions 6.1 and 6.2. In both cases, the basis is orthogonal and the corresponding coefficients can be expressed using (linear combinations of) partial derivatives of the corresponding function. The derivatives used in the two cases look different but they are trivially related (at least for monogenic functions) to each other. In the formulation using spinor valued functions, the Appell property is true even with respect to all three variables. Hence in this case application of any of the three basic derivatives maps any basis element to a multiple of another basis element. For quaternion valued functions, this is not the case.

Applying a simple normalization (see, i.e., [6], [7]) to each element (5.7) of the Appell basis, explicitly given by the relation

$$(6.3)\quad \varphi^l_{n,\mathbb{H}} = \frac{1}{2^{l+1}\, n!}\sqrt{\frac{(2n+3)\,(n-l)!\,(n+l+1)!}{\pi}}\, A^l_n, \qquad l = 0,\ldots,n,\ n\in\mathbb{N}_0,$$

yields directly:

Corollary 6.3 ([6], [7]). *The system of inner solid spherical monogenics*

$$(6.4)\qquad \left\{\varphi^l_{n,\mathbb{H}} \,:\, l = 0,\ldots,n\right\}_{n\in\mathbb{N}_0}$$

is an orthonormal basis in $L_2(\mathbb{B}_3,\mathbb{H}) \cap \ker D$.

Using the orthogonality and the completeness of the orthonormal system (6.4), we can state the Fourier series expansion in $L_2(\mathbb{B}_3,\mathbb{H}) \cap \ker D$.

Corollary 6.4 (Fourier series in $L_2(\mathbb{B}_3,\mathbb{H}) \cap \ker D$). *Let* $f \in L_2(\mathbb{B}_3,\mathbb{H}) \cap \ker D$. *Then* f *can be uniquely represented in terms of the orthonormal system* (6.4), *that is:*

$$(6.5)\qquad f := \sum_{n=0}^{\infty}\sum_{l=0}^{n} \varphi^l_{n,\mathbb{H}}\, \boldsymbol{\alpha}_{n,l}, \quad \textit{with} \quad \boldsymbol{\alpha}_{n,l} = \int_{\mathbb{B}_3} \overline{\varphi^l_{n,\mathbb{H}}}\, f\, d\lambda^3.$$

Here it should be emphasized that in contrast to the complex case the order of $\varphi^l_{n,\mathbb{H}}$ and f in the inner products has to be respected. As a direct consequence of relation (6.3) and the orthogonality of both series expansions, each Fourier coefficient (6.5) of a function $f \in L_2(\mathbb{B}_3,\mathbb{H}) \cap \ker D$ can be explicitly expressed in terms of the corresponding Taylor coefficient (6.1) and vice versa by

$$\boldsymbol{\alpha}_{n,l} = 2^{l+1}\sqrt{\frac{\pi}{(2n+3)\,(n-l)!\,(n+l+1)!}}\, D^l_{\mathbb{C}}\,\overline{D}_0^{\,n-l}\, f(\mathbf{x})\Big|_{\mathbf{x}=\mathbf{0}},$$

where $l = 0,\ldots,n$ and $n \in \mathbb{N}_0$. This important analytic property of the series expansions is analogous to the complex one-dimensional case.

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Sebastian Bock: Institute of Mathematics and Physics, Bauhaus University, 99423 Weimar, Germany.
E-mail: sebastian.bock@uni-weimar.de

Klaus Gürlebeck: Institute of Mathematics and Physics, Bauhaus University, 99423 Weimar, Germany.
E-mail: klaus.guerlebeck@uni-weimar.de

Roman Lávička: Mathematical Institute, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic.
E-mail: lavicka@karlin.mff.cuni.cz

Vladimír Souček: Mathematical Institute, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic.
E-mail: soucek@karlin.mff.cuni.cz

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