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Diagonalizing "Compact" Operators on Hilbert W*-Modules

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Abstract. For W*-algebras A and self-dual Hilbert A-modules $\mathcal{M}$ we show that every self-adjoint, "compact" module operator on $\mathcal{M}$ is diagonalizable. Some specific properties of the eigenvalues and of the eigenvectors are described.

Keywords: *Diagonalization of "compact" operators, Hilbert W*-modules, W*-algebras, eigenvalues, eigenvectors*

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The goal of the present short note is to consider self-adjoint, "compact" module operators on self-dual Hilbert W*-modules (which can be supposed to possess a countably generated W*-predual Hilbert W*-module, in general) with respect to their diagonalizability. Some special properties of their eigenvalues and eigenvectors are described.

A partial result in this direction was recently obtained by V. M. Manuilov [10, 11] who proved that every such operator on the standard countably generated Hilbert W*-module $l_2(A)$ over *finite* W*-algebras A can be diagonalized on the respective A-dual Hilbert A-module $l_2(A)'$. The same was shown to be true for every self-adjoint bounded module operator on finitely generated Hilbert C*-modules over general W*-algebras by R. V. Kadison [5 - 7] and over commutative AW^*-algebras by K. Grove and G. K. Pedersen [4] sometimes earlier. M. Frank has made an attempt to find a generalized version of the Weyl-Berg theorem in the $l_2(A)'$ setting for some (abelian) monotone complete C*-algebras which should satisfy an additional condition, as well as a counterexample (cf. [2]). Further results on generalizations of the Weyl-von Neumann-Berg theorem can be found, e.g., in papers of G. J. Murphy [12], S. Zhang [15, 16] and H. Lin [9].

We go on to investigate situations where *non-finite* W*-algebras appear as coefficients of the special Hilbert W*-modules under consideration (Proposition 5), and where arbitrary self-dual Hilbert W*-modules are considered (Theorem 9). The applied techniques are rather different from that in [10, 11]. By the way, the results of V. M. Manuilov in [10, 11] are obtained to be valid for arbitrary self-adjoint, "compact" module operators on the self-dual Hilbert A-module $l_2(A)'$ over finite W*-algebras (Proposition 3). This generalizes [10] since in the situation of finite W*-algebras A the set of "compact" operators on $l_2(A)$ may be definitely smaller than that on $l_2(A)'$, and the latter

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may not contain all bounded module operators on $l_2(A)'$, in general. We characterize the role of self-duality for getting adequate results in the finite W*-case (Proposition 4). The final result of our investigations is Theorem 9 describing the diagonalizability of "compact" operators on self-dual Hilbert W*-modules in great generality.

We consider Hilbert W*-modules $\{\mathcal{M}, \langle \cdot, \cdot \rangle\}$ over general W*-algebras A , i.e. (left) A-modules $\mathcal{M}$ together with an A-valued inner product $\langle \cdot, \cdot \rangle : \mathcal{M} \times \mathcal{M} \to A$ satisfying the following conditions:

(i) $\langle x, x \rangle \geq 0$ for every $x \in \mathcal{M}$

(ii) $\langle x, x \rangle = 0$ if and only if $x = 0$

(iii) $\langle x, y \rangle = \langle y, x \rangle^*$ for any $x, y \in \mathcal{M}$

(iv) $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$ for any $a, b \in A$ and $x, y, z \in \mathcal{M}$

(v) $\mathcal{M}$ is complete with respect to the norm $\|x\| = \|\langle x, x \rangle\|_A^{1/2}$.

We always suppose that the linear structures of the W*-algebra A and of the (left) A-module $\mathcal{M}$ are compatible, i.e. $\lambda(ax) = (\lambda a)x = a(\lambda x)$ for every $\lambda \in \mathbf{C}$, $a \in A$ and $x \in \mathcal{M}$.

Let us denote the A-dual Banach A-module of a Hilbert A-module $\{\mathcal{M}, \langle \cdot, \cdot \rangle\}$ by

$$\mathcal{M}' = \Big\{ r : \mathcal{M} \to A |\ r \text{ is } A-\text{linear and bounded} \Big\}.$$

Hilbert W*-modules have some very nice properties in contrast to general Hilbert C*-modules. First of all, the A-valued inner product can always be lifted to an A-valued inner product on the A-dual Hilbert A-module $\mathcal{M}'$ via the canonical embedding of $\mathcal{M}$ into $\mathcal{M}'$, $x \to \langle \cdot, x \rangle$, turning $\mathcal{M}'$ into a (left) self-dual Hilbert A-module, $(\mathcal{M}' = (\mathcal{M}')')$. Moreover, one has the following criterion on self-duality.

Proposition 1 (see [1: Theorem 3.2]). *Let A be a W*-algebra and $\{\mathcal{M}, \langle \cdot, \cdot \rangle\}$ be a Hilbert A-module. Then the following conditions are equivalent:*

(i) *$\mathcal{M}$ is self-dual.*

(ii) *The unit ball of $\mathcal{M}$ is complete with respect to the topology τ_1 induced by the semi-norms $\{f(\langle \cdot, \cdot \rangle)^{1/2}\}$ on $\mathcal{M}$, where f runs over the normal states of A.*

(iii) *The unit ball of $\mathcal{M}$ is complete with respect to the topology τ_2 induced by the linear functionals $\{f(\langle \cdot, x \rangle)\}$ on $\mathcal{M}$ where f runs over the normal states of A and x over $\mathcal{M}$.*

Furthermore, on self-dual Hilbert W*-modules every bounded module operator has an adjoint, and the Banach algebra of all bounded module operators is actually a W*-algebra. And last but not least, every bounded module operator on a Hilbert W*-module $\{\mathcal{M}, \langle \cdot, \cdot \rangle\}$ can be continued to a unique bounded module operator on its A-dual Hilbert W*-module $\mathcal{M}'$ preserving the operator norm. (Cf. [13].)

We want to consider (self-adjoint) "compact" module operators on Hilbert W*-modules. By G. G. Kasparov [8] an A-linear bounded module operator K on a Hilbert

A-module $\{\mathcal{M}, \langle\cdot,\cdot\rangle\}$ is "compact" if it belongs to the norm-closed linear hull of the elementary operators

$$\left\{\theta_{x,y} : \theta_{x,y}(z) = \langle z, x\rangle y \ \ (x, y \in \mathcal{M})\right\}.$$

The set of all "compact" operators on $\mathcal{M}$ is denoted by $K_A(\mathcal{M})$. By [13: Theorem 15.4.2] the C*-algebra $K_A(\mathcal{M})$ is a two-sided ideal of the set of all bounded, adjointable module operators $\mathrm{End}_A^*(\mathcal{M})$ on $\mathcal{M}$, and both these sets coincide if and only if $\mathcal{M}$ is algebraically finitely generated as an A-module (cf. also [3: Appendix]). This will be used below. Since we are going to investigate single "compact" operators we make the useful observation that both the range of a given "compact" operator and the support of it are Hilbert C*-modules generated by countably many elements with respect to the norm topology or at least with respect to the τ_1-topology (cf. Proposition 1). Hence, without loss of generality we can restrict our attention to countably generated Hilbert W*-modules and their W*-dual Hilbert W*-modules.

We are especially interested in the Hilbert W*-module

$$l_2(A) = \left\{ \{a_i\}_{i\in\mathbf{N}} \subset A \,\middle|\, \sum_{i=1}^{\infty} a_i a_i^* \text{ converges with respect to } \|\cdot\|_A \right\}$$

$$\langle\{a_i\},\{b_i\}\rangle = \|\cdot\|_A\text{-}\lim_{N\in\mathbf{N}} \sum_{i=1}^{N} a_i b_i^*$$

and in its A-dual Hilbert W*-module

$$l_2(A)' = \left\{ \{a_i\}_{i\in\mathbf{N}} \subset A \,\middle|\, \sup_{N\in\mathbf{N}} \left\| \sum_{i=1}^{N} a_i a_i^* \right\| < \infty \right\}$$

$$\langle\{a_i\},\{b_i\}\rangle = \mathrm{w}^*\text{-}\lim_{N\in\mathbf{N}} \sum_{i=1}^{N} a_i b_i^*$$

because of G. G. Kasparov's stabilization theorem [8], stating that every countably generated Hilbert C*-module over a unital C*-algebra A is a direct summand of $l_2(A)$.

Definition 2. Let A be a W*-algebra and let $\{\mathcal{M}, \langle\cdot,\cdot\rangle\}$ be a self-dual Hilbert A-module possessing a countably generated Hilbert A-module as its A-predual. A bounded module operator T on $\mathcal{M}$ is *diagonalizable* if there exists a sequence $\{x_i\}_{i\in\mathbf{N}} \subset \mathcal{M}$ of non-trivial elements such that the following conditions are fulfilled:

(i) $T(x_i) = \Lambda_i x_i$ for some elements $\Lambda_i \in A$.

(ii) The Hilbert A-submodule generated by $\{x_i\}_{i\in\mathbf{N}}$ inside $\mathcal{M}$ has a trivial orthogonal complement.

(iii) The elements of $\{x_i\}_{i\in\mathbf{N}}$ are pairwise orthogonal and $p_i = \langle x_i, x_i\rangle$ are projections in A.

(iv) The equality $\Lambda_i p_i = \Lambda_i$ holds for the projection p_i.

Note that the eigenvalues and the eigenvectors are not uniquely determined for the operator T since $T(x) = \Lambda x$ implies $T(y) = \Lambda' y$ for $\Lambda' = u\Lambda u^*$ and $y = ux$ for all unitaries $u \in A$. Moreover, the eigenvalues of T do not belong to the center $Z(A)$ of A, in general. Consequently, $T(ax) = a(\Lambda x) \neq \Lambda(ax)$, in general. That is, eigenvectors are often not one-to-one related to T-invariant A-submodules of the Hilbert A-module $\mathcal{M}$ under consideration.

Now, we start our investigations decomposing A into components of prescribed type with respect to its direct integral representation. Denote by p that central projection of A dividing A into a finite part pA and infinite part $(1-p)A$. That means, with respect to the direct integral decomposition of A the fibers are almost everywhere factors of type I_n $(n < \infty)$ or II_1 inside pA and almost everywhere factors of type I_∞ or II_∞ or III inside $(1-p)A$. Analogously, the Hilbert A-module $l_2(A)$ decomposes into the direct sum of two Hilbert A-modules $l_2(A) = l_2(pA) \oplus l_2((1-p)A)$, and every bounded A-linear operator T on $l_2(A)$ splits into the direct sum $T = pT \oplus (1-p)T$, where each part acts only on the respective part of the Hilbert A-module non-trivially and at the same time as an A-linear operator.

Consequently, we can proceed considering W*-algebras A of coefficients of prescribed type. Our first goal is to revise the case of finite W*-algebras investigated by V. M. Manuilov. There the set $K_A(l_2(A)')$ does not coincide with the set $\mathrm{End}_A(l_2(A)')$, and there are always self-adjoint, bounded module operators T on $l_2(A)'$ which can not be diagonalized. For example, consider a self-adjoint, bounded linear operator T_0 on a separable Hilbert space H being non-diagonalizable (cf. Weyl's theorem). Using the decomposition $l_2(A) = \overline{A \otimes H}$ one obtains a self-adjoint, bounded module operator T on $l_2(A)$ by the formula $T(a \otimes h) = a \otimes T_0(h)$ $(a \in A, h \in H)$. The operator T extends to an operator on $l_2(A)'$, and T can not be diagonalizable by assumption. Surprisingly, V. M. Manuilov proved that every self-adjoint, "compact" operator on the standard countably generated Hilbert W*-module $l_2(A)$ over finite W*-algebras A can be diagonalized on the respective A-dual Hilbert A-module $l_2(A)'$. A careful study of his detailed proofs at [10, 11] brings to light that for finite W*-algebras with infinite center the continuation of the "compact" operators to the respective A-dual Hilbert A-module is not only a proof-technical necessity, but it is of principal character. Self-duality has to be supposed to warrant the diagonalizability of all self-adjoint "compact" module operators on $\mathcal{M} \subseteq l_2(A)'$ in the finite case, and the key steps of the proof can be repeated one-to-one. Consequently, we give the generalized formulation of V. M. Manuilov's diagonalization theorem for the finite case, and we show additionally that self-duality is an essential property of Hilbert W*-modules for finding a (well-behaved) diagonalization of arbitrary "compact" module operators on them, in general.

Proposition 3 (cf. [10] and [11: Theorem 4.1]). *Let A be a W*-algebra of finite type. Then every self-adjoint, "compact" module operator K on $l_2(A)'$ is diagonalizable. The sequence of eigenvalues $\{\Lambda_n\}_{n\in\mathbb{N}}$ of K has the property $\lim_{n\to\infty} \|\Lambda_n\| = 0$. The eigenvalues Λ_n can be chosen in such a way that $\Lambda_2 \leq \Lambda_4 \leq \ldots \leq 0 \leq \ldots \leq \Lambda_3 \leq \Lambda_1$. Moreover, for positive operators K without kernel the eigenvectors x_n may possess the property $\langle x_n, x_n\rangle = 1_A$, in addition.*

For the detailed (but extended) proof of this proposition see [11] (see also [10]). The proving technique relies mainly on spectral decomposition theory of operators and on the center-valued trace on the finite W*-algebra A.

Proposition 4. *Let A be a finite W^*-algebra with infinite center. Consider a Hilbert A-module $\mathcal{M}$ such that $l_2(A) \subset \mathcal{M} \subseteq l_2(A)'$. Then the following two statements are equivalent.*

(i) $\mathcal{M} = l_2(A)'$, *i.e. $\mathcal{M}$ is self-dual.*

(ii) *Every positive "compact" module operator is diagonalizable inside $\mathcal{M}$ with eigenvalues being comparable inside the positive cone of A.*

Proof. Note that $l_2(A) \neq l_2(A)'$ by assumption. Denote the standard orthonormal basis of $l_2(A)$ by $\{e_n\}_{n\in N}$. If the center of A is supposed to be infinite dimensional, then one finds a sequence of pairwise orthogonal non-trivial projections $\{p_n\}_{n\in N} \subset Z(A)$ summing up to 1_A in the sense of w*-convergence. Fix a sequence of positive non-zero numbers $\{\alpha_n\}_{n\in N}$ monotonically converging to zero. The bounded module operator K defined by

$$K(e_1) = \left(\sum_{n=1}^{\infty} \alpha_n p_n e_n \right), \qquad K(e_j) = \alpha_j p_j e_1 \quad \text{for } j \neq 1$$

is a "compact" operator on $l_2(A)$. It can be easily continued to a "compact" operator on $\mathcal{M}$. As an exercise one checks that the eigenvalues of K are $\alpha_1 p_1, \alpha_2 p_2, \cdots, 0, \cdots, -\alpha_2 p_2$ (ordering by sign and norm and taking into account conditions (iii) and (iv) of Definition 2), and that the appropriate eigenvectors are

$$p_1 e_1, \frac{1}{\sqrt{2}} p_2(e_1 + e_2), \frac{1}{\sqrt{2}} p_3(e_1 + e_3), \cdots,$$
$$\{(1_A - p_n)e_n\}_{n\in N}, \cdots, \frac{1}{\sqrt{2}} p_3(e_1 - e_3), \frac{1}{\sqrt{2}} p_2(e_1 - e_2).$$

The only way of making the eigenvalues comparable inside the positive cone of A preserving Definition 2/(iii)-(iv) is to sum up the positive and the negative eigenvalues separately. But then the resulting eigenvector

$$x = \left(1_A + \left(1 + \frac{1}{\sqrt{2}} \right) (1_A - p_1), \frac{1}{\sqrt{2}} p_2, \frac{1}{\sqrt{2}} p_3, \cdots, \frac{1}{\sqrt{2}} p_n, \cdots \right)$$

corresponding to the only positive eigenvalue $\sum_{n=1}^{\infty} \alpha_n p_n$ of K does not belong to $\mathcal{M}$ any longer by assumption. This shows one implication. The converse implication follows from Proposition 6 ∎

The second big step is to investigate the case of infinite W*-algebras as coefficients of the Hilbert W*-modules under consideration. The result is characteristic for the situation in self-dual Hilbert W*-modules over infinite W*-algebras, and quite different from that in the finite W*-case, and elsemore, from the classical Hilbert space situation.

Proposition 5. *Let A be a W^*-algebra which possesses infinitely many pairwise orthogonal, non-trivial projections p_i $(i \in \mathbf{N})$ equivalent to 1_A and summing up to 1_A in the sense of w^*-convergence of the sum $\sum_i p_i = 1_A$. Then the Hilbert A-module $l_2(A)'$ equipped with its standard A-valued inner product is isomorphic to the Hilbert A-module $\{A, \langle \cdot, \cdot \rangle_A\}$, where $\langle a, b \rangle_A = ab^*$.*

Proof. Suppose, the equivalence of the projections p_i $(i \in \mathbf{N})$ with 1_A is realized by partial isometries $u_i :$ $p_i = u_i u_i^*$ and $1_A = u_i^* u_i$. Then the mapping

$$S : l_2(A)' \to A, \qquad \{a_i\} \to \mathrm{w}^* - \lim \sum_{\substack{\text{finite number} \\ \text{of summands}}} a_i u_i^*$$

with the inverse mapping

$$S^{-1} : A \to l_2(A)', \qquad a \to \{au_i\}$$

realizes the isomorphism of $l_2(A)'$ and A as Hilbert A-modules due to Proposition 1 ∎

Corollary 6. *Let A be a W^*-algebra of infinite type. Then every bounded module operator T on $l_2(A)'$ is diagonalizable, and the formula*

$$T(\{a_i\}) = \langle \{a_i\}, \{u_i\} \rangle \Lambda_T \{u_i\}$$

holds for every $\{a_i\} \in l_2(A)'$, some $\Lambda_T \in A$ and the partial isometries $u_i \in A$ described in the previous proof.

Proof. Every W^*-algebra of type I_∞, II_∞ or III possesses a set of partial isometries with properties described in Proposition 3. The same is true for W^*-algebras consisting only of parts of these types. Now, translate the operator T on $l_2(A)'$ into an operator STS^{-1} on A and vice versa using Proposition 3, and take into account that every bounded module operator on A is a multiplication operator with a concrete element (from the right) ∎

Corollary 7. *Let A be a W^*-algebra without any fibers of type I_n $(n < \infty)$ and II_1 in its direct integral decomposition. Let $\mathcal{M}$ be a self-dual Hilbert A-module possessing a countably generated A-predual Hilbert A-module. Then every bounded module operator T on $\mathcal{M}$ is diagonalizable, and the formula*

$$T(x) = \langle x, u \rangle \Lambda_T u$$

holds for every $x \in \mathcal{M}$, some $\Lambda_T \in A$ and an eigenvector $u \in \mathcal{M}$ being universal for all T.

Proof. Since $\mathcal{M}$ has a countably Hilbert A-module as its A-predual, $\mathcal{M}$ is a direct summand of the Hilbert A-module $l_2(A)'$ by G. G. Kasparov's stabilization theorem [8]. Hence, one has to show the assertion for the self-dual Hilbert A-module $l_2(A)'$ only. For further use denote the projection from $l_2(A)'$ onto $\mathcal{M}$ by P. Consider the direct integral decomposition of A over its center. Therein every fiber is a W^*-factor of type I_∞, II_∞ or III by assumption. Putting it into the $l_2(A)'$-context one obtains that A is isomorphic to $l_2(A)'$ either applying Corollary 6 fiberwise or constructing a suitable set of partial isometries $u_i \in A$ to make use of Proposition 5. Then in the same way as there the diagonalization result turns out for arbitrary bounded module operators T on $l_2(A)'$. To get the formula $T(x) = \langle x, u \rangle \Lambda_T u$ one has only to set $u = P(\{u_i\})$ ∎

Remark. Let A be an I_∞-factor, for example. Then there are self-adjoint elements Λ_T in A which can not be diagonalized in a stronger sense. More precisely, there is no way of representing any such operator as a sum $\sum \lambda_i P_i$ with $\lambda_i \in \mathbf{C} = Z(A)$ and $P_i = P_i^* = P_i^2 \in A$ because of Weyl's theorem. Therefore, the Corollaries 6 and 7 are the strongest results one could expect.

Example 8. Consider the C*-algebra A of all 2×2-matrices on the set of complex numbers. Set $\mathcal{M} = A^2$ with the usual A-valued inner product. Consider the ("compact") bounded module operator $K = \theta_{x,x} + \theta_{y,y}$ for

$$x = \left(\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) \quad \text{and} \quad y = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right).$$

Eigenvectors of K are $x, y \in A^2$, for example, and the respective eigenvalues are

$$\Lambda_x = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix} \quad \text{and} \quad \Lambda_y = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}.$$

Remark that one can not compare these eigenvalues as elements of the positive cone of A. But, making another choice one arrives at that situation described in Proposition 6:

$$x_1 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \quad \text{and} \quad x_2 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right).$$

Then the respective eigenvalues are

$$\Lambda_1 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \quad \text{and} \quad \Lambda_2 = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}$$

and they can be ordered, as well as the eigenvectors x_1, x_2 are units. Last but not least, dropping out condition (iv) of Definition 2 one can correlate K-invariant submodules of $\mathcal{M}$ and eigenvectors of K. Simply, set

$$x_1 = \left(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right) \quad \text{and} \quad x_2 = \left(\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right).$$

In this case the corresponding eigenvalues are

$$\Lambda_1 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \quad \text{and} \quad \Lambda_2 = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}.$$

They can be ordered in the positive cone of A. But, the eigenvectors corresponding to the K-invariant submodules of $\mathcal{M}$ can not be selected to be units any longer.

Theorem 9. *Let A be a W^*-algebra and $\mathcal{M}$ be a self-dual Hilbert A-module. Then every self-adjoint, "compact"-module operator on $\mathcal{M}$ is diagonalizable. The sequence of eigenvalues $\{\Lambda_n\}_{n \in \mathbf{N}}$ of K has the property $\lim_{n\to\infty} \|\Lambda_n\| = 0$. The eigenvalues Λ_n $(n \in \mathbf{N})$ of K can be chosen in such a way that $\Lambda_2 \leq \Lambda_4 \leq \ldots \leq 0 \leq \ldots \leq \Lambda_3 \leq \Lambda_1$, and that Λ_n $(n \geq 3)$ are contained in the finite part of A.*

Proof. Both the τ_1-closure of the range and of the support of K are self-dual Hilbert C*-modules possessing countably generated A-predual Hilbert A-modules because of the "compact"ness of K. Hence, without loss of generality one can restrict the attention to self-dual Hilbert W*-modules with countably generated W*-predual Hilbert W*-modules formed as the τ_1-completed direct sum of range and support of K. As usual, on the kernel of K one has the eigenvalue zero and a suitable system of eigenvectors.

Now, gluing Corollary 4 and Proposition 6 together the theorem turns out to be true in the special case $\mathcal{M} = l_2(A)'$ (cf. the remarks at the beginning of the present note). The only loss may be that the eigenvectors are not units, in general. Because of G. G. Kasparov's stabilization theorem [8] $\mathcal{M}$ possesses an embedding into $l_2(A)'$ as a direct summand by assumption. Therefore, every self-adjoint, "compact" module operator K on $\mathcal{M}$ can be continued to a unique such operator on $l_2(A)'$ preserving the norm, simply applying the rule $K|_{\mathcal{M}^\perp} = 0$. The eigenvectors of this extension are elements of $\mathcal{M}$. The Hilbert A-module $\mathcal{M}^\perp$ belongs to its kernel. This shows the theorem ∎

Remark. For commutative AW*-algebras A Theorem 9 is still true by [4]. The general AW*-case is open at present because of two crucial unsolved problems in the AW*-theory:

(i) Are the self-adjoint elements of $M_n(A)$ $(n \geq 2)$ diagonalizable for arbitrary (monotone complete) AW*-algebras A, or not?

(ii) Does every finite (monotone complete) AW*-algebra possess a center-valued trace, or not?

Remark. One can extend the statements of Theorem 9 to the case of normal, "compact" module operators dropping out only the ordering of the eigenvalues. To see this note that for normal elements K of the C*-algebra $K_A(\mathcal{M})$ there exists always a self-adjoint element $K' \in K_A(\mathcal{M})$ such that K is contained in the C*-subalgebra of $\mathrm{End}_A(\mathcal{M})$ generated by K' and the identity operator. Applying functional calculus inside the W*-algebra $\mathrm{End}_A(\mathcal{M})$ the result turns out. Beside this, it would be interesting to investigate some more general variants of the Weyl-von Neumann-Berg theorem for appropriate bounded module operators on (self-dual) Hilbert W*-submodules over (finite) W*-algebras A as those obtained by H. Lin, G. J. Murphy and S. Zhang.

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